Proportional and Derivative State-Feedback Decoupling of Linear Systems
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Abstract—We consider here the row-by-row decoupling of linear time-invariant systems by proportional and derivative state feedback. Our contribution, with respect to previous results, is that our procedure is based on simple operator (say matrices) manipulations, without any need to use a canonical form. The only assumptions for applying such a decoupling strategy are that the system is right invertible (which is a necessary condition to ensure solvability) and minimum phase. An illustrative example is proposed.

Index Terms—Decoupling, geometric approach, implicit systems, linear system theory, proportional and derivative feedback.

I. INTRODUCTION

Initiated by works on the proportional state-feedback decoupling (see, for instance, [7]), a quite rich literature on various versions of that control problem evolved. We consider here proportional and derivative state-feedback laws. Before entering the details of our contribution, without recalling all of the obvious merits of a decoupling strategy (SISO systems are easier to control), let us just mention [16] (and references therein) where decoupling has been shown to impose no additional performance limitations, as compared with a synthesis without decoupling. Also note that, although the compensation scheme in [16] (unity output feedback) is different from the one chosen here, the authors there also accept nonproper compensators for achieving decoupling and good performances (in terms of measures involving the sensitivity and complementary sensitivity functions).

Let us consider the linear time-invariant system:

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)
\]

(1)

where

- \(x(t)\) state
- \(u(t)\) input
- \(y(t)\) output.

A: \( \mathcal{X} \to \mathcal{X} \), B: \( \mathcal{U} \to \mathcal{X} \), and C: \( \mathcal{X} \to \mathcal{Y} \) are linear operators, with \( \dim \mathcal{X} = n \), \( \dim \mathcal{U} = m \), and \( \dim \mathcal{Y} = p \), respectively.

For (1), Tan and Vandewalle proposed in [20] a proportional and derivative state feedback

\[
u(t) = F_\alpha \dot{x}(t) + F_\beta x(t) + r(t)
\]

(2)

such that the output of the system exactly matches its reference, namely

\[
y(t) = r(t).\tag{3}
\]

This PD feedback was found assuming, first, that (1) is a controllable and square invertible system, carrying, next, the system into its \(m\)-block diagonal Brunovsky canonical form [6], and proposing, finally, the PD state feedback

\[
u_s(t) = \dot{x}(t) + c_i^T \gamma x(t)\tag{4}
\]

(4)

where the \(c_i^T\) are the rows of \(C\). Following the same procedure of Tan and Vandewalle, Malabre and Velasquez [18] relaxed the controllability assumption, and replaced the square invertibility by a right-invertibility assumption.

But in those papers, a fundamental question arising when dealing with nonproper control laws was not considered, namely, the solvability, which is the existence and unicity of state solutions for the closed-loop system [2], [13]. Since a PD control law is a nonproper transformation, it has to be verified that the closed-loop system remains solvable [15] in order to avoid pathological situations, such as, for instance, (3) satisfied, but only for \(r(t) \equiv 0\). As an example, if, for the following system,

\[
\dot{x}_1(t) = 0, \quad \dot{x}_2(t) = u(t), \quad y(t) = x_1(t)
\]

(5)

we propose the following control law

\[
u(t) = \dot{x}_2(t) - y(t) + r(t)
\]

(6)

we exactly obtain from the second part of (4) and (5) that \(y(t) = r(t)\). But from the last equation [the first and third parts of (4)], we must satisfy \(r(t) = 0\) with \(r(0) = y(0)\), namely, the closed-loop system is not solvable (cannot accept any reference input).

The aim of this paper is to propose conditions which guarantee solvability on the generalization of the PD control law of Tan and Vandewalle, without using any canonical form.

Before ending this introductory section, let us recall some useful results.

