

# Extreme-Value Copulas

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- Let  $\mathbf{X}_i = (X_{i1}, \dots, X_{id})_{\{1, \dots, n\}}$  i.i.d. random vectors with continuous distribution function  $F$  and marginal distribution functions  $F_1, \dots, F_d$  and copula  $C_F$ .

$$F(x_1, \dots, x_d) = C_F(F_1(x_1), \dots, F_d(x_d)) \quad (1)$$

- Vector of componentwise maxima:

$$\mathbf{M}_n = (M_{n,1}, \dots, M_{n,d}), \quad \text{where } M_{n,j} = \bigvee_{i=1}^n X_{ij}, \quad (2)$$

with ' $\bigvee$ ' denoting maximum,  $F_{(n)}$  the distribution function of  $\mathbf{M}_n$  and  $F_{(n)1}, \dots, F_{(n)d}$  the marginals associated to  $M_{n,1}, \dots, M_{n,d}$

The copula  $C_{(n)}$  associated to  $\mathbf{M}_n$  is given by

$$C_{(n)}(u_1, \dots, u_d) = C_F(u_1^{1/n}, \dots, u_d^{1/n})^n, \quad (u_1, \dots, u_d) \in [0, 1]^d.$$

**Proof.**

$$\begin{aligned} C_{(n)}(F_{(n)1}(x_1), \dots, F_{(n)d}(x_d)) &= P(M_{n,1} \leq x_1, \dots, M_{n,d} \leq x_d) \\ &= P(X_{11} \leq x_1, \dots, X_{1d} \leq x_d)^n \\ &= C_F(F_1(x_1), \dots, F_d(x_d))^n \\ &= C_F(F_{(n)1}(x_1)^{1/n}, \dots, F_{(n)d}(x_d)^{1/n})^n \end{aligned}$$

## Definition 1

A copula  $C$  is called an **extreme-value copula** if there exists a copula  $C_F$  such that

$$C_F(u_1^{1/n}, \dots, u_d^{1/n})^n \rightarrow C(u_1, \dots, u_d) \quad (n \rightarrow \infty) \quad (3)$$

for all  $(u_1, \dots, u_d) \in [0, 1]^d$ . The copula  $C_F$  is said to be in the **domain of attraction** of  $C$ .

## Definition 2

A  $d$ -variate copula  $C$  is **max-stable** if it satisfies the relationship

$$C(u_1, \dots, u_d) = C(u_1^{1/m}, \dots, u_d^{1/m})^m \quad (4)$$

for every integer  $m \geq 1$  and all  $(u_1, \dots, u_d) \in [0, 1]^d$ .

### Theorem 3

*A copula is an extreme-value copula if and only if it is max-stable.*

**Proof.**

⇐ : Trivial

⇒ : for fixed integer  $m \geq 1$  and for  $n = mk$ , write

$$\begin{aligned} C_{(n)}(u_1, \dots, u_d) &= C_F(u_1^{1/n}, \dots, u_d^{1/n})^n = \\ C_F(u_1^{1/mk}, \dots, u_d^{1/mk})^{mk} &= C_F((u_1^{1/m})^{1/k}, \dots, (u_d^{1/m})^{1/k})^{mk} = \\ C_{(k)}(u_1^{1/m}, \dots, u_d^{1/m})^m \end{aligned}$$



### Theorem 4

A  $d$ -variate copula  $C$  is an extreme-value copula if and only if there exists a finite Borel measure  $H$  on  $\Delta_{d-1}$ , called *spectral measure*, such that  $(u_1, \dots, u_d) \in (0, 1]^d$ ,

$$C(u_1, \dots, u_d) = \exp(-\ell(-\log u_1, \dots, -\log u_d)),$$

where the *tail dependence function*  $\ell : [0, \infty)^d \rightarrow [0, \infty)$  is given by

$$\ell(x_1, \dots, x_d) = \int_{\Delta_{d-1}} \bigvee_{j=1}^d (w_j x_j) dH(w_1, \dots, w_d), \quad (5)$$

with  $(x_1, \dots, x_d) \in [0, \infty)^d$ . The spectral measure  $H$  is arbitrary except for the  $d$  moment constraints

