Solidarity in Preference Aggregation: Improving on a Status Quo

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Abstract

Working in the Arrovian framework, we search for preference aggregation rules with desirable solidarity properties. In a fixed-population setting, we formulate two versions of the solidarity axiom *welfare dominance under preference replacement*. Although the stronger proves incompatible with *efficiency*, the combination of *efficiency* and our second version leads to an important class of rules which improve upon a "status quo" order. These rules are also *strategy-proof*, which reveals a further connection between solidarity and incentive properties. Allowing the population to vary, we again characterize the status quo rules by *efficiency* and a different solidarity axiom, *population monotonicity*. This extends a similar characterization of a subclass of these rules by Bossert and Sprumont (2014).

Keywords: Welfare dominance under preference replacement; preference aggregation; status quo rules.

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1 Introduction

Groups make decisions, choose projects, and embark on courses of action that affect each of their members. The search for decision methods that can accommodate diversity of preferences has a long tradition. Broadly, the social choice literature investigates two types of rules: Choice rules and preference aggregation rules. Choice rules, which assign a single choice to each profile of preferences, are appropriate when choosing from among a fixed and known set of alternatives, so identifying the top alternative is sufficient. This setting applies when, for example, a group selects a president from among a list of available candidates. Other times, there may be some ambiguity about which alternatives will ultimately be available. For example, a board of directors searching for a CEO may have a list of prospective candidates but be uncertain which will accept an offer. Rather than simply identifying the best candidate, the board requires a complete ordering of the candidates. Preference aggregation rules, which assign an order to each profile of preferences, fill this need. As the Arrow (Arrow, 1963) and Gibbard-Satterthwaite (Gibbard, 1973; Satterthwaite, 1975) Theorems make stark, one quickly encounters trade-offs along either route.

Following Bossert and Sprumont (2014), we adopt the second approach. The data of our problem consist of agents' preferences over a finite set of alternatives. However, the outcome of a preference aggregation rule is an order rather than a selection. Consequently, we must infer agents' preferences over orders. Again following Bossert and Sprumont (2014), we take the "prudent" approach: We assume only that an agent prefers one order to another if the agents' preferences over alternatives agree with all pairwise differences from the second order to the first. In other words, the first order lies "between" the agent's preferences and the second order, representing an unambiguous improvement for the agent. This defines a partial order over social orders which we call the "prudent extension" of the agents preferences over alternatives to preferences over orders.[1]

The prudent extension is sensible when agents are uncertain about which subset of alternatives will ultimately be available. For example, considering two orders, the second may improve the rank of an agent's most preferred alternative and also improve the rank of her least preferred alternative. Depending on which alternatives appear in the eventual choice set, the agent may be better or worse off, and a prudent agent may be unwilling to "trade off" these changes. Similarly, as modelers, it is prudent to limit our inference to unambigu-

[1]Extension of preferences is also an important consideration in probabilistic assignment where most properties are formulated in terms of first-order stochastic dominance. See Bogomolnaia and Moulin (2001) and Cho (2014b).

ous comparisons. The prudent extension withholds judgement in these cases. The cost, of course, is that many orders will not be comparable.

Whether a rule chooses a single alternative or a social order, it must be perceived as fair. Focusing on a particular aspect of fairness, we search for preference aggregation rules with desirable solidarity properties. Solidarity embodies the idea of a common endeavor and shared outcome. As a general principle, solidarity requires that when the environment changes and no agent is responsible for the change, all agents should be affected in the same direction: Either all gain together or all lose together. Relevant changes may include changes in resources, changes in technology, or the arrival or departure of agents.

In our model with abstract alternatives, one's environment includes the other agents and their preferences. Contemplating changes in these components suggests two properties, both common in the literature. "Welfare dominance under preference replacement", *welfare dominance* for short, requires that when the preferences of one agent change, the welfares of the agents whose preferences are fixed move in the same direction: Either all gain together or all lose together. When the population may vary, a second property also applies. "Population monotonicity" requires that when one agent departs, those who remain be affected in the same direction.

While solidarity principles have been extensively studied in social choice, much less is known about the solidarity properties of preference aggregation rules. In fact, Bossert and Sprumont (2014) are the first to formulate *population monotonicity* in this context, and *welfare dominance* has yet to be stated precisely. Ambiguity arises because agents' (extended) preferences over orders are incomplete. As formulated by Bossert and Sprumont (2014), *population monotonicity* requires that agents be able to compare the orders chosen before and after the departure of another agent. Consequently, *population monotonicity* is a strong requirement. Our first version of *welfare dominance* takes the same approach and requires that agents whose preferences are fixed be able to compare the orders chosen before and after another agent's preferences change. Unfortunately, this version proves too strong: It is incompatible with *efficiency* and even restricts choices when all agents begin with identical preferences and solidarity should be moot. This leads us to formulate a restricted notion: *Adjacent welfare dominance* requires the same conclusion as *welfare dominance*, but applies only to the "smallest" change in preferences, reversal of a single pair of adjacently ranked alternatives. Importantly, *adjacent welfare dominance* is compatible with *efficiency.*

Our analysis leads to the "status quo" rules. Each status quo rule is defined by a reference order which it improves upon as much as possible. Intuitively, the improvement process makes all changes to the reference order that meet with the unanimous approval of the agents. We describe the improvement process formally and in detail in the next section. Bossert and Sprumont (2014) introduced an important subclass of the status quo rules defined by strict reference orders, the "strict-order status quo" rules. Our definition extends this class to allow for weak reference orders.

So that our solidarity requirements are meaningful, we consider problems with at least three agents and at least two alternatives. In our main results, we characterize the status quo rules by *efficiency* and *adjacent welfare dominance* (Theorem 2) and the strict-order status quo rules by the same axioms and the requirement that the rule select a strict order (Theorem 1). Allowing the population to vary, we also characterize the status quo rules on the basis of *efficiency* and *population monotonicity* (Theorem 3). This extends Theorem 2 of Bossert and Sprumont (2014) which characterizes the subset of these rules defined by a strict reference order. Notably, the status quo rules satisfy additional properties that we did not impose. For instance, each status quo rule is "anonymous", meaning that the names of the agents do not matter. Each status quo rule is also "strategy-proof", meaning that no agent has an unambiguous incentive to report false preferences,[2] and in fact "group strategy-proof", which extends *strategy-proofness* to groups. Our results also highlight the tradeoffs involved in allowing indifferences in the social order.

In Section 2 we introduce the model, our axioms, and the status quo rules. We analyze fixed populations in Section 3 and variable populations in Section 4. We conclude in Section 5 and collect omitted proofs in the Appendix. The remainder of this section reviews related literature.

Related literature

Our results contribute to the growing literature on solidarity. Moulin (1987) is first to apply the principle to preference changes, introducing *welfare dominance* in a binary choice model. Versions of *welfare dominance* have since been studied for allocating a divisible resource (Thomson, 1997), assigning objects and money (Thomson, 1998), and choosing the location of one or more facilities (Thomson, 1993; Miyagawa, 2001; Umezawa, 2012; Harless, 2014).[3] *Population monotonicity* originated in the axiomatic theory of bargaining (Thomson, 1983a,b) and has now been adapted to our context (Bossert and Sprumont, 2014). Gordon

[2]By *unambiguous*, we mean according to the prudent extension of the agent's true preferences. In particular, changing the outcome to an order incomparable to the original does not constitute a violation. Bossert and Sprumont (2014) elaborate on this modeling choice.

[3]See Thomson (1999) for a survey.

(2007a,b, 2015) and Harless (2015) derive and compare results of both properties. Our restriction of *welfare dominance* to *adjacent welfare dominance* parallels the restriction of *strategy-proofness* to "adjacent strategy-proofness" studied by Sato (2013) and Cho (2014b).[4]

Most closely, our conclusions complement those Bossert and Sprumont (2014) obtain in the same setting, particularly their characterization of the strict status quo rules by *efficiency* and *population monotonicity.* We extend this characterization to the full family of status quo rules and offer a new characterization based on fixed-population principles. Our results also reinforce conclusions Gordon (2007a) draws in a general social choice framework: Once again, *welfare dominance* is "stronger" than *population monotonicity.* Gordon (2007b) provides another point of reference. The six orders over three alternatives can be viewed as a cycle, a case where he shows that *efficiency* and *welfare dominance* are incompatible. Our positive result highlights the role of incomplete preferences: Completeness effectively strengthens *efficiency* to what we call "strong efficiency" which continues to be incompatible with our solidarity principles.

Nehring and Puppe (2007) introduce a generalized model of social choice and Gordon (2015) studies solidarity in this framework.[5] These studies find that *efficiency* is essentially incompatible with *strategy-proofness* or *welfare dominance.*[6] The positive results we and Bossert and Sprumont (2014) obtain again reflect the incompleteness of the prudent extension of preferences.

2 Model

There is a finite set of social alternatives $\boldsymbol{A}$, $|A| \geq 2$, and a finite population of agents $\boldsymbol{N}$, $|N| \geq 3$, each with strict preferences over the alternatives. The set of strict orders over A is $\boldsymbol{\mathcal{R}}$ and the set of weak orders over A is $\bar{\boldsymbol{\mathcal{R}}}$.[7] When a is at least as good as b according to R_0, we write $\boldsymbol{a\ R_0\ b}$ or $\boldsymbol{(a,b) \in R_0}$, and when a is preferred to b, we write $\boldsymbol{a\ P_0\ b}$. The indifference class of a in R_0 is $\bar{\boldsymbol{a}} \equiv \{a' \in A : a\ I_0\ a'\}$. With slight abuse of notation, we

[4]Other studies parameterize normative axioms (Moulin and Thomson, 1988; Piacquadio, 2014; Harless, 2014) and incentive properties (Carroll, 2012; Cho, 2014a).

[5]Nehring and Puppe (2010) draw similar conclusions about *strategy-proofness* specific to preference aggregation. In additional to modeling complete preferences, however, they restrict attention to rules satisfying Arrows "independence of irrelevant alternatives" and "monotonicity", two properties that the status quo rules violate.

[6]More specifically, their analyses identify a class of "median spaces" on which these axioms are compatible. However, as we define betweenness, our model does not fall in this class. For example, the Condorcet triple *abc–bca–cab* has no median.