Let \(\mathcal{V}^*\) and \(\mathcal{S}_r\) denote the supremal \((A, B)\) invariant subspace contained in \(\ker C\) and the infimal \((C, A)\) invariant subspace containing \(\text{Im} B\), respectively, namely (see [19]),

\[
\begin{align*}
\mathcal{V}^* := & \sup \{ \mathcal{T} \subseteq \ker C \mid \exists F \text{ s.t. } (A + BF)\mathcal{T} \subseteq \mathcal{T} \} \\
\mathcal{S}_r := & \inf \{ \mathcal{T} \cap \text{Im} B \mid \exists K \text{ s.t. } (A + KC)\mathcal{T} \subseteq \mathcal{T} \}
\end{align*}
\]

(7)

where the nonunique \(F\) and \(K\) which make \(\mathcal{V}^*\) and \(\mathcal{S}_r\) invariant are called friends of \(\mathcal{V}^*\) and \(\mathcal{S}_r\), respectively, and are identified by \(F \in \mathcal{F}(A, B; \mathcal{V}^*)\) and \(K \in \mathcal{F}(C, A; \mathcal{S}_r)\).

Lemma 1—Lebret and Loiseau [14]: The finite elementary divisors of the pencil,

\[
\begin{bmatrix}
\mathcal{H}(\lambda I - A) & -C \\
\end{bmatrix}
\]

are those of the map \(A + BF \parallel \mathcal{V}^*/(\mathcal{V}^* \cap \mathcal{S}_r)\), the map induced by the restriction of \((A + BF)\) to \(\mathcal{V}^*\) in the quotient space \(\mathcal{V}^*/(\mathcal{V}^* \cap \mathcal{S}_r)\), where \(\mathcal{H}: \mathcal{X} \to \mathcal{X}/\text{Im} B\) is the canonical projection and \(F\) is any element of \(\mathcal{F}(A, B; \mathcal{V}^*)\).

Lemma 2—Bonilla et al. [5]: The pencil

\[
\begin{bmatrix}
\mathcal{H}(\lambda I - A) & -C \\
\end{bmatrix}
\]

is monic, i.e., \(\ker \mathcal{H}(\lambda I - A) \cap \ker C = \{0\}\) when \(\mathcal{V}^* \cap \mathcal{S}_r = \{0\}\).

1In [5, Sect. 2], it is assumed that \(\mathcal{U} \approx \mathcal{Y}\) and \(\mathcal{V}^* + \mathcal{S}_r = \mathcal{X}\), which imply that \(\mathcal{V}^* \cap \mathcal{S}_r = \{0\}\). And, in fact, in the proof of Fact 3.1, only the condition \(\mathcal{V}^* + \mathcal{S}_r = \{0\}\) is used.
II. PD STATE FEEDBACK

The PD control law of Tan and Vandewalle can be generalized as follows:

\[ u(t) = L(P(\dot{x}(t) - Ax(t)) + M(Cx(t) - r(t))) \]  

(7)

where \( \lambda_0 \) being any complementary subspace of \( \text{Im } B \), i.e., \( X = \text{Im } B \oplus \lambda_0 \):  
\[ P : X \rightarrow \text{Im } B, \text{ natural projection along } \lambda_0 \]  
\[ L : \text{Im } B \rightarrow \mathcal{U}, \text{ isomorphism such that: } PBL = I \]  
\[ M : \text{Im } C \rightarrow \text{Im } B, \text{ any monic map (ker } M = \{0\}) \].

(8)

Indeed, applying the control law (7) to (1), we obtain

\[ \dot{x}(t) = Ax(t) + BL(P(\dot{x}(t) - Ax(t)) + M(Cx(t) - r(t))) \]

namely (just premultiply it by \( P \) and recall that \( PBL = I \) and \( \text{ker } M = \{0\})\),

\[ y(t) = r(t). \]

(9)

The advantage of this generalization is that the control law is defined in terms of operators directly deduced from the matrices describing the system \((A, B, C)\), avoiding a pass through the Brunovsky canonical form.

In order to study the solvability [8] of the closed-loop system, let us rewrite (7) as

\[
\begin{align*}
N_0 \dot{z}(t) &= \zeta(t) + \Gamma x(t) \\
u(t) &= -L((PA - MC)x(t) + Mr(t) - H \zeta(t))
\end{align*}
\]

(10)

where \( N_0 \) is a nilpotent operator, and \( \Gamma \) and \( H \) are maps such that \((\lambda \text{ is a complex number})

\[ H(\lambda N_0 - I)^{-1} \Gamma = \lambda P \]

namely,\(^2\)

\[ H \Gamma = 0, \quad H_0 \Gamma = -P, \quad H_0^{2j} \Gamma = 0 \quad \forall j > 1. \]