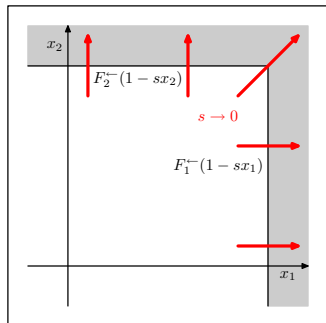
$$\int_{\Delta_{d-1}} w_j dH(w_1, \dots, w_d) = 1, \quad j \in \{1, \dots, d\}. \quad (6)$$

For  $(x_1, \dots, x_d) \in [0, \infty)^d$ :

$$\lim_{t \downarrow 0} t^{-1} (1 - C_F(1 - tx_1, \dots, 1 - tx_d)) \\ = \ell(x_1, \dots, x_d),$$

see for instance

DREES & HUANG (1998).



## Definition 5 (PICKANDS (1981))

The restriction of  $\ell$  to the unit simplex

$$\Delta_d = \{(w_1, \dots, w_d) : \sum_{i=1}^d w_i = 1, w_j \geq 0, \forall j = 1, \dots, d\}$$

is called **Pickands dependence function**  $A$ .

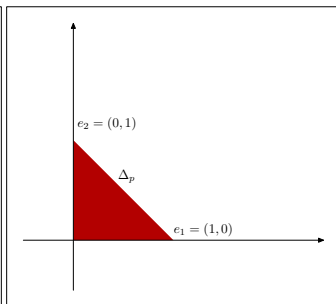
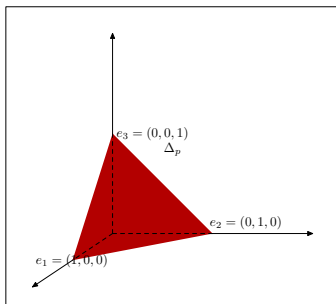
$$\begin{aligned} C(\mathbf{u}) &= \exp\{-\ell(-\log u_1, \dots, -\log u_d)\} \\ &= \exp\left\{\left(\sum_{j=1}^d \log u_j\right) A\left(\frac{\log u_1}{\sum_{j=1}^d \log u_j}, \dots, \frac{\log u_{d-1}}{\sum_{j=1}^d \log u_j}\right)\right\} \end{aligned}$$

for  $0 < u_j \leq 1$ .



For dimension  $d = 3$ :

$$\Delta_3 = \left\{ (w_1, w_2, w_3) : \sum_{i=1}^3 w_i = 1, w_j \geq 0, \forall j = 1, \dots, 3 \right\}$$



$A$  satisfies the following properties:

- 1  $A$  is convex;
- 2  $\max(w_1, \dots, w_d) \leq A(\mathbf{w}) \leq 1$ .

$\Rightarrow A(\mathbf{e}_j) = 1$ , for  $\mathbf{e}_j = (0, \dots, 1, \dots, 0)$  being the  $d$  vertices.

### Remark

*Functions satisfying the previous conditions do not characterize the class of multivariate extreme value copulas except in dimension  $d = 2$ .*

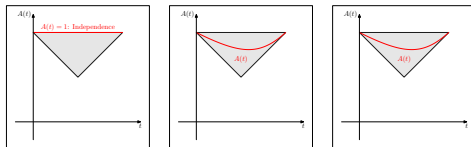
## Theorem 6

A bivariate copula  $C$  is an extreme-value copula if and only if

$$C(u, v) = (uv)^{A(\log(v)/\log(uv))}, \quad (u, v) \in (0, 1]^2 \setminus \{(1, 1)\}, \quad (7)$$

where  $A : [0, 1] \rightarrow [1/2, 1]$  is convex and satisfies  $t \vee (1 - t) \leq A(t) \leq 1$  for all  $t \in [0, 1]$ .