[7]A **weak order** is complete, reflexive, and transitive; a **strict order** is also anti-symmetric.

extend the comparisons to sets of alternatives and agents: Given $B, B' \subseteq A$ and $S \subseteq N$, we write $\boldsymbol{B}\ \boldsymbol{P_S}\ \boldsymbol{B'}$ if for each $b \in B$, each $b' \in B'$, and each $i \in S$, $b\ P_i\ b'$. An **economy** is a profile $\boldsymbol{R} \in \mathcal{R}^N$. A **rule** is a mapping $\boldsymbol{F}\colon \mathcal{R}^N \to \bar{\mathcal{R}}$. A rule is **strict** if for each $N \in \mathcal{N}$ and each $R \in \mathcal{R}^N$, $F(R) \in \mathcal{R}$.

Since agents' preferences are defined only over the set of alternatives, we must extend them to preferences over orders. Adopting a conservative approach, we use betweenness to extend preferences over alternatives to preferences over orders. For each triple $R_0, R_0', R_0'' \in \mathcal{R}$, R_0' is **between** R_0 and R_0'', written $\boldsymbol{R_0' \in [R_0, R_0'']}$, if $R_0 \cap R_0'' \subseteq R_0' \subseteq R_0 \cup R_0''$. For each $R_0 \in \mathcal{R}$, the **prudent extension** of R_0, $\boldsymbol{R_0^e}$, is the partial order over $\mathcal{R}$ such that for each pair $R_0', R_0'' \in \mathcal{R}$, $R_0'\ R_0^e\ R_0'' \iff R_0' \in [R_0, R_0'']$.

Remark 1. Betweenness implies a transitivity property for strict orders. For each quadruple $R_0, R_0', R_0'', R_0''' \in \mathcal{R}$, if $R_0' \in [R_0, R_0''']$ and $R_0'' \in [R_0', R_0''']$, then $R_0' \in [R_0, R_0'']$.[8]

Finally, we distinguish adjacent and inverse orders. Strict orders $R_0, R_0' \in \mathcal{R}$ are **adjacent** if one is formed from the other by reversing adjacently ranked alternatives. That is, $|R_0 \backslash R_0'| = |R_0' \backslash R_0| = 1$. For each $R_0 \in \mathcal{R}$, the **inverse** of R_0, $\boldsymbol{R_0^{-1}}$, reverses the rankings of R_0: For each pair $a, b \in A$, $(a, b) \in R_0^{-1} \iff (b, a) \in R_0$. An economy $R \in \mathcal{R}^N$ contains an **inverse pair** if there is a pair $i, j \in N$ such that $R_i = R_j^{-1}$.

2.1 Status quo rules

Next we introduce a family of rules. Each rule in the family begins from a reference order, which is a parameter of the rule. In a given economy, the rule Pareto improves upon the reference order until reaching an efficient order. We first consider rules with strict reference orders. For each $R^* \in \mathcal{R}$, $(\mathcal{R}, R^{*e})$ is a lattice[9] and so $\bigcup_{i \in N} R_i$ has a unique least upper bound in $(\mathcal{R}, R^{*e})$. We define a rule that selects in each economy this least upper bound.

Strict-order status quo rule with reference order $\boldsymbol{R_0 \in \mathcal{R}}$, $\boldsymbol{F^{R^*}}$: For each $R \in \mathcal{R}^N$,

$$F^{R^*}(R) \equiv \left\{ R_0 \in \mathcal{R} : \begin{array}{l} \text{(i) } R_0 \in \bigcap_{i \in N} [R_i, R^*] \text{ and} \\ \text{(ii) } \forall R_0' \in \mathcal{R},\ R_0' \in \bigcap_{i \in N} [R_i, R^*] \Rightarrow R_0' \in [R_0, R^*] \end{array} \right\}.$$

Condition (i) says that $F^{R^*}(R)$ is an upper bound on $\bigcup_{i \in N} R_i$, and condition (ii) says that it is the least upper bound.

[8]To see this, suppose that $R_0' \notin [R_0, R_0'']$. Since the orders are strict, there is a pair $a, b \in A$ such that $(a, b) \in R_0 \cap R_0''$ and $(b, a) \in R_0'$. But $R_0' \in [R_0, R_0''']$ implies $(b, a) \in R_0'''$, which contradicts $R_0'' \in [R_0', R_0''']$.

[9]See Bossert and Sprumont (2014) for additional discussion of this point.

To extend the definition to weak reference orders, we introduce an alternative description that is more explicit about the improvement process. Beginning from a reference order, we choose a pair of adjacent alternatives and ask the agents whether they prefer to reverse their ranking. If there is unanimous agreement in favor of reversal, we reverse the alternatives. Continuing in this fashion until no further unanimously approved reversals are possible, we reach a final order which is the outcome of the status quo rule. The lattice structure ensures that we reach the same final order regardless of the sequence in which we propose pairs for reversal, provided we always propose adjacent pairs. Example 1 illustrates the process.

Example 1. Illustrating the process of improving on a strict reference order. Let $N \equiv \{1, 2\}$, $A \equiv \{a, b, c, d\}$, and $R^*, R_1, R_2 \in \mathcal{R}$ be as specified in the table.

R^*	R_1	R_2
a	c	d
b	d	c
c	b	a
d	a	b

Let $R \equiv (R_1, R_2)$. One improvement path is:

R^*		R'_0		R''_0		R'''_0		R_0
a		a		c		c		c
b	$\rightarrow$	c	$\rightarrow$	a	$\rightarrow$	a	$\rightarrow$	d
c		b		b		d		a
d		d		d		b		b

No further exchanges are possible, so $F^{R^*}(R) = R_0$. While other improvement paths are possible, all exhaustive improvement paths lead to R_0.

To accommodate "thick" indifference classes, we generalize the improvement process in two ways. First, to "break" an indifference class, the agents must unanimously prefer each alternative moved up to each alternative moved down; if even one agent disagrees with one comparison, then the indifference class cannot be broken as proposed. Second, to exchange adjacent indifference classes, the agents must unanimously prefer each alternative in the lower indifference class to each alternative in the higher indifference class. Example 2 illustrates the process.

Example 2. Illustrating the process of improving on a weak reference order. Let $N \equiv \{1, 2\}$, $A \equiv \{a, b, c, d, e, f, g\}$, and $R^*, \hat{R}^*, \bar{R}^*, R_0, \hat{R}_0, \bar{R}_0, R_1, R_2 \in \mathcal{R}$ be as specified in the tables.

R^*	$\hat{R}^*$	$\hat{R}^*$	R_1	R_2
a	a	$abcdefg$	g	b
b	$bcde$		f	g
c	fg		b	c
d			e	d
e			c	f
f			d	e
g			a	a

R_0	$\hat{R}_0$	$\hat{R}^*$
b	a	$bcdefg$
g	$bcde$	a
c	fg	
d		
f		
e		
a		

Let $R \equiv (R_1, R_2)$. We compare the outcomes of F^{R^*}, $F^{\hat{R}^*}$, and $F^{\bar{R}^*}$. First, following an exhaustive improvement process as in the previous example, $F^{R^*}(R) = R_0$.

Now consider the reference order $\hat{R}^*$. Beginning with the indifference class ($bcde$), all agents prefer b to each of the other alternatives, so this class can be broken with b above (cde). Similarly, the indifference class (fg) can be broken with g above f. The indifference class (cde) cannot be further divided because no alternative is unanimously preferred to the remaining two alternatives. In particular, even though c is unanimously preferred to d, this cannot be reflected in the social order. Now g is unanimously preferred to all of the alternatives in the class (cde) and so these classes can be exchanged in the order. However, f and (cde) cannot be exchanged. Again, although f is unanimously preferred to e, this is not sufficient to move f up in the order. Additional reversals consist of moving a down in the order and $F^{\hat{R}^*}(R) = \hat{R}_0$. The complete improvement path is summarized by:

R^*		R_0^i		R_0^{ii}		R_0^{iii}		R_0^{iv}		R_0^{v}		R_0^{vi}		R_0
a		a		a		a		b		b		b		b
$bcde$	$\to$	b	$\to$	b	$\to$	b	$\to$	a	$\to$	g	$\to$	g	$\to$	g
fg		cde		cde		g		g		a		cde		cde
		fg		g		cde		cde		cde		a		f
				f		f		f		f		f		a

Although g is eventually raised above (cde), this is not possible at the first step; improving the position of g in R^* would additionally require that g be unanimously preferred to b, which is not the case. After the indifference class ($bcde$) is broken, we might consider exchanging the indifference classes (cde) and (fg). However, this is not possible because f is not unanimously preferred to e.

Finally consider the complete indifference reference order $\bar{R}^*$. Since agent 1 ranks g first and agent 2 ranks b first, a division of the indifference class must include both g and b in the top group. Since $f\ P_1\ b$, f must be included as well. Then, since $c\ P_2\ f$ and $d\ P_2\ g$, these alternatives must be included. Finally, $e\ P_1\ c$, so e must be included. $F^{\bar{R}^*}(R) = \bar{R}_0$.

Importantly, the process never adds indifferences; the final order is at least as "resolute" as the reference order. To formalize this process,[10] we proceed in two steps. Let $R^* \in \bar{\mathcal{R}}$ and $R \in \mathcal{R}^N$.

Step 1: Determining indifference classes. Let $A_1^*, \ldots, A_{K^*}^*$ be the indifference classes of R^* ordered so that $A_1^*\ P^*\ A_2^*\ P^* \cdots P^*\ A_{K^*}^*$. For each $k \in \{1, \ldots, K^*\}$ and each $a \in A_l^*$, let

$$\bar{a} \equiv \left\{ a' \in A_k^* : \begin{array}{l} \exists\, i_1, \ldots, i_s \in N \text{ and } a_1, \ldots, a_s \in A_k^* \text{ s.t. } a^1 = a, \\ a^s = a', \text{ and } \forall\, l \in \{1, \ldots, s-1\},\ a_{l+1}\ P_{i_l}\ a_l \end{array} \right\}.$$

That is, $\bar{a}$ is the "transitive closure" of a with respect to R restricted to A_k^*. By construction, for each $k \in \{1, \ldots, K^*\}$ and each pair $a, a' \in I_k^*$, if $\bar{a} \neq \bar{a}'$, then either $\bar{a}\ P_N\ \bar{a}'$ or $\bar{a}'\ P_N\ \bar{a}$. Let $A_1, \ldots, A_K$ be the indifference classes formed in this way and ordered so that $A_1\ R^*\ A_2\ R^* \cdots R^*\ A_K$.