(11)

Let us now define the operator:

\[ \mathcal{N} : X \rightarrow X/\text{Im } B \quad \text{the canonical projection.} \]

(12)

Then, the closed-loop system (1), (10) can be described by the following implicit description (recall that \( PBL = I \) and \( \mathcal{N} B = 0 \)),

\[
\begin{bmatrix}
\dot{x} \\
\dot{\zeta}
\end{bmatrix} =
\begin{bmatrix}
\mathcal{N} A & 0 \\
0 & \mathcal{N} C
\end{bmatrix}
\begin{bmatrix}
x \\
\zeta
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix} r
\]

(13)

\[^2\]Just iteratively apply the inversion lemma [11].

where \([r^T(t) \ \zeta^T(t)]^T \in X \oplus \mathcal{Z} \). In a pencil setting, (13) can also be written as

\[
\begin{bmatrix}
\lambda E - H \ | \ B
\end{bmatrix} =
\begin{bmatrix}
\mathcal{N} (\lambda I - A) & 0 \\
\lambda P - MC & -\Gamma \\
0 & \lambda N_0 - I
\end{bmatrix}
\begin{bmatrix}
0 \\
(M)
\end{bmatrix}.
\]

(14)

Premultiplying (14) by the epic map

\[
L =
\begin{bmatrix}
I & 0 \\
0 & M^T
\end{bmatrix}
\begin{bmatrix}
M^T H(\lambda N_0 (\lambda N_0 - I)^{-1} - I) + H \Gamma = \lambda P, \text{ we get}
\end{bmatrix}
\]

\[
L [\lambda E - H \ | \ B] =
\begin{bmatrix}
\mathcal{N} (\lambda I - A) & 0 \\
-C & -\Gamma \\
0 & \lambda N_0 - I
\end{bmatrix}
\begin{bmatrix}
0 \\
(I)
\end{bmatrix}.
\]

(15)

We are now in a position to state the main contribution of this paper.

\textit{Theorem 1:} Let us suppose that system (1) satisfies the standard assumptions

\[ \ker B = \{0\}; \quad \text{Im } C = \mathcal{Y}; \quad \text{dim } \text{Im } C \leq \text{dim } \text{Im } B. \]

Then, the proportional and derivative state feedback law (7) completely decouples (i.e., decouples and leads to a closed-loop transfer function matrix equal to the identity) the linear system (1), guaranteeing solvability on the closed-loop system (i.e., the system has solution trajectories for every reference signal \( r(t) \) if and only if the system is right invertible, namely,

\[ \mathcal{V}^* + \mathcal{S}^* = X. \]

\textit{Proof of Theorem 1:}

2) Let us first note that, the solvability of (14) is equivalent to the solvability of the pencil

\[ \Lambda_0 (\lambda) = \begin{bmatrix} \mathcal{N} (\lambda I - A) & 0 \\ -C & -\Gamma \end{bmatrix} \]

(18)

mapping from the space \( X \oplus \mathcal{Y} \) to the space \( X/\text{Im } B \oplus \mathcal{Y} \).

3) Let us next decompose the state space \( X \) and the quotient space \( X/\text{Im } B \) as follows:

\[ X = X_0 \oplus \ker C; \quad X/\text{Im } B = \mathcal{X}_0 \oplus \mathcal{X}/(\lambda I - A) \ker C \]

(19)

where \( X_0 \) and \( \mathcal{X}_0 \) are some complementary subspaces. Then, based on (19), the pencil \( \Lambda_0 (\lambda) \) takes the following form:

\[ \Lambda_0 (\lambda) = \begin{bmatrix} \mathcal{X}_0 (\lambda) \\ \mathcal{X}_2 (\lambda) \mathcal{X}_0 (\lambda) \\ -C_0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \]

(20)

where \( C_0 \) is the restriction of the map \( C \) to \( X_0 \).

4) Now, pencil (20) is solvable if and only if \( \mathcal{X}_1 (\lambda) \equiv 0 \). Indeed, if this is not the case, the reference signal \( r(t) \) has to satisfy

\[ 0 \equiv \mathcal{X}_1 (\nu) C_0^{-1} r(t) \]

\[^3\nu \text{ is the nilpotent index of the map } N_0.\]
where $C_0^{-1}$ is the inverse map of $C_0$ and $p$ is the derivative operator $dt/dt$.