Pickands dependence  
function  $A$  together with  
the region  
 $t \vee (1 - t) \leq A(t) \leq 1$ .



# Archimedean copula

- Archimedean copula with generator  $\phi : [0, 1] \rightarrow [0, \infty]$ .

$$C_\phi(u_1, \dots, u_d) = \phi^{\leftarrow}(\phi(u_1) + \dots + \phi(u_d)), \quad (u_1, \dots, u_d) \in [0, 1]^d \quad (8)$$

- If the following limit exists,

$$\theta = -\lim_{s \downarrow 0} \frac{\phi(1-s)}{s \phi'(1-s)} \in [0, 1] \quad (9)$$

then

$$C_\phi(u_1^{1/n}, \dots, u_p^{1/n})^n \rightarrow C_\theta(u_1, \dots, u_d)$$

with

$$C_\theta(u_1, \dots, u_d) = \exp\left\{-\left((-\log u_1)^{1/\theta} + \dots + (-\log u_d)^{1/\theta}\right)^\theta\right\},$$

# Archimedean copula

- Gumbel–Hougaard or logistic copula

$$C_\theta(u_1, \dots, u_d) = \exp\left\{-\left((-\log u_1)^{1/\theta} + \dots + (-\log u_d)^{1/\theta}\right)^\theta\right\},$$

- $\theta$ : measures the degree of dependence

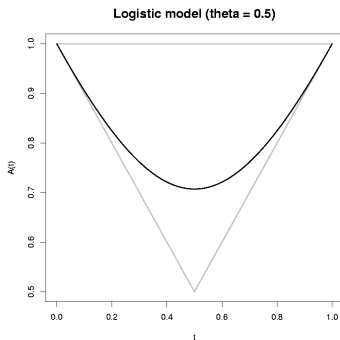
$\theta = 1$  : independence

$\theta = 0$  : complete dependence.

$$\ell(x_1, \dots, x_d) = \begin{cases} (x_1^{1/\theta} + \dots + x_d^{1/\theta})^\theta & \text{if } 0 < \theta \leq 1, \\ x_1 \vee \dots \vee x_d & \text{if } \theta = 0, \end{cases} \quad (10)$$

# Example: Logistic or Gumbel copula

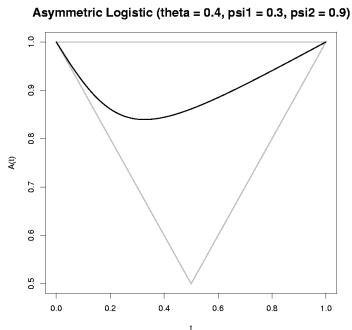
$$\text{GUMBEL (1961): } A(t) = [t^{1/\theta} + (1 - t)^{1/\theta}]^\theta$$



# Example: Asymmetric logistic copula

TAWN (1988):

$$A(t) = (1 - \psi_1)(1 - t) + (1 - \psi_2)t + [(\psi_1 t)^{1/\theta} + \{\psi_2(1 - t)\}^{1/\theta}]^\theta$$



# Example: Archimedean survival copulas

- Survival copula of an Archimedean copula:

$$C_\phi(u_1, u_2) = u_1 + u_2 - 1 + \phi^\leftarrow(\phi(1 - u_1) + \phi(1 - u_2))$$

- If

$$\theta = -\lim_{s \downarrow 0} \frac{\phi(s)}{s\phi'(s)} \in [0, \infty]$$

then  $C_\phi(u_1^{1/n}, u_2^{1/n})^n \rightarrow C(u_1, u_2)$  where

$$A(t) = 1 - \{t^{-1/\theta} + (1 - t)^{-1/\theta}\}^{-\theta}$$

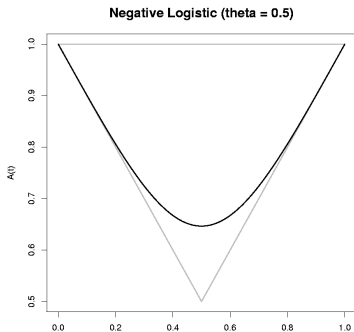
- $C$  is the **negative logistic** or **Galambos** copula