Step 2: Ordering indifference classes. Let $\mathcal{A} \equiv \{A_1, \ldots, A_K\}$ and define a binary relation $T(R^*, R) \subseteq \mathcal{A} \times \mathcal{A}$ by setting for each pair $A, B \in \mathcal{A}$,

$$(A, B) \in T(R^*, R) \iff \begin{cases} \text{(i)} & A\ R_N\ B \text{ or} \\ \text{(ii)} & A\ R^*\ B \text{ and } \exists\, i \in N,\ a \in A,\ b \in B \text{ s.t. } a\ R_i\ b \end{cases}.$$

Then $T(R^*, R)$ is complete and antisymmetric,[11] though not necessarily transitive. To build a transitive relation from $T(R^*, R)$, we compare pairs of indifference classes in order according to their "distance" in R^*. Formally, define an order $\succ_{R^*}$ over $\bar{A}$ such that

$$\begin{aligned} \{A_1, A_2\} \succ_{R^*} \{A_2, A_3\} \succ_{R^*} \{A_3, A_4\} \succ_{R^*} \cdots \succ_{R^*} \{A_{K-1}, A_K\} \\ \succ_{R^*} \{A_1, A_3\} \succ_{R^*} \{A_2, A_4\} \succ_{R^*} \cdots \succ_{R^*} \{A_{K-2}, A_K\} \\ \vdots \qquad\qquad\qquad\qquad \\ \succ_{R^*} \{A_1, A_K\} \qquad\qquad \end{aligned}$$

[10] Our technique adapts the "lexicographic alteration" procedure introduced by Bossert and Sprumont (2014).

[11] To see that it is antisymmetric, suppose $A \neq B$, $A\ R^*\ B$, and $(B, A) \in T(R^*, R)$. If $A\ P^*\ B$, then because agents preferences are strict, $B\ P_N\ A$ and (A, B) satisfies neither (i) nor (ii). If $A\ I^*\ B$, then by the construction in Step 1, $B\ P_N\ A$ and again (A, B) satisfies neither (i) nor (ii).

Let $T^0 \equiv \{(A_l, A_l) : l = 1, \ldots, K\}$. For each $k = 1, \ldots, \frac{K(K-1)}{2}$, let $\{B_k, B'_k\}$ be the k^{th}-listed pair according to $\succ_{R^*}$ and define T^k by

$$T^k \equiv \begin{cases} T^{k-1} \cup \{(B_k, B'_k)\} & \text{if } \exists\, B'' \text{ s.t. } \{(B_k, B''), (B'', B'_k)\} \subseteq T^{k-1} \\ T^{k-1} \cup \{(B_k, B'_k)\} & \text{if } (B_k, B'_k) \in T(R^*, R) \text{ and } \nexists\, B'' \text{ s.t. } \{(B'_k, B''), (B'', B_k)\} \subseteq T^{k-1} \\ T^{k-1} \cup \{(B'_k, B_k)\} & \text{otherwise} \end{cases}.$$

In particular, T^k is formed from T^{k-1} by adding exactly one of (B_h, B'_h) and (B'_h, B_h) to avoid intransitivities. Let $T^*(R^*, R) \equiv T^{K(K-1)/2}$. Then $T^*(R^*, R)$ is a complete, reflexive, and transitive order over the indifference classes $\{A_1, \ldots, A_K\}$. To define the corresponding status quo rule, we recover a relation over A from $T^*(R^*, R)$.

Status quo rule with reference order $\boldsymbol{R^*} \in \bar{\mathcal{R}}$, $\boldsymbol{F^{R^*}}$: For each $R \in \mathcal{R}^N$ and each pair $a, b \in A$,

$$(a, b) \in F^{R^*}(R) \iff (\bar{a}, \bar{b}) \in T^*(R^*, R).$$

While our formal description of the status quo rules is somewhat complicated, the definition simply structures the "myopic" improvement process described in Examples 1 and 2. Also, when $R^* \in \mathcal{R}$, our definition reduces to our earlier definition for strict-order status quo rules.

2.2 Axioms

We now introduce desirable properties of rules. Let F be a rule.

Our first axioms adapt the standard requirement of Pareto efficiency. An order is **efficient** if there is no other order that all agents prefer according to the prudent extensions of their preferences. We require that a rule always select an efficient order.

Efficiency: For each $R \in \mathcal{R}^N$, $R \in \mathcal{R}^N$, $\bigcap_{i \in N} [R_i, F(R)[= \emptyset$.

Because the prudent extensions are incomplete, *efficiency* is a weak requirement. *Efficiency* takes a holistic viewpoint, using the prudent extension of preferences to compare entire orders. In contrast with this "global" perspective, our next axiom applies to pairwise comparisons. We require that if all agents rank one alternative above a second alternative, then the first alternative should be ranked above the second alternative in the social order.[12]

[12]Bossert and Sprumont (2014) call this requirement "local unanimity".

Strong efficiency: For each $R \in \mathcal{R}^N$, each $\bigcap_{i \in N} R_i \subseteq F(R)$.

Strong efficiency is common in the literature (see Arrow (1963)) and implies *efficiency*.[13] While *strong efficiency* is often desirable, *efficiency* is the appropriate translation of the traditional notion of Pareto dominance to our setting.[14]

We now turn to solidarity. Our next axiom requires solidarity when the preferences of one agent change: Either all agents whose preferences are fixed find the new order at least as good as the old order or all agents whose preferences are fixed find the old order at least as good as the new order.

Welfare dominance: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, either $F(R'_i, R_{-i}) \in \bigcap_{j \in N \setminus \{i\}} [R_j, F(R)]$ or $F(R) \in \bigcap_{j \in N \setminus \{i\}} [R_j, F(R'_i, R_{-i})]$.

To satisfy *welfare dominance*, all agents whose preferences are fixed must be able to compare the orders chosen in the two economies. Since the prudent extension only partially orders $\mathcal{R}$, this is a strong condition. In fact, through the comparability requirement, *welfare dominance* imposes restrictions unrelated to solidarity, such as when moving from an economy of identical preferences. Additionally, *welfare dominance* is incompatible with *efficiency* (Proposition 2).

The incompatibility of *welfare dominance* with *efficiency* motivates us to consider a weaker notion of solidarity. This time, we limit the conclusion to changes in preferences in which an agent reverses a single pair of adjacently ranked alternatives.

Adjacent welfare dominance: For each $R \in \mathcal{R}^N$, each $i \in N$, each $R'_i \in \mathcal{R}$, if $|R_0 \setminus R'_0| + |R'_0 \setminus R_0| \leq 2$, then either $F(R'_i, R_{-i}) \in \bigcap_{j \in N \setminus \{i\}} [R_j, F(R)]$ or $F(R) \in \bigcap_{j \in N \setminus \{i\}} [R_j, F(R'_i, R_{-i})]$.

Our next property, a consequence of *adjacent welfare dominance* (Lemma 1), will facilitate our arguments. Again consider a situation in which the preferences of one agent change. The property says that if the social ranking of a pair of alternatives is reversed, then all agents whose preferences are fixed rank those alternatives in the same way.

Pairwise welfare dominance: For each $R \in \mathcal{R}^N$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $(a, b) \in (F(R'_i, R_{-i}) \setminus F(R)) \cup (F(R) \setminus F(R'_i, R_{-i}))$, either $a \; P_{N \setminus \{i\}} \; b$ or $b \; P_{N \setminus \{i\}} \; a$.

[13] To see this, let $R \in \mathcal{R}^N$ and suppose there is $R_0 \in \bigcap_{i \in N} [R_i, F(R)[$. Then there is a pair $a, b \in A$ such that $(a, b) \in R_0 \setminus F(R)$ and $a \; P_N \; b$. But then $\bigcap_{i \in N} R_i \not\subseteq F(R)$.

[14] See Bossert and Sprumont (2014) for further discussion on this point.

Our final axiom concerns incentives. We require that no agent be able to gain by misreporting her preferences.

Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, each $R'_i \in \mathcal{R}$, if $F(R'_i, R_{-i}) \in [R_i, F(R)]$, then $F(R'_i, R_{-i}) = F(R)$.

Strategy-proofness prevents misrepresentations that lead to unambiguous gains according to the prudent extension of preferences. Because the prudent extension of preferences is incomplete, *strategy-proofness* is fairly mild. See Bossert and Sprumont (2014) for further discussion. A stronger requirement, **group strategy-proofness**, requires the same conclusion for misrepresentations by groups.

3 Fixed populations

Our main results characterize the strict-order status quo rules (Theorem 1) and the full class of status quo rules (Theorem 2). We also show that strengthening either our efficiency or our solidarity requirement leads to an impossibility (Proposition2). As preliminaries, we show that *adjacent welfare dominance* implies *pairwise welfare dominance* and derive an invariance condition as a consequence of *pairwise welfare dominance*.

Lemma 1. *If a rule satisfies* adjacent welfare dominance*, then it satisfies* pairwise welfare dominance.

Proof. Let F satisfy *adjacent welfare dominance*. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. There is a sequence of adjacent preference relations $R_i^{(0)}, \ldots, R_i^{(k)}$ such that $R_i^{(0)} = R_i$ and $R_i^{(k)} = R'_i$. For each $l = 1, \ldots, k$, let $R_0^{(l)} \equiv F(R_i^{(l)}, R_{-i})$. Also let $R_0 \equiv R_0^{(0)}$ and $R'_0 \equiv R_0^{(k)}$. Suppose that $(a, b) \in R_0 \backslash R'_0$. Then there is $l \in \{1, \ldots, k\}$ such that $(a, b) \in R_0^{(l-1)} \backslash R_0^{(l)}$. By *adjacent welfare dominance*, either $R_0^{(l)} \in \bigcap_{j \in N \backslash \{i\}} [R_j, R_0^{(l-1)}]$ or $R_0^{(l-1)} \in \bigcap_{j \in N \backslash \{i\}} [R_j, R_0^{(l)}]$. Therefore, either $a \ P_{N \backslash \{i\}} \ b$ or $b \ P_{N \backslash \{i\}} \ a$. Since this is true for each $(a, b) \in R_0 \backslash R'_0$, *pairwise welfare dominance* is satisfied. □

Lemma 2. *If a rule satisfies* pairwise welfare dominance*, then the rule selects the same order in each economy that contains an inverse pair.*

Proof. Let F satisfy *pairwise welfare dominance* and $R_0, R_0^{-1} \in \mathcal{R}$ be inverse orders. Let $i, j \in N$, $R_{-ij} \in \mathcal{R}^{N \backslash \{i,j\}}$, and $R^* \equiv F(R_0, R_0^{-1}, R_{-ij})$. Let $k \in N \backslash \{i, j\}$, $R'_k \in \mathcal{R}$, and $\hat{R}^* \equiv F(R_0, R_0^{-1}, R'_k, R_{-ijk})$. By *pairwise welfare dominance*, for each $(a, b) \in \hat{R}^* \backslash R^*$, either

$a\ P_N\ b$ or $b\ P_N\ a$. Since R_0 and R_0^{-1} are inverse orders, this implies that $\hat{R}^*\backslash R^* = \emptyset$. Similarly, $R^*\backslash\hat{R}^* = \emptyset$ and so $R^* = \hat{R}^*$.

