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Quintic Spline Method for Solving Linear and Nonlinear Boundary Value Problems

(Kaedah Splin Kuintik untuk Menyelesaikan Masalah Nilai Sempadan Linear dan Tak Linear)

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ABSTRACT

In this article, a fourth order quintic spline method has been developed to obtain numerical solutions for second order boundary value problems with Dirichlet boundary conditions. The developments of the quintic spline method and convergence analysis were presented. Three test problems have been considered for comparison purposes. The numerical results showed that the quintic spline method is more accurate compared to existing cubic spline method when solving nonlinear second order boundary value problems but vice versa when solving linear second order boundary value problems.

Keywords: Boundary value problem; quintic spline method; shooting method; spline interpolation

ABSTRAK

Dalam artikel ini, satu kaedah splin kuintik peringkat keempat telah dibangunkan untuk memperoleh penyelesaian berangka bagi masalah nilai sempadan peringkat kedua dengan syarat sempadan Dirichlet. Pembangunan kaedah splin kuintik dan analisis penumpuan telah dibentangkan. Tiga masalah ujian telah diambil kira untuk tujuan perbandingan. Keputusan berangka menunjukkan bahawa kaedah splin kuintik adalah lebih tepat berbanding dengan kaedah splin kubik sedia ada apabila menyelesaikan masalah nilai sempadan tak linear peringkat kedua, tetapi sebaliknya apabila menyelesaikan masalah nilai sempadan linear peringkat kedua.

Kata-kata kunci: Interpolasi splin; kaedah splin kuintik; kaedah tembakan; masalah nilai sempadan

INTRODUCTION

The theory of spline functions is a very active field in approximation theory and ordinary differential equations (ODEs). Bickley first suggested the idea of using spline methods to solve boundary value problems (BVPs) numerically in 1968 (Hamid et al. 2011). Since then, many scholars began to use the spline methods to approximate the solution of BVPs. Recently, many scholars have used different degrees of splines to obtain approximate solution for second order BVPs (Al-Said et al. 2011; Ala'yed et al. 2013; Caglar et al. 2006; Fauzi & Sulaiman 2012; Hamid et al. 2012, 2011; Ramadan et al. 2007; Rashidinia et al. 2008). Most of these spline methods have been used to approximate special cases of BVPs such as linear BVPs and BVPs with the absence of first derivative.

In our work, however, we are considering the general second order BVPs of the form

$$u'' = f(x, u, u'),\ a \leq x \leq b, \tag{1}$$

subject to the Dirichlet boundary conditions given by

$$u(a) = \alpha,\ u(b) = \beta. \tag{2}$$

Keller (1968) proved that problem (1) together with the Dirichlet boundary conditions in (2), has a unique solution if $f(x, u, u')$ satisfy the following conditions:

$f(x, u, u')$ is continuous on the domain Ω where the domain Ω is defined as

$$\Omega = \{(x, u, u') \,|\, a \leq x \leq b, -\infty < u < \infty, -\infty < u' < \infty\},$$

$\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial u'}$ exist and continuous for all $(x,u,u') \in \Omega$; and

$\dfrac{\partial f}{\partial u} > 0$, and $\left|\dfrac{\partial f}{\partial u'}\right| \leq W$ for some positive constant W.

However, in the rest of this discussion we have to assume that $u \in C^6[a,b]$. The main objective for our research was to introduce a new quintic spline method to approximate the second order BVPs as in (1).

This paper is organized as follows. In the next section, a quintic spline method is constructed. Next, we discussed the convergent for the proposed method. To show the performance of the proposed algorithm and for comparison purposes, some numerical examples are given in the following section. Finally, the conclusion in the last section.

QUINTIC SPLINE

Let P be the partition for the interval $[a, b]$ such that

$$P : a = x_0, x_1, \dots, x_n = b,$$

where $x_i = a + ih$, and $h = \frac{b-a}{n}$. Every quintic spline function has to satisfy the following conditions:

$S(x) = s_i(x),\ x \in [x_i, x_{i+1}]$,

$S(a) = u(a)$ and $S(b) = u(b)$,

$s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1}),\ r = 0, 1, 2, 3, 4.$

We let $u(x)$ be the exact solution of problem (1) and s_i be the approximate solution to $u_i = u(x_i)$ obtained by the quintic spline $s_i(x)$ on the interval $[x_i, x_{i+1}]$. Since our spline is of degree five, the fourth derivative is a linear polynomial, which can be written as follows

$$s_i^{(4)}(x) = Z_{i+1}\frac{(x-x_i)}{h} + Z\frac{(x-x_{i+1})}{h}, \tag{3}$$

where $Z_i = s_i^{(4)}(x)$ and $x \in [x_i, x_{i+1}]$. On integrating (3) four times, we obtain

$$s_i(x) = Z_{i+1}\frac{(x-x_i)^5}{120h} + Z_i\frac{(x_{i+1}-x)^5}{120h} + A_i(x-x_i)^3 + B_i(x_{i+1}-x)^2 + C_i(x-x_i) + D_i(x_{i+1}-x), \tag{4}$$