5) $\bar{X}_0(\lambda) \equiv 0$ if and only if $\bar{N}(\lambda I - A) \c 0 \leq \bar{N}(\lambda I - A) \c 0,$ namely, if and only if

$$\begin{align*}
(\lambda I - A) \c 0 + \text{Im} B &= (\lambda I - A) \c 0 + \text{Im} B.
\end{align*}
$$

6) Let us now compute the annihilators of the left-hand side of (21) and of the right invertibility condition [remember assumption (17)]:

$$\begin{align*}
((\lambda I - A) \c 0 + \text{Im} B)^\perp \\
&= (\lambda I - A)^{-1} \text{Im} C' \cap \ker B' \\
&= \ker \left[ \begin{array}{c}
\bar{N}'(\lambda I - A') \\
B'
\end{array} \right] \\
(\mathcal{V}')^\perp \cap (S_c)^\perp &= (\mathcal{V}' + S_c)^\perp = (\mathcal{X}')^\perp = \{0'\}
\end{align*}
$$

where $(A', B', C')$ are the dual maps of $(A, B, C)$ and $\bar{N}' : \mathcal{X}' \rightarrow \mathcal{V}' / \text{Im} C'$ is the canonical projection.

7) Hence, from Lemma 2, (22), and (23), we get $((\lambda I - A) \c 0 + \text{Im} B)^\perp = \{0'\}$, namely

$$\begin{align*}
(\lambda I - A) \c 0 + \text{Im} B &= \mathcal{X}
\end{align*}
$$

which implies

$$\begin{align*}
\mathcal{X} &= (\lambda I - A) \c 0 + \text{Im} B \subset (\lambda I - A) \c 0 + \text{Im} B \subset \mathcal{X}.
\end{align*}
$$

And thus, condition (21) is fulfilled, implying in this way the solvability of (14).

From this proof, we have the following immediate corollary.

**Corollary 1**: If, in addition to the conditions of Theorem 1, we assume that (1) is a minimum phase system, namely, spectrum $\{A + BF[\mathcal{V}' / (\mathcal{V}' + S_c)]\} \subset C^{-1}$ for all $F \in \text{F}(A, B; \mathcal{V}')$, then the closed-loop system has no unstable modes.

This corollary follows from (15) and Lemma 1.

Let us finish this section by considering the following illustrative example.

**Illustrative Example**: Let us consider the following system:

$$\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & c_3 & c_4 \end{bmatrix} x(t)
\end{align*}
$$

for which we assume that $c_1 c_4 - c_2 c_3 \neq 0$, $c_1 \neq 0$, and $c_3 \neq 0$. This system satisfies

$$\begin{align*}
\ker B &= \{0\}, \\
\text{Im} C &= \mathcal{X}, \\
\dim \text{Im} C &\leq \dim \text{Im} B \\
\mathcal{V}' &= \{0\}, \\
\text{and } S_c &= \mathcal{X}, \\
\text{i.e., } \mathcal{V}' + S_c &= \mathcal{X}.
\end{align*}
$$

So this system fulfills all of the assumptions (17), and is thus decouplable by PD state feedback.

Before illustrating it, let us note that (E1) is not decouplable by using proportional state feedback alone since the corresponding “Falb and Wolovich matrix” is not invertible (see [7]). Indeed, this matrix is equal to $CB = \begin{bmatrix} c_1 & c_0 \end{bmatrix}$. The maps needed for the control law can be taken as [see (7) and (8)]

$$\begin{align*}
P &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
L &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \\
M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{align*}
$$

obtaining the control law

$$\begin{align*}
u_1(t) &= \dot{x}_1(t) - \dot{x}_3(t) + ((c_1 - c_3) x_1(t) \\
+ (c_2 - c_4) x_2(t) - (r_1(t) - r_2(t)) \\
u_2(t) &= \dot{x}_3(t) + (c_3 x_1(t) + c_4 x_2(t)) - r_2(t).
\end{align*}
$$