Repeating the previous argument, for each $R_{-ij} \in \mathcal{R}^{N\backslash\{i,j\}}$, $F(R_0, R_0^{-1}, R_{-ij}) = R^*$. Let $k \in N\backslash\{i,j\}$. Then $F(R_0, R_0^{-1}, R_0, R_{-ijk}) = R^*$ and $F(R_0, R_0^{-1}, R_0^{-1}, R_{-ijk}) = R^*$. Therefore, by *pairwise welfare dominance*, $F(R_0^{-1}, R_0, R_{-jk}) = R^*$ and $F(R_0, R_0^{-1}, R_{-ik}) = R^*$. Moreover, for each $R_{-jk} \in \mathcal{R}^{N\backslash\{j,k\}}$, $F(R_0^{-1}, R_0, R_{-jk}) = R^*$ and for each $R_{-ik} \in \mathcal{R}^{N\backslash\{i,k\}}$, $F(R_0, R_0^{-1}, R_{-ik}) = R^*$. Repeating these arguments, for each $i', j' \in N$ and each $R_{-i'j'} \in \mathcal{R}^{N\backslash\{i',j'\}}$, $F(R_0, R_0^{-1}, R_{-i'j'}) = R^*$.

Finally, let $\bar{R}_0, \bar{R}_0^{-1} \in \mathcal{R}$ be inverse orders. Let $i, j, k \in N$, $R_{-ijk} \in \mathcal{R}^{N\backslash\{i,j,k\}}$, and $R^{**} \equiv F(\bar{R}_0, \bar{R}_0^{-1}, R_{-ijk})$. Then, by previous arguments and *pairwise welfare dominance*,

$$F(R_0, R_0^{-1}, \bar{R}_0, R_{-ijk}) = R^*, \tag{1}$$

$$F(R_0, \bar{R}_0^{-1}, \bar{R}_0, R_{-ijk}) = R^{**}, \text{ and} \tag{2}$$

$$F(\bar{R}_0^{-1}, R_0^{-1}, \bar{R}_0, R_{-ijk}) = R^{**}. \tag{3}$$

Suppose by way of contradiction that $R^* \neq R^{**}$ and let $(a, b) \in (R^*\backslash R^{**}) \cup (R^{**}\backslash R^*)$. Without loss of generality, suppose $(a, b) \in (R^*\backslash R^{**})$. By *pairwise welfare dominance* comparing (1) with (2) and (3) respectively,

(a) Either (i) $a\ P_0\ b$ and $a\ \bar{P}_0\ b$ or (ii) $b\ P_0\ a$ and $b\ \bar{P}_0\ a$.

(b) Either (i) $a\ P_0^{-1}\ b$ and $a\ \bar{P}_0\ b$ or (ii) $b\ P_0^{-1}\ a$ and $b\ \bar{P}_0\ a$.

These conditions are incompatible: If $a\ P_0\ b$, then by (a), $a\ \bar{P}_0\ b$ and so by (b), $a\ P_0^{-1}\ b$, which contradicts $a\ P_0\ b$. If instead $b\ P_0\ a$, then by (a), $b\ \bar{P}_0\ a$ and so by (b), $b\ P_0^{-1}\ a$, which contradicts $b\ P_0\ a$. Instead, $R^* = R^{**}$. □

Interestingly, the converse of Lemma 1 is false, although the properties are equivalent under *efficiency*.[15] Lemma 2 shows that rules satisfying *pairwise welfare dominance* distinguish a "default" or "status quo" alternative. Versions of this result are familiar from related models.[16] In fact, the existence of a distinguished alternative is a general consequence of solidarity properties in social choice models (Gordon, 2007a).

Next, we identify properties of the status quo rules.[17]

[15]Equivalence under *efficiency* is a consequence of Theorem 2 and Remark 2. An example showing that converse of Lemma 1 fails in general is available upon request.

[16]See, for example, Thomson (1993), Miyagawa (2001), Gordon (2007b), Umezawa (2012), and Bossert and Sprumont (2014).

[17]Bossert and Sprumont (2014) show that the strict-order status quo rules are *efficient* and *strategy-proof*.

Proposition 1. *Each status quo rule satisfies* efficiency*,* adjacent welfare dominance*, and* group strategy-proofness*.*

Proof. Let $R^* \in \bar{\mathcal{R}}$ and consider F^{R^*}.

Efficiency. Let $R \in \mathcal{R}^N$ and $R_0 \equiv F^{R^*}(R)$ and suppose by way of contradiction that R_0 is not efficient at R. Then there is $R'_0 \in \bar{\mathcal{R}}$ such that $R'_0 \in \bigcap_{i \in N}[R_i, R_0[$. In particular, there is a pair $a, b \in A$ such that $(a,b) \in R_0$, $(a,b) \notin R'_0$, and $b \; P_N \; a$. Without loss of generality, we may choose a and b so they are in the same or adjacent indifference classes in R_0. Let $\bar{a}$, $\bar{b}$, $\bar{a}'$, and $\bar{b}'$ denote the indifference classes of a and b in R_0 and R'_0 respectively.

First suppose $\bar{a} = \bar{b}$. By the definition of f^{R^*}, since $(a,b) \in F^{R^*}(R)$, there are sequences $a_0, \dots, a_l \in \bar{a}$ and $i_1, \dots, i_l \in N$ such that $a_0 = a$, $a_l = b$, and for each $s = 1, \dots, l$, $a_{s-1} \; P_{i_s} \; a_s$. By the assumption $R'_0 \in \bigcap_{i \in N}[R_i, R_0[$, for each $s = 1, \dots, l$, $a_{s-1} \; P_{i_s} \; a_s$ implies $(a_{s-1}, a_s) \in R'_0$. In particular, $a_{l-1} \; P_{i_l} \; b$ implies $(a_{l-1}, b) \in R'_0$. But then by transitivity, $(a,b) \in R'_0$, which is a contradiction.

Suppose instead that $a \; P_0 \; b$. By the previous argument, $\bar{a} \subseteq \bar{a}'$ and $\bar{b} \subseteq \bar{b}'$. Since $(b,a) \in R'_0$, R'_0 either joins $\bar{a}$ and $\bar{b}$ or raises $\bar{b}$ above $\bar{a}$. By the definition of f^{R^*}, since $(b,a) \notin F^{R^*}(R)$, there are $a' \in \bar{a}$, $b' \in \bar{b}$, and $i \in N$ such that $a' \; P_i \; b'$. But by transitivity, $(b', a') \in R'_0$, which contradicts $R'_0 \in [R_i, R_0[$.

Adjacent welfare dominance. Let $R \in \mathcal{R}^N$ and $i \in N$. Let $R'_i \in \mathcal{R}$ be adjacent to R_i and $R' \equiv (R'_i, R_{-i})$. Let a and b be the alternatives reversed with $a \; P_i \; b$ and $b \; P'_i \; a$. Also let $T(R^*, R)$ and $T(R^*, R')$ be as in the definition of F^{R^*}. Without loss of generality, suppose that $(a,b) \in R^*$. There are two cases.

Case 1: *a P* b*. Then the same equivalence classes define $T(R^*, R)$ and $T(R^*, R')$. Since $(a,b) \in R^*$, $(\bar{a}, \bar{b}) \in T(R^*, R)$. If $(\bar{a}, \bar{b}) \in T(R^*, R')$, then $(\bar{b}, \bar{a}) \notin T(R^*, R')$ and $(\bar{b}, \bar{a}) \notin T(R^*, R)$. In this case, $T(R^*, R) = T(R^*, R')$ and so $F^{R^*}(R) = F^{R^*}(R')$.

Suppose instead that $(\bar{a}, \bar{b}) \notin T(R^*, R')$. Then by construction of the equivalence classes, $(\bar{b}, \bar{a}) \in T(R^*, R')$. Moreover, $T(R^*, R) \backslash \{(\bar{a}, \bar{b})\} = T(R^*, R) \backslash \{(\bar{b}, \bar{a})\}$. From the definition of F^{R^*}, $b \; P_{N \backslash \{i\}} \; a$. Following the adjustment process according to $\succ_{R^*}$ as in the definition of F^{R^*}, each change made to $T(R^*, R')$ is also made to $T(R^*, R)$. Each additional change to $T(R^*, R)$ overrules a unanimous preference, so each additional change makes all agents whose preferences are fixed are worse off. Therefore, $F^{R^*}(R') \in \bigcap_{j \in N \backslash \{i\}}[R_i, F^{R^*}(R)]$.

Case 2: *a I* b*. If a and b are in the same equivalence class in $T(R^*, R')$, then the same equivalence classes define $T(R^*, R)$ and $T(R^*, R')$ and the analysis from Case 1 applies. Suppose instead that a and b are in different equivalence classes in $T(R^*, R')$. Then $T(R^*, R')$

contains b as a singleton indifference class. Moreover, $T(R^*, R')|_{A\backslash\{b\}} = T(R^*, R)$. From the definition of F^{R^*}, for each $j \in N\backslash\{i\}$, $b\ P_j\ a$. Following the adjustment process according to $\succ_{R^*}$ as in the definition of F^{R^*}, each change made to $T(R^*, R')|_{A\backslash\{b\}}$ is also made to $T(R^*, R)$. Since b remains ranked above the indifference class containing a in $F^{R^*}(R')$ and $b\ P'_N\ a$, each change made to preserve transitivity with b in $T(R^*, R')$ is also made to preserve transitivity with $\bar{a}$ in $T(R^*, R)$. Again, additional changes make all agents whose preferences are fixed worse off, so $F^{R^*}(R') \in \bigcap_{j\in N\backslash\{i\}}[R_i, F^{R^*}(R)]$.