# Example: Negative logistic (Galambos) copula

GALAMBOS (1975):

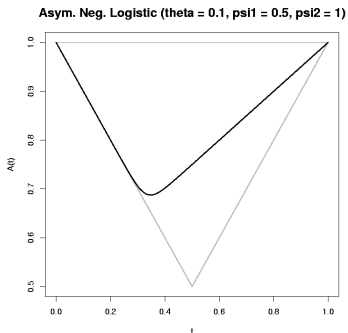
$$A(t) = 1 - \{t^{-1/\theta} + (1-t)^{-1/\theta}\}^{-\theta}$$



# Example: Asym. negative logistic copula

JOE (1990)

$$A(t) = 1 - [\{\psi_1(1-t)\}^{-1/\theta} + (\psi_2 t)^{-1/\theta}]^{-\theta}$$



$C$  admits the following properties:

- **Positive quadrant dependent**,  $A \leq 1$  implies that  $C(u, v) \geq uv$  for all  $(u, v) \in [0, 1]^2$ .
- **Monotone regression dependent**, that is, the conditional distribution of  $U$  given  $V = v$  is stochastically increasing in  $v$  and *vice versa*; see GARRALDA-GUILLEM (2000).
- **Kendall's  $\tau$**  and **Spearman's  $\rho_S$**

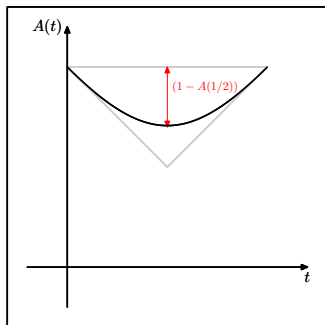
$$\tau = 4 \iint_{[0,1]^2} C(u, v) dC(u, v) - 1 = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t),$$

$$\rho_S = 12 \iint_{[0,1]^2} uv dC(u, v) - 3 = 12 \int_0^1 \frac{1}{(1+A(t))^2} dt - 3.$$

**Coefficient of upper tail dependence:** For a bivariate copula  $C_F$  in the domain of attraction of an extreme-value copula with tail dependence function  $\ell$  and Pickands dependence function  $A$ , we find

$$\begin{aligned}\lambda_U &= \lim_{u \uparrow 1} P(U > u \mid V > u) \\ &= \lim_{t \downarrow 0} t^{-1} (2t - 1 + C_F(1-t, 1-t)) \\ &= 2 - \ell(1, 1) = 2(1 - A(1/2)) \in [0, 1].\end{aligned}$$

$$\begin{aligned}\lambda_L &= \lim_{u \downarrow 0} P(U \leq u \mid V \leq u) \\ &= \lim_{u \downarrow 0} u^{(2A(1/2)-1)}\end{aligned}$$



# Estimation

- Parametric estimation: not treated here
- Nonparametric estimation
  - Known margins: apply *probability integral transform* (i.i.d. sample from  $C$ )

$$(U_{i,1}, \dots, U_{i,d}) := (F_1(X_{i,1}), \dots, F_d(X_{i,d}))$$

Unknown margins: use *ranks* (empirical copula)

$$(\hat{U}_i, \dots, \hat{U}_{i,d}) := \left( \frac{R_{i,1}}{n+1}, \dots, \frac{R_{i,d}}{n+1} \right)$$

, where  $R_{i,1} = \sum_{k=1}^n I(X_{k,1} \leq X_{i,1})$ .