Applying (E2) to (E1), we get $0 = (c_1 x_1(t) + c_2 x_2(t)) - r_1(t)$, $\dot{x}_2(t) = x_3(t), 0 = (c_3 x_1(t) + c_4 x_2(t)) - r_2(t), y_1(t) = (c_1 x_1(t) + c_2 x_2(t))$, and $y_2(t) = (c_3 x_1(t) + c_4 x_2(t))$. Namely,

$$\begin{align*}
y_1(t) &= r_1(t), \\
y_2(t) &= r_2(t) \tag{E3}
\end{align*}
$$

$$\begin{align*}
x_1(t) &= \frac{c_1 x_1(t) - c_2 x_2(t)}{c_1 c_4 - c_2 c_3} \\
x_2(t) &= \frac{c_3 x_1(t) + c_4 x_2(t)}{c_1 c_4 - c_2 c_3} \\
x_3(t) &= \frac{c_3 x_1(t) + c_4 x_2(t)}{c_1 c_4 - c_2 c_3}.
\end{align*}
$$

That is to say, the system is decoupled and completely solvable.

As concerns the practical synthesis of the control law (E2), we can follow any approximation procedure for nonproper systems, as, for example, the one proposed in [4]. This particular procedure leads to the following approximation (we took, for illustration, $c_1 = c_3 = c_4 = 1$ and $r_2 = 0$):

$$\begin{align*}
u(t) &= \begin{bmatrix} -\frac{1}{\varepsilon_0} & 0 & 0 \\ 0 & -\frac{1}{\varepsilon_0} & -\frac{2}{\varepsilon_0} \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tau(t) \tag{E5}
\end{align*}
$$

where $\varepsilon_0$ is a positive constant. The closed-loop system is

$$\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -\frac{1}{\varepsilon_0} & -\frac{1}{\varepsilon_0} & -\frac{2}{\varepsilon_0} \\ 0 & 0 & 1 \\ 0 & -\frac{1}{\varepsilon_0} & -\frac{2}{\varepsilon_0} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tau(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} x(t). \tag{E6}
\end{align*}
$$

Let us note that its characteristic polynomial $\pi(\lambda, \varepsilon_0)$ is equal to $(\lambda + 1/\varepsilon_0^2) (\lambda + 1/\varepsilon_0)^2$, and the transfer function matrix $T(\lambda, \varepsilon_0)$ is equal to

$$\begin{align*}
T(\lambda, \varepsilon_0) &= \frac{2 \varepsilon_0^2 \lambda^2 + \varepsilon_0^2 \lambda + 1}{(\varepsilon_0^2 + 1)(\varepsilon_0^2 + 1)^2} \quad \frac{\varepsilon_0^2 \lambda^2 + \varepsilon_0 \lambda + 1}{(\varepsilon_0^2 + 1)(\varepsilon_0^2 + 1)^2} \\
\end{align*}
$$

And thus, $\lim_{\varepsilon_0 \to 0} T(\lambda, \varepsilon_0) = I$. In Fig. 1, we show some simulations with $\varepsilon_0 = 0.01$.

\footnote{Remember that $c_1 c_4 - c_2 c_3 \neq 0$.}
III. Conclusion

We have proposed an alternative for decoupling linear time-invariant, right-invertible, and minimum phase systems by proportional and derivative state feedback. Our method is based on simple matricial operations, and no longer relies on Brunovsky’s canonical form. As for the previously published contributions, our control strategy amounts to using, for the dynamic compensator which is equivalent to our proportional and derivative state feedback, some right inverse of the open-loop system. An important feature of our control strategy is that solvability of the compensated system is guaranteed. The main reason for our minimum phase assumption has been to guarantee stability for the decoupled system. We are convinced that it should be possible to relax this minimum phase assumption since unstable zeros may be acceptable as long as they do not have to be canceled by the compensator. In the “classical” context of decoupling by proportional state feedback, it is quite well known (see, for instance, [9], [10], and [12]) that a necessary and sufficient condition for decoupling with internal stability is that the so-called “interconnection zeros” (say, roughly, the global zeros which are not zeros of row subsystems) be stable. This property should also hold in our present case, but is not yet available. We are now exploring this more general case. Another important extension of our result will be to look for the minimal assumptions which will ensure the existence of decoupling solutions by proportional and derivative output feedback.

References