Group strategy-proofness. Let $R \in \mathcal{R}^N$, $S \subseteq N$, and $R'_S \in \mathcal{R}$. Also let $R_0 \equiv F^{R^*}(R)$ and $R'_0 \equiv F^{R^*}(R'_S, R_{-S})$ and suppose that $R_0 \neq R'_0$. There is a pair $a, b \in A$ such that either $(a, b) \in R'_0\backslash R_0$ or $(b, a) \in R_0\backslash R'_0$. Let $T_0 \equiv T^0(R^*, R)$ and $T'_0 \equiv T^0(R^*, (R'_i, R_{-i}))$, the (possibly intransitive) base relations as in the definition of F^{R^*}. Also let $\bar{a}$, $\bar{b}$, $\bar{a}'$, and $\bar{b}'$, represent the indifference classes of a and b in T_0 and T'_0 respectively. Then $(\bar{a}', \bar{b}') \in T'_0$ and $(\bar{b}, \bar{a}) \in T_0$.

Case 1: $\boldsymbol{(a, b) \in R'_0\backslash R_0}$. First suppose $(\bar{a}, \bar{b}) \notin T_0$. Since $(\bar{a}', \bar{b}') \in T'_0$, the definition of F^{R^*} implies $b\ P_S\ a$ and so $R'_0 \notin \bigcap_{i\in S}[R_i, R_0]$. Suppose instead that $(\bar{a}, \bar{b}) \in T_0$. By the definition of F^{R^*}, $(\bar{b}, \bar{a}) \in T_0$ and $(\bar{a}, \bar{b}) \in T_0$ imply $a\ I^*\ b$. Also by the definition of F^{R^*}, $(a, b) \notin R_0$ implies $b\ P_N\ a$. In particular, $b\ P_S\ a$ and so $R'_0 \notin \bigcap_{i\in S}[R_i, R_0]$.

Case 2: $\boldsymbol{(b, a) \in R_0\backslash R'_0}$. First suppose $(\bar{b}', \bar{a}') \notin T'_0$. Since $(\bar{b}, \bar{a}) \in T_0$, the definition of F^{R^*} implies $b\ P_S\ a$ and so $R'_0 \notin \bigcap_{i\in S}[R_i, R_0]$. Suppose instead that $(\bar{b}', \bar{a}') \in T'_0$. By the definition of F^{R^*}, $(\bar{b}, \bar{a}) \in T'_0$ and $(\bar{a}, \bar{b}) \in T'_0$ imply $a\ I^*\ b$ and also $a\ I_0\ b$.

Since $(\bar{b}, \bar{a}) \in T_0$, there are $c_a \in \bar{a}'$, $c_b \in \bar{b}'$, and $j \in N$ such that $c_b\ P_j\ c_a$. If $j \notin S$, then $(\bar{b}', \bar{a}') \in T'_0$ and $(b, a) \in R'_0$, a contradiction. Instead, there is $k \in S$ such that $c_b\ P_k\ c_a$. But by the definition of F^{R^*}, $c_a\ I'_0\ a\ P'_0\ b\ I'_0\ c_b$ and so again $R'_0 \notin [R_k, R_0] \subseteq \bigcap_{i\in S}[R_i, R_0]$. □

We now present our characterizations based on *efficiency* and *adjacent welfare dominance*. Our first result adds the assumption of *strictness* and characterizes the strict-order status quo rules. Requiring *strictness* permits an intuitive proof using the lattice structure of $(\mathcal{R}, R^{*e})$.

Theorem 1. *For a fixed population with at least three agents, a rule satisfies* strictness*,* efficiency*, and* adjacent welfare dominance *if and only if it is a strict-order status quo rule.*

Proof. We have seen that each status quo rule satisfies *efficiency* and *adjacent welfare dominance*, and the strict-order status quo rules satisfy *strictness* by definition. For the converse, let F be a rule satisfying the axioms of the theorem. By Lemma 1, F also satisfies *pairwise welfare dominance*. We first calibrate the rule by constructing a candidate reference order:

Let $\tilde{R} \in \mathcal{R}^N$ be an economy with an inverse pair and let $R^* \equiv F(\tilde{R})$. Now let $R \in \mathcal{R}^N$ and $R_0 \equiv F(R)$. Since $(\mathcal{R}, R^{*e})$ is a lattice, $\bigcup_{i \in N} R_i$ has a unique least upper bound, namely $F^{R^*}(R)$. To conclude, we show that R_0 is also the least upper bound on $\bigcup_{i \in N} R_i$.

Step 1: For each $(a, b) \in R_0 \backslash R^*$, either $a\ P_N\ b$ or $b\ P_N\ a$. Let $(a, b) \in R_0 \backslash R^*$ and let $i, j, k \in N$ be distinct. Let $R'_j \equiv R_i^{-1}$ and $R'_k \equiv R_i^{-1}$. By Lemma 2, $F(R'_j, R_{-j}) = F(R'_k, R_{-k}) = R^*$. By *pairwise welfare dominance*, either $a\ P_{N\backslash\{j\}}\ b$ or $b\ P_{N\backslash\{j\}}\ a$. Also by *pairwise welfare dominance*, either $a\ P_{N\backslash\{k\}}\ b$ or $b\ P_{N\backslash\{k\}}\ a$. Since $i \in N\backslash\{j\}$ and $i \in N\backslash\{k\}$, these conditions together imply that either $a\ P_N\ b$ or $b\ P_N\ a$.

Step 2: R_0 is an upper bound. Again let $(a, b) \in R_0 \backslash R^*$. By Step 1, either $a\ P_N\ b$ or $b\ P_N\ a$, so suppose by way of contradiction that $b\ P_N\ a$. We may suppose that there is no pair of alternatives between a and b in R_0 with the same properties: If there is a pair $a', b' \in A$ such that $a\ P_0\ a'\ P_0\ b'\ P_0\ b$, $(a', b') \in R'_0 \backslash R^*$, and $b'\ P_N\ a'$, then we pass to that pair. Let $S \equiv \{c \in A : a\ P_0\ c\ P_0\ b\}$ and

$$\begin{aligned} S_1 &\equiv \{c \in S : c\ P^*\ b\}, \\ S_2 &\equiv \{c \in S : b\ P^*\ c\ P^*\ a\}, \text{ and} \\ S_3 &\equiv \{c \in S : a\ P^*\ c\}. \end{aligned}$$

The partition is illustrated in Figure 1. By Step 1, we may partition these sets as $S_1 \equiv S_1^+ \cup S_1^-$, $S_2 \equiv S_2^+ \cup S_2^-$, and $S_3 \equiv S_3^+ \cup S_3^-$ so that $S_1^+\ P_N\ a\ P_N\ S_1^-$, $S_3^+\ P_N\ b\ P_N\ S_3^-$, and $S_2^+\ P_N\ b\ P_N\ a\ P_N\ S_2^-$. Let $S^+ \equiv S_1^+ \cup S_2^+ \cup S_3^+ \cup \{b\}$ and $S^- \equiv S_1^- \cup S_2^- \cup S_3^- \cup \{a\}$ and define $R'_0 \in \mathcal{R}$ by

$$R'_0 \equiv R_0 \cup \{(c^+, c^-) : c^+ \in S^+,\ c^- \in S^-\} \backslash \{(c^-, c^+) : c^+ \in S^+,\ c^- \in S^-\}.$$

Comparing S^+ and S^-, only elements from the sets S_1^+ and S_3^- may be incomparable. If either $S_1^+ = \emptyset$ or $S_3^- = \emptyset$, then $R'_0 \in \bigcap_{i \in N} [R_0, R_i[$. Similarly, if for each $c^+ \in S_1^+$ and each $c^- \in S_3^-$, either $c^+\ P_0\ c^-$ or $c^+\ P_N\ c^-$, then $R'_0 \in \bigcap_{i \in N} [R_0, R_i[$. Either case violates *efficiency*. Suppose instead that there are $a' \in S_1^+$, $b' \in S_3^-$, and $i \in N$ such that $a'\ P_0\ b'$ and $b'\ P_i\ a'$. Since $b'\ P^*\ a'$, by Step 1, $b'\ P_i\ a'$ implies $b'\ P_N\ a'$. But now $a\ P_0\ a'\ P_0\ b'\ P_0\ b$ contradicts our assumption that no pair between a and b in R_0 with the same properties. Instead, $a\ P_N\ b$ and R_0 is an upper bound on $\bigcup_{i \in N} R_i$.

Step 3: R_0 is the least upper bound. Let $R'_0 \equiv F^{R^*}(R)$ so R'_0 is the least upper bound. Then $R'_0 \in \bigcap_{i \in N} [R^*, R_i]$. Since R_0 is an upper bound, $R_0 \in [R^*, R'_0]$. Moreover,

R_0	R^*
	S_1
a	b
$S_1 \cup S_2 \cup S_3$	S_2
b	a
	S_3

Figure 1: Illustrating the preferences hypothesized in Step 2 of the proof of Theorem 1. The alternatives between a and b in R_0 are partitioned according to their location in R^*. By Step 1, alternatives from S_1 and S_2 are unanimously comparable to a and alternatives from S_3 and S_2 are unanimously comparable to b.

$R_0 \in \bigcap_{i \in N} [R^*, R_i]$, so $R_0' \in \bigcap_{i \in N} [R_0, R_i]$ (see Remark 1). then by *efficiency*, $R_0 = R_0'$. $\square$

Theorem 2 drops the assumption of *strictness*. In this case, the full class of status quo rules emerges.

Theorem 2. *For a fixed population with at least three agents, a rule satisfies* efficiency*, and* adjacent welfare dominance *if and only if it is a status quo rule.*

The proof of Theorem 2 is in the appendix. Compared with the proof of Theorem 1, the additional difficultly consists of appropriately handling indifference classes. In particular, analysing the conditions under which indifference classes may be split or merged requires delicate treatment.

Remark 2. The characterizations in Theorems 1 and 2 continue to hold if *adjacent welfare dominance* is replaced by *pairwise welfare dominance.*

Together with Proposition 1, we have an immediate relationship among axioms.

Corollary 1. Adjacent welfare dominance *and* efficiency *imply* group strategy-proofness.

This link between solidarity and incentive properties is familiar from other public decision models.[18] Intuitively, solidarity properties align the interests of all agents and efficiency prevents the rule from behaving perversely.

Our final results are, unfortunately, negative: strengthening either *adjacent welfare dominance* to *welfare dominance* or *efficiency* to *strong efficiency* in Theorem 2 leads to an impossibility.

[18] For example, see Thomson (1993), Miyagawa (2001), and Gordon (2007a,b).

Proposition 2. *With at least three alternatives, (i) no rule satisfies* efficiency *and* welfare dominance*; and (ii) no rule satisfies* strong efficiency *and* adjacent welfare dominance.