where A_i, B_i, C_i, and D_i, $i = 0, 1, \ldots, n-1$, are coefficients which need to be determined in terms of u_{i+1}, μ_i, η_i, and Z_i. In order to derive explicit expressions for the four coefficients of equation (4), we define the following relations:

$$u_i = s_i(x_i), \tag{5}$$

$$u_{i+1} = s_i(x_{i+1}), \tag{6}$$

$$\mu_i = s_i''(x_i), \text{ and} \tag{7}$$

$$\eta_i = s_i'''(x_i). \tag{8}$$

From (5)-(8) and by using straightforward calculation, we obtain the following expressions

$$A_i = \frac{h}{12}Z_i + \frac{\eta_i}{6}, \tag{9}$$

$$B_i = -\frac{h^2}{12}Z_i + \frac{\mu_i}{2}, \tag{10}$$

$$C_i = \frac{u_{i+1}}{h} - \frac{h^3}{12}Z_i - \frac{h^3}{120}Z_{i+1} - \frac{h^2}{6}\eta, \text{ and} \tag{11}$$

$$D_i = \frac{u_i}{h} + \frac{9h^3}{120}Z_i - \frac{h}{2}\mu_i. \tag{12}$$

Now, we impose the first, second and third continuity conditions of quintic spline $s_i(x)$ at the point x_{i+1}, i.e. $s_i^{(r)}(x_{i+1}) = s_{i+1}^{(r)}(x_{i+1})$, $r = 1, 2, 3$, and the following relations were obtained

$$\frac{11h^3}{120}Z_i + \frac{8h^3}{120}Z_{i+1} + \frac{h^3}{120}Z_{i+2} + \frac{2h^2}{6}\eta_i + \frac{h^2}{6}\eta_{i+1} + \frac{h}{2}\mu_i + \frac{h}{2}\mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h}, \tag{13}$$

$$\frac{4h^2}{12}Z_i + \frac{2h^2}{12}Z_{i+1} + h\eta_i + \mu_i - \mu_{i+1} = 0, \text{ and} \tag{14}$$

$$\frac{h}{2}Z_i + \frac{h}{2}Z_{i+1} + \eta_i - \eta_{i+1} = 0. \tag{15}$$

From (14), we obtain

$$\eta_i = -\frac{4h}{12}Z_i - \frac{2h}{12}Z_{i+1} - \frac{\mu_i}{h} + \frac{\mu_{i+1}}{h}. \tag{16}$$

On substituting (16) into (15) and (13), we obtain

$$\frac{h}{6}Z_i + \frac{4h}{6}Z_{i+1} + \frac{h}{6}Z_{i+2} - \frac{\mu_i}{h} + \frac{2\mu_{i+1}}{h} - \frac{\mu_{i+2}}{h} = 0, \text{ and} \tag{17}$$

$$-\frac{7h^3}{360}Z_i - \frac{16h^3}{360}Z_{i+1} - \frac{7h^3}{360}Z_{i+2} + \frac{h}{6}\mu_i + \frac{4h}{6}\mu_{i+1} + \frac{h}{6}\mu_{i+2} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h}, \tag{18}$$

respectively. On eliminating μ_i and μ_{i+2} from (17) and (18) and after some computations, we obtain

$$\mu_{i+1} = \frac{u_i - 2u_{i+1} + u_{i+2}}{h^2} - \frac{3h^2}{360}Z_i - \frac{24h^2}{360}Z_{i+1} - \frac{3h^2}{360}Z_{i+2}. \tag{19}$$

On substituting (19) into (17), we obtain the following main recurrence relation given by

$$Z_{i-1} + 26Z_i + 66Z_{i+1} + 26Z_{i+2} + Z_{i+3} = \frac{120}{h^4}(u_{i-1} - 4u_i + 6u_{i+1} - 4u_{i+2} + u_{i+3}),\ i = 1, \ldots, n-3. \tag{20}$$

Equation (20) forms a system of $n-3$ equations with $n+1$ unknowns, which were the Z_i, where $i = 0, \ldots, n$. To solve this system uniquely, we have to add four more conditions at the end points x_0 and x_n. Hence, we choose $Z_0 = Z_n = \eta_0 = \eta_n = 0$ to obtain the last two equations.
We substitute $i = 0$ in (16) and obtain

$$\frac{h^2}{6}Z_1 = -\mu_0 + \mu_1. \tag{21}$$

In order to find μ_0, we replace μ_{i+1} and μ_{i+2} in (18) by their values obtained from (19) and then substitute $i = 0$ in (18) to obtain

$$\mu_0 = \frac{65h^2}{120}Z_1 + \frac{26h^2}{120}Z_2 + \frac{h^2}{120}Z_3 - \frac{-u_0 + 5u_1 - 5u_2 + u_3}{h^2}. \tag{22}$$

Finally, we substitute μ_0 from (22) and μ_1 from (19) into (21) and obtain the second to last equation as

$$93Z_1 + 27Z_2 + Z_3 = \frac{120}{h^4}(-u_0 + 3u_1 - 3u_2 + u_3). \tag{23}$$

Now, we substitute $i = n - 1$ in (15) and then substitute the value of η_{n-1} from (