# Pickands estimator

For  $t \in [0, 1]$ , define

$$\xi_i(t) = \min\left(\frac{\log U_{i,1}}{1-t}, \frac{\log U_{i,2}}{t}\right),$$

with the obvious conventions for division by zero.

$$\begin{aligned} P[\xi_i(t) > x] &= P[U_i < e^{-(1-t)x}, V_i < e^{-tx}] \\ &= C(e^{-(1-t)x}, e^{-tx}) = e^{-x A(t)}. \end{aligned} \quad (11)$$

PICKANDS (1981) proposed the estimator:

$$\frac{1}{\hat{A}^P(t)} = \frac{1}{n} \sum_{i=1}^n \xi_i(t). \quad (12)$$

# Deheuvels estimator:

DEHEUVELS (1991):



$$\frac{1}{\hat{A}^D(t)} = \frac{1}{n} \sum_{i=1}^n \xi_i(t) - t \frac{1}{n} \sum_{i=1}^n \xi_i(1) - (1-t) \frac{1}{n} \sum_{i=1}^n \xi_i(0) + 1. \quad (13)$$

- Endpoint constraints verified:  $\hat{A}^D(0) = \hat{A}^D(1) = 1$
- Weights  $(1-t)$  and  $t$  rather pragmatic choices:

# Deheuvels estimator:

SEGERS (2007):

$$\frac{1}{\hat{A}^D(t)} = \frac{1}{n} \sum_{i=1}^n \xi_i(t) - \beta_1(t) \frac{1}{n} \sum_{i=1}^n \xi_i(1) - \beta_2(t) \frac{1}{n} \sum_{i=1}^n \xi_i(0) + 1. \quad (14)$$

Variance-minimizing weight functions via a linear regression:

$$\xi_i(t) = \beta_0(t) + \beta_1(t) \{\xi_i(0) - 1\} + \beta_2(t) \{\xi_i(1) - 1\} + \epsilon_i(t).$$

$\hat{\beta}_0(t)$ : minimum-variance estimator for  $1/A(t)$  in the class of Deheuvels estimators.



## CFG-estimator

CAPÉRAÀ, FOUGÈRES AND GENEST (1997):

$$\log \hat{A}^{CFG}(t) = -\frac{1}{n} \sum_{i=1}^n \log \xi_i(t) - (1-t) \frac{1}{n} \sum_{i=1}^n \log \xi_i(0) - t \frac{1}{n} \sum_{i=1}^n \log \xi_i(1) \quad (15)$$

for  $t \in [0, 1]$ , using

$$E[-\log \xi_i(t)] = \log A(t) + \gamma, \quad t \in [0, 1],$$

with the Euler–Mascheroni constant  $\gamma = 0.5772\dots$

$$\log \hat{A}^{CFG}(t) = -\frac{1}{n} \sum_{i=1}^n \log \xi_i(t) + \beta_1(t) \frac{1}{n} \sum_{i=1}^n \log \xi_i(0) + \beta_2(t) \frac{1}{n} \sum_{i=1}^n \log \xi_i(1).$$

# Estimation

- HALL-TAJVIDI (2000)
- Optimize weight functions  $\beta_1(\mathbf{w}), \dots, \beta_d(\mathbf{w})$ ; see SEGERS (2007) and GUDENDORF & SEGERS (2009).
- Estimation using unknown margins; see GENEST & SEGERS (2009).
- Other estimators . . .

# Shape constraints

- Even after endpoint correction, the estimators still do not satisfy the **shape constraints**
  - $A$  is convex
  - $\max(t, 1 - t) \leq A(t) \leq 1$
- Possible solutions:
  - Convex minorant to  $\{A(t) \vee t \vee (1 - t)\} \wedge 1$  (DEHEUVELS 1991)
  - Spline smoothing (HALL & TAJVIDI 2000)
  - $L_2$  projection (FILS-VILLETARD, GUILLOU & SEGERS 2008)

# Reading

References can be found in:

- G. GUDENDORF & J. SEGERS (2009)  
“Extreme-Value Copulas”  
IS - DP 2009-26