Proof. Since *welfare dominance* implies *adjacent welfare dominance* and *strong efficiency* implies *efficiency*, it suffices to show that no status quo rule satisfies the stronger properties. In each case, we provide an example with exactly three alternatives and a strict-order status quo rule. The examples embed in larger problems by distinguishing top three alternatives in the reference order and considering an economy in which all agents' preferences below these three alternatives coincide with the reference order. The examples also apply to weak-order status quo rules because, in each case, *efficiency* requires that indifferences be broken to yield the same final orders.

Let $A \equiv \{a, b, c\}$ and let $R^*, R_0, R_0', R_0'' \in \mathcal{R}$ be as specified in the table.

R^*	R_0	R_0'	R_0''	$\hat{R}_0$	$\hat{R}_0'$
a	c	a	b	b	c
b	b	c	a	c	a
c	a	b	c	a	b

We show that F^{R^*} satisfies neither *welfare dominance* nor *strong efficiency.*

Welfare dominance. Let $R \equiv (R_0', R_0, \ldots, R_0)$ and $R' \equiv R_0'', R_0, \ldots, R_0)$. Then $F^{R^*}(R) = R_0'$ and $F^{R^*}(R') = R_0''$. However, $R_0' \notin [R_0, R_0'']$ and $R_0'' \notin [R_0, R_0']$ so R_0' and R_0'' are not comparable according to R_0. This violates *welfare dominance.*

Strong efficiency. Let $\hat{R} \equiv (\hat{R}_0, \hat{R}_0', \ldots, \hat{R}_0')$. Then $F^{R^*}(\hat{R}) = R^*$. However, $(c, a) \in \bigcap_{j \in N} \hat{R}_j$ while $(c, a) \notin R^*$. This violates *strong efficiency.* □

4 Variable populations

To accommodate variable populations, we generalize our definition of an economy. Let $\boldsymbol{\mathcal{N}}$ be a countable set of potential agents. An **economy** is now a pair $N \subset \mathcal{N}$ and $R \in \mathcal{R}^N$, denoted by $\boldsymbol{(N, R)}$.

In a variable-population model, it is natural to require solidarity with respect to changes in the population.[19] When one agent leaves, we require that all remaining agents are no worse off than initially.

[19]Population monotonicity was first introduced for bargaining problems (Thomson, 1983a,b). The adaptation to our setting is due to Bossert and Sprumont (2014).

Population monotonicity: For each $N \in \mathcal{N}$, each $R \in \mathcal{R}^N$, and each $i \in N$, $F(N\backslash\{i\}, R_{-i}) \in \bigcap_{j\in N\backslash\{i\}}[R_j, F(N,R)]$.

Solidarity alone would also allow $f(N,R) \in \bigcap_{j\in N\backslash\{i\}}[R_j, F(N\backslash\{i\}, R_{-i})]$. In our model, this possibility violates *efficiency* and is arguably perverse. Since we will only impose *population monotonicity* in conjunction with *efficiency*, we follow Bossert and Sprumont (2014) and state the axiom in the simpler directed form. We may also be interested in the departure of groups of agents. Because we use the directed form, *population monotonicity* immediately implies the stronger conclusion that for each $S \subseteq N$, $F(N\backslash S, R_{-S}) \in \bigcap_{j\in N\backslash S}[R_j, F(N,R)]$.

The status quo rules generalize naturally to this setting. Given $R^* \in \bar{\mathcal{R}}$, the **variable-population status quo rule with reference order $\boldsymbol{R^*}$, $\boldsymbol{F^{R^*}}$,**[20] is defined so that for each $N \in \mathcal{N}$ and each $R \in \mathcal{R}^N$, $F^{R^*}(N,R) \equiv F^{R^*}(R)$. Each rule applies a single reference order in all populations. Conceivably, the reference order could depend on the set of agents who are present, but this is inconsistent with *population monotonicity.* In fact, *efficiency* and *population monotonicity* characterize the variable-population status quo rules.[21] To verify this, we first relate *population monotonicity* and *pairwise welfare dominance.*

Lemma 3. *If a variable-population rule satisfies* population monotonicity*, then it satisfies* pairwise welfare dominance*.*

Proof. Let F satisfy *population monotonicity.* Let $N \in \mathcal{N}$, $R \in \mathcal{R}^N$, $i \in \mathcal{N}\backslash N$, and $R_i, R_i' \in \mathcal{R}$. Let $R_0 \equiv F(N\cup\{i\}, (R, R_i))$, $R_0' \equiv F(N \cup \{i\}, (R, R_i'))$, and $R_0'' \equiv F(N,R)$. Suppose by way of contradiction that there are $(a,b) \in R_0\backslash R_0'$ and $j,k \in N$ such that $(a,b) \in R_j$ and $(b,a) \in R_k$. By *population monotonicity,* $(a,b) \in R_0 \cap R_j$ implies $(a,b) \in R_0''$. But also by *population monotonicity,* $(b,a) \in R_0' \cap R_k$ implies $(b,a) \in R_0''$, which is a contradiction. □

Interestingly, *population monotonicity* does not directly imply *adjacent welfare dominance.*[22] Nevertheless, together with Theorem 2 and Remark 2, Lemma 3 provides a second characterization of the status quo rules.

Theorem 3. *A variable-population rule satisfies* efficiency *and* population monotonicity *if and only if it is a variable-population status quo rule.*

[20] With slight abuse of notation, we directly extend our naming convention to emphasize the relationship to the (fixed-population) status quo rules.

[21] Bossert and Sprumont (2014) characterize the strict-order status quo rules on the basis of *efficiency*, *population monotonicity*, and *strictness* (their Theorem 2). Our Theorem 3 generalizes this result.

[22] Example available upon request.

5 Conclusion

We have studied the solidary properties of preference aggregation rules in fixed and variable populations. Based on the familiar notion of welfare dominance under preference replacement, we formulated and compared new axioms for this setting, *welfare dominance* and *adjacent welfare dominance*. Our analysis uncovered and characterized a new class of status quo rules which generalizes the class of strict-order status quo rules identified by Bossert and Sprumont (2014). Our analysis further revealed an important relationship between solidarity and incentive properties, showing that *strategy-proofness* is a consequence of *adjacent welfare dominance* and *efficiency*.

Our model includes the universal domain of strict preferences over a finite set of abstract alternatives. Modifying these assumptions suggests several avenues for future work. For example, we may enrich the preference domain to allow agents to express indifference or further restrict the domain to a special class of preferences, such as preferences which are single-peaked with respect to a reference order. Similarly, with a specific economic environment in mind, we may impose a topological structure on the space of alternatives. Alternatively, following Gordon (2015), we may study solidarity in a general attribute space and introduce incomplete preferences. Finally, we have used the prudent extension to derive preferences over orders. Other extensions as possible, perhaps based on distance. Comparing extensions may yield additional insight.

A Proof of Theorem 2

Theorem 2 characterizes the status quo rules on the basis of *efficiency* and *adjacent welfare dominance*. We have seen that each status quo rule satisfies these axioms, so we turn to the converse. In fact, we prove a stronger claim: Each rule satisfying *efficiency* and *pairwise welfare dominance* is a status quo rule. We divide the proof into several lemmas. Lemma 2, proved earlier, shows that each rule satisfying *pairwise welfare dominance* selects the same reference order in each economy that contains an inverse pair.

For ease of presentation, we fix the rule, economy, and notation for all subsequent results. Let F be a rule satisfying *efficiency* and *pairwise welfare dominance*. To calibrate the candidate reference order, let $i, j \in N$, $R_0 \in \mathcal{R}$, and $R_{-ij} \in \mathcal{R}^{N\setminus\{i,j\}}$ and define $R^* \equiv F(R_0, R_0^{-1}, R_{-ij})$. Now let $R \in \mathcal{R}^N$, $R_0 \equiv F^{R^*}(R)$, and $R_0' \equiv F(R)$. For each $a \in A$, let $\bar{a}$, $\bar{a}'$, and $\bar{a}^*$ denote the equivalence classes of a in R_0, R_0', and R^* respectively.

Lemma 4 shows that each alteration F makes to the reference order meets with unanimous approval or unanimous disdain. This is a consequence of *pairwise welfare dominance* alone.

Lemma 4. *For each pair $a, b \in A$, if $(a,b) \in R_0' \backslash R^*$ or $(b,a) \in R^* \backslash R_0'$, then either $a\ P_N\ b$ or $b\ P_N\ a$.*

Proof. Let $a, b \in A$ be such that either $(a,b) \in R_0' \backslash R^*$ or $(b,a) \in R^* \backslash R_0'$ and let $i, j, k \in N$ be distinct. By Lemma 2, $F(R_i, R_i^{-1}, R_{-ij}) = F(R_i, R_i^{-1}, R_{-ik}) = R^*$. By *pairwise welfare dominance*, either $a\ P_{N\backslash\{j\}}\ b$ or $b\ P_{N\backslash\{j\}}\ a$. Also by *pairwise welfare dominance*, either $a\ P_{N\backslash\{k\}}\ b$ or $b\ P_{N\backslash\{k\}}\ a$. Since $i \in N\backslash\{j\}$ and $i \in N\backslash\{k\}$, these conditions together imply that either $a\ P_N\ b$ or $b\ P_N\ a$. $\square$

Next we show that F does not add indifferences to the reference order. Interestingly, this requires *efficiency*.

Lemma 5. *For each $a \in A$, $\bar{a}' \subseteq \bar{a}^*$.*

Proof. Suppose by way of contradiction that there is a pair $a, b \in A$ such that $\bar{a}' = \bar{b}'$ and $\bar{a}^* \neq \bar{b}^*$. By Lemma 4, either $a\ P_N\ b$ or $b\ P_N\ a$. Without loss of generality, suppose $a\ P_N\ b$. Let $S_a \equiv \bar{a}' \cap \bar{a}^*$, $S_b \equiv \bar{a}' \cap \bar{b}^*$, and $S_0 \equiv \bar{a}' \backslash (S_a \cup S_b)$ so $\bar{a}' = S_a \cup S_b \cup S_0$. By Lemma 4, these sets may be partitioned as $S_a \equiv S_a^+ \cup S_a^-$, $S_b \equiv S_b^+ \cup S_b^= \cup S_b^-$, and $S_0 \equiv S_0^+ \cup S_0^-$ so that

$$S_b^+\ P_N\ S_a^+\ P_N\ S_b^=\ P_N\ S_a^-\ P_N\ S_b^-$$

and $S_0^+\ P_N\ b\ P_N\ S_0^-$. Let $S^+ \equiv S_a^+ \cup S_b^+ \cup S_0^+$ and $S^- \equiv S_a^- \cup S_b^= \cup S_b^- \cup S_0^-$. Then $a \in S^+$, $b \in S^-$, $\bar{a}' = S^+ \cup S^-$, and $S^+\ P_N\ S^-$. But now we can construct a Pareto improvement by breaking the indifference class: Let

$$R_0'' \equiv R_0' \backslash \big\{ (c^-, c^+) : c^+ \in S^+ \text{ and } c^- \in S^- \big\}$$

so $R_0'' \in \bigcap_{i \in N} [R_i, R_0'[$, which violates *efficiency*. Instead, $a\ I^*\ b$ and $\bar{a}' \subseteq \bar{a}^*$. $\square$

Lemma 6 extends Lemma 4 to indifference classes.

Lemma 6. *For each pair $a, b \in A$, if $(a,b) \in R_0' \backslash R^*$ or $(b,a) \in R^* \backslash R_0'$, then either $\bar{a}\ P_N\ \bar{b}$ or $\bar{b}\ P_N\ \bar{a}$.*

Proof. Let $a, b \in A$ be such that either $(a,b) \in R_0' \backslash R^*$ or $(b,a) \in R^* \backslash R_0'$. By Lemma 5, $\bar{a}' \subseteq \bar{a}^*$ and $\bar{b}' \subseteq \bar{b}^*$. Therefore, for each $a' \in \bar{a}'$ and each $b' \in \bar{b}'$, either $(a', b') \in R_0' \backslash R^*$ or $(b', a') \in R^* \backslash R_0'$. By Lemma 4, either $a'\ P_N\ b'$ or $b'\ P_N\ a'$.

R_0	R^*	
	S_1	
$\bar{a}$	$\bar{b}^*$	
$S_1 \cup S_2 \cup S_3$		$\Big\}\ S_2$
$\bar{b}$	$\bar{a}^*$	
	S_3	

Figure 2: Illustrating the preferences hypothesized in Lemma 7. The alternatives between a and b in R_0 are partitioned according to their location in R^*. By Lemma 4, alternatives from S_1 and S_2 are unanimously comparable to a and alternatives from S_3 and S_2 are unanimously comparable to b.

We may partition $\bar{a}'$ and $\bar{b}'$ as $\bar{a}' \equiv S_a^+ \cup S_a^-$ and $\bar{b}' \equiv S_b^+ \cup S_b^-$ so that $S_a^+ \ P_N \ b \ P_N \ S_a^-$ and $S_b^+ \ P_N \ a \ P_N \ S_b^-$. If $S_a^+ \neq \emptyset \neq S_a^-$, then breaking $\bar{a}'$ by raising S_a^+ above S_a^- is a Pareto improvement. Similarly, if $S_b^+ \neq \emptyset \neq S_b^-$, then breaking $\bar{b}'$ by raising S_b^+ above S_b^- is a Pareto improvement. Instead, at least one set in each partition is empty. If $\bar{a}' = S_a^+$, then $b \in S_b^-$ and so $\bar{b}' = S_b^-$. In this case, $\bar{a}' \ P_N \ \bar{b}'$. If instead $\bar{a}' = S_a^-$, then $b \in S_b^+$ and so $\bar{b}' = S_b^+$. In this case, $\bar{b}' \ P_N \ \bar{a}'$. $\square$

We can now extend Lemma 4 to show that, in fact, agents unanimously approve all changes to the reference order.

Lemma 7. *For each pair $a, b \in A$, if $(a,b) \in R_0' \backslash R^*$ or $(b,a) \in R^* \backslash R_0'$, then $\bar{a}' \ P_N \ \bar{b}'$.*

Proof. Let $a, b \in A$ be such that either $(a,b) \in R_0' \backslash R^*$ or $(b,a) \in R^* \backslash R_0'$ and suppose by way of contradiction that there is $i \in N$ such that $b \ P_i \ a$. We may choose a and b to be as near as possible in R_0. More precisely, if there are $a', b' \in A$ and $i' \in N$ such that $b' \ P_{i'} \ a'$, $a \ P_0 \ a' \ P_0 \ b' \ P_0 \ b$, and either $(a', b') \in R_0' \backslash R^*$ or $(b', a') \in R^* \backslash R_0'$, then we pass to a' and b' instead. By Lemma 4, $b \ P_N \ a$.

We distinguish the (possibly empty) set of alternatives between a and b in R_0. Let $S \equiv \{c \in A : a \ P_0' \ c \ P_0' \ b\}$ and

$$
\begin{aligned}
S_1 &\equiv \{c \in S : c \ P^* \ b\}, \\
S_2 &\equiv \{c \in S : b \ R^* \ c \ R^* \ a\} \\
S_3 &\equiv \{c \in S : a \ P^* \ c\}.
\end{aligned}
$$

The partition is illustrated in Figure 2. By Lemmas 4 and 6, we may partition these sets as $S_1 \equiv S_1^+ \cup S_1^-$, $S_2 \equiv S_2^+ \cup S_2^-$, and $S_3 \equiv S_3^+ \cup S_3^-$ so that $S_1^+ \ P_N \ \bar{a}' \ P_N \ S_1^-$, $S_2^+ \ P_N \ \bar{b}' \ P_N$

$\bar{a}'\ P_N\ S_2^-$, and $S_3^+\ P_N\ \bar{b}'\ P_N\ S_3^-$. Let $S^+ \equiv S_1^+ \cup S_2^+ \cup S_3^+ \cup \{a\}$ and $S^- \equiv S_1^- \cup S_2^- \cup S_3^- \cup \{b\}$ and define

$$R_0'' \equiv R_0' \cup \big(\{(c^+, c^-) : c^+ \in S^+ \text{ and } c^- \in S^-\}\big) \backslash \{(c^-, c^+) : c^+ \in S^+ \text{ and } c^- \in S^-\}.$$

Except possibly the alternatives from S_1^+ and S_3^-, the alternatives in S^+ are unanimously comparable with the alternatives in S^-. By Lemma 5, for each $a' \in S_1^+$ and each $b' \in S_3^-$, $\bar{a'}' \neq \bar{b'}'$. If either $S_1^+ = \emptyset$ or $S_3^- = \emptyset$, then R_0'' is a Pareto improvement over R_0'. Similarly, if for each $a' \in S_1^+$ and each $b' \in S_3^-$ such that $a'\ P_0\ b'$, we have $a'\ P_N\ b'$, then R_0'' is a Pareto improvement over R_0'. Instead, there are $a' \in S_1^+$, $b' \in S_3^-$, and $i' \in N$ such that $a'\ P_0\ b'$ and $b'\ P_{i'}\ a'$. But $a\ P_0\ a'\ P_0\ b'\ P_0\ b$, so this contradicts the choice of a and b. Instead, $\bar{a}'\ P_N\ \bar{b}'$. □

We now show that F and F^{R^*} include the same indifference classes.

Lemma 8. *For each $a \in A$, $\bar{a}' = \bar{a}$.*

Proof. First we show $\bar{a}' \subseteq \bar{a}$. Let $a, b \in A$ with $\bar{a}' = \bar{b}'$. By Lemma 5, $\bar{a}^* = \bar{b}^*$. Suppose by way of contradiction that $\bar{a} \neq \bar{b}$ and, without loss of generality, $a\ P_0\ b$. Let $S_a \equiv \{a' \in \bar{a}^* : a'\ R_0\ a\}$ and $S_b \equiv \{b' \in \bar{a}^* : a\ P_0\ b'\}$. Then $a \in S_a$, $b \in S_b$, and $\bar{a}^* = S_a \cup S_b$. Moveover, by the definition of F^{R^*}, $\bar{a}\ P_N\ \bar{b}$. Now let

$$R_0'' \equiv R_0' \backslash \{(b', a') : b' \in \bar{a}' \cap S_b \text{ and } a' \in \bar{a}' \cap S_a\}.$$

Then R_0'' is a Pareto improvement over R_0'. Instead, $\bar{a} = \bar{b}$ and $\bar{a}' \subseteq \bar{a}$.

Next we show $\bar{a} \subseteq \bar{a}'$. Let $a, b \in A$ with $\bar{a} = \bar{b}$. Suppose by way of contradiction that $\bar{a}' \neq \bar{b}'$ and, without loss of generality, $a\ P_0'\ b$. By definition of F^{R^*}, $\bar{a} \subseteq \bar{a}^*$ and so $(b, a) \in R^* \backslash R_0'$. If there are $a', b' \in \bar{a}$ and $i, j \in N$ such that $a' \in \bar{a}'$, $b' \in \bar{b}'$, $a'\ P_i\ b'$, and $b'\ P_j\ a'$, then we will have a contradiction with Lemma 6. The following claim shows that such combination exists.

Claim: There are $a', b' \in \bar{a}$ and $i, j \in N$ such that $a'\ I_0'\ a$, $b'\ I_0'\ b$, $a'\ P_i\ b'$, and $b'\ P_j\ a'$. Let $S_a \equiv \{c \in \bar{a} : c\ R_0'\ a\}$ and $S_b \equiv \{c \in \bar{a} : a\ P_0'\ c\}$. Then $a \in S_a$, $b \in S_b$, and $\bar{a} = S_a \cup S_b$.

Suppose $S_a = \{a\}$. Since $\bar{a} = \bar{b}$, by the definition of F^{R^*}, there are $c \in S_b$ and $i, j \in N$ such that $a\ P_i\ c$ and $c\ P_j\ a$. However, by Lemma 6, either $a\ P_N\ c$ or $c\ P_N\ a$, so this is a contradiction. Similarly, if $S_b = \{b\}$, then there are $c' \in S_a$ and $i', j' \in N$ such that $c'\ P_{i'}\ b$ and $b\ P_{j'}\ c'$. This contradicts Lemma 6.

Suppose instead that $|S_a| \geq 2$ and $|S_b| \geq 2$. By the definition of F^{R^*}, there are $a', a'' \in S_a$, $b', b'' \in S_b$, and $i, j \in N$ such that $a' \; P_i \; b'$ and $b'' \; P_j \; a''$. Then by Lemmas 6 and 7,

$$\begin{aligned}
&\text{(i)} && a' \; P_N \; b' \\
&\text{(ii)} && b'' \; P_N \; a'' \\
&\text{(iii)} && a' \; P_N, b'' \quad \text{or} \quad b'' \; P_N \; a' \\
&\text{(iv)} && a'' \; P_N \; b' \quad \text{or} \quad b' \; P_N \; a''.
\end{aligned}$$

If the first conditions hold in (iii) and (iv), then $a' \; P_N \; b'' \; P_N \; a'' \; P_N \; b'$. By Lemma 6, all agents rank the remaining alternatives in $\bar{a}$ the same with respect to these four distinguished alternatives. More precisely, for each $c \in \bar{a} \backslash \{a', a'', b', b''\}$,

$$\begin{aligned}
&\text{(v)} && c \; P_N \; a' \quad \text{or} \quad a' \; P_N \; c \\
&\text{(vi)} && c \; P_N \; a'' \quad \text{or} \quad a'' \; P_N \; c.
\end{aligned}$$

Then $\bar{a}$ can be partitioned into more preferred and less preferred sets: Let $\hat{S}_a \equiv \{c \in \bar{a} : c \; R_N \; a'\}$ and $\hat{S}_b \equiv \bar{a} \backslash \hat{S}_a$. But $a' \; I_0 \; a''$, so this contradicts the definition of F^{R^*}.

The remaining combinations of conditions in (iii) and (iv) yield similar partitions. If the first condition of (iii) and second condition of (iv) hold, then $a' \; P_N \; b' \; P_N \; a''$ and $a' \; P_N \; b'' \; P_N \; a''$. Again we can find a unanimous partition separating a' and a'' which contradicts $a' \; I_0 \; a''$. If the second condition of (iii) and first condition of (iv) hold, then $b'' \; P_N \; a' \; P_N \; b'$ and $b'' \; P_N \; a'' \; P_N \; b'$. In this case, we can find a unanimous partition separating b' and b'' which contradicts $b' \; I_0 \; b''$. Finally, if the second conditions hold in both (iii) and (iv), then $b'' \; P_N \; a' \; P_N \; a'' \; P_N \; b'$. Again we can find a unanimous partition separating b' and b'' which contradicts $b' \; I_0 \; b''$. This verifies the claim. □

To complete the proof of Theorem 2, we show that R_0 and R'_0 order their common indifference classes in the same way. Since the indifference classes are the same, we drop the prime notation and let $\bar{a}$ refer to the common indifference class of a in R_0 and R'_0.

Lemma 9. *The orders R'_0 and R_0 coincide.*

Proof. By Lemma 8, R_0 and R'_0 contain the same indifference classes. Suppose by way of contradiction that the orders differ. Then there is a pair $a, b \in A$ such that $a \; P_0 \; b$ and $b \; P'_0 \; a$. Without loss of generality, we may assume that $\bar{a}$ and $\bar{b}$ are adjacent in R'_0, passing to a different pair of indifference classes if necessary. If $b \; R^* \; a$, then by the definition of

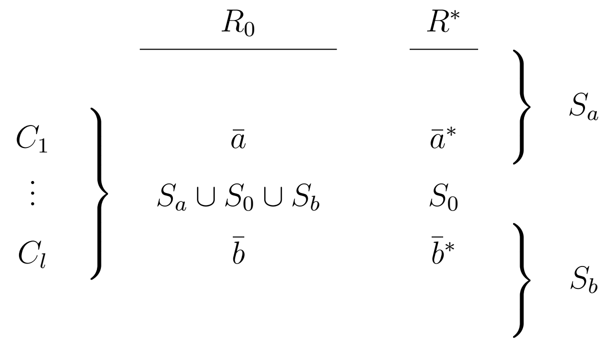

Figure 3: Illustrating the preferences hypothesized in Lemma 9. The alternatives between a and b in R_0 are partitioned according to their location in R^*. The indifference classes of R_0 constituting $S_a \cup S_0 \cup S_b$ are enumerated as $C_1, \ldots, C_l$.

F^{R^*}, $\bar{a}\ P_N\ \bar{b}$. But then switching the order of these indifference classes in R_0' is a Pareto improvement: Let

$$R_0'' \equiv R_0' \cup \left(\left\{(a', b') : a' \in \bar{a} \text{ and } b' \in \bar{b}\right\}\right) \setminus \left\{(b', a') : a' \in \bar{a} \text{ and } b' \in \bar{b}\right\}.$$

so $R_0'' \in \bigcap_{i \in N} [R_i, R_0'[$, which violates *efficiency.*

Suppose instead that $a\ P^*\ b$. By Lemma 7, $\bar{b}\ P_N\ \bar{a}$. Let

$$\begin{aligned}
S_a &\equiv \{c \in A : c\ R^*\ a \text{ and } a\ P_0\ c\ P_0\ b\}, \\
S_0 &\equiv \{c \in A : a\ P^*\ c\ P^*\ b \text{ and } a\ P_0\ c\ P_0\ b\}, \\
S_b &\equiv \{c \in A : b\ R^*\ c \text{ and } a\ P_0\ c\ P_0\ b\},
\end{aligned}$$

and $S \equiv S_a \cup S_0 \cup S_b$. The partition is illustrated in Figure 3. By the definition of F^{R^*}, $S_b\ P_N\ \bar{b}\ P_N\ \bar{a}\ P_N\ S_a$.

We claim that $S_0 \neq \emptyset$. Suppose by way of contradiction that $S_0 = \emptyset$. First, if $S = \emptyset$, then $\bar{a}$ and $\bar{b}$ are adjacent in R_0. But then $a\ P_0\ b$ violates *efficiency*, so instead $S \neq \emptyset$. Let $c \in S$ be in the indifference class immediately below $\bar{a}$ in R_0. If $c \in S_b$, then $\bar{c}\ P_N\ \bar{a}$ in violation of *efficiency.* Instead, $c \in S_a$. Let $c' \in S$ be in the indifference class immediately below $\bar{c}$ in R_0. If $c' \in S_b$, then $\bar{c}'\ P_N\ \bar{c}$ in violation of *efficiency.* Instead, $c' \in S_a$. Continuing in this fashion, $S_b = \emptyset$. But now there is $c'' \in S_a$ in the indifference class immediately above b. Since $\bar{b}\ P_N\ S_a$, this violates *efficiency.* Instead, $S_0 \neq \emptyset$.

For reference, we enumerate the indifference classes in R_0 that constitute S. Let $C_1, \ldots, C_l \subseteq S$ be such that $\bigcup_{k=1}^{l} C_k = S$ and for each pair $k, k' \in \{1, \ldots, l\}$ with $k < k'$, $C_k\ P_0\ C_{k'}$. By Lemma 5, for each $k \in \{1, \ldots, l\}$, either $C_k \subseteq S_a$, $C_k \subseteq S_b$, or $C_k \subseteq S_0$.

Let $C^+, C^- \in \{C_1, \ldots, C_l\}$ be the highest and lowest indifference classes that intersect S_0: $C^+, C^- \subseteq S_0$ and for each $c \in S_0$, $C^+ \; R_0 \; \bar{c} \; R_0 \; C^-$.

We now compare the locations of the alternatives of S_0 in R_0 and R_0'. First consider C^+ and suppose that $C^+ \; P_0' \; \bar{b}$. Then by Lemma 7, $C^+ \; P_N \; \bar{b} \; P_N \; \bar{a} \; P_N \; S_a$. By the definition of F^{R^*}, $C^+ \; P_N \; \bar{a}$ implies $C^+ \neq C_1$ and so $C_1 \nsubseteq S_0$. Similarly, $S_b \; P_N \; \bar{a}$ implies $C_1 \nsubseteq S_b$ so $C_1 \subseteq S_a$. Since $C^+ \; P_N \; C_1$, by the definition of F^{R^*}, $C^+ \neq C_2$. Repeating the same arguments, $C_2 \subseteq S_a$. Continuing in this fashion, $\bigcup_{i=1}^{l} C_k \subseteq S_a$. But then $S_0 = \emptyset$, which is a contradiction. Instead, $\bar{b} \; P_0' \; C^+$. Since $\bar{a}$ and $\bar{b}$ are adjacent in R_0', $\bar{a} \; P_0' \; C^+$.

Next consider C^- and suppose that $\bar{a} \; P_0' \; C^-$. Then by Lemma 7, $S_b \; P_N \; \bar{b} \; P_N \; \bar{a} \; P_N \; C^-$. By the definition of F^{R^*}, $\bar{b} \; P_N \; C^-$ implies $C^- \neq C_l$ and so $C_l \nsubseteq S_0$. Similarly, $\bar{b} \; P_N \; S_a$ implies $C_l \nsubseteq S_a$ so $C_l \subseteq S_b$. But now $C_l \; P_N \; C^-$, so $C^- \neq C_{l-1}$. Repeating the same arguments, $C_{l-1} \subseteq S_b$. Continuing in this fashion, $\bigcup_{i=1}^{l} C_k \subseteq S_b$. But then $S_0 = \emptyset$, which is a contradiction. Instead, $C^- \; P_0' \; \bar{a}$. Since $\bar{a}$ and $\bar{b}$ are adjacent in R_0', $C^- \; P_0' \; \bar{b}$.

Altogether, $C^- \; P_0' \; \bar{b} \; P_N \; \bar{a} \; P_0' \; C^+$, so by Lemma 7, $C^- \; P_N \; \bar{b} \; P_N \; \bar{a} \; P_N \; C^+$. Also, $S_b \; P_N \; C^+$ and $C^- \; P_N \; S_a$. Since $C^- \; P_N \; C^+$, by the definition of F^{R^*}, C^+ and C^- are not adjacent in R_0. Moreover, repeating the same arguments as before with C^+ in the place of $\bar{a}$ and C^- in the place of $\bar{b}$, there is a pair $\tilde{C}^+, \tilde{C}^- \subseteq S_0$ such that $C^+ \; P_0 \; \tilde{C}^+ \; P_0 \; \tilde{C}^- \; P_0 \; C^-$. These arguments further imply that $\tilde{C}^- \; P_0' \; C^- \; P_0' \; \bar{b} \; P_N \; \bar{a} \; P_0' \; C^+ \; P_0' \; \tilde{C}^+$. But then $\tilde{C}^- \; P_N \; \tilde{C}^+$, so $\tilde{C}^+$ and $\tilde{C}^-$ are not adjacent in R_0. Continuing in this fashion, we obtain two sequences of indifferences classes contained in S_0, but this is impossible as $\{C_1, \ldots, C_l\}$ contains finitely many indifference classes. Instead, we conclude that $\bar{a} \; P_0' \; \bar{b}$ and $R_0' = R_0$. □

Combining the results of Lemmas 2 through 9, we conclude that $F = F^{R^*}$. Altogether, each rule satisfying *efficiency* and *pairwise welfare dominance* is a status quo rule.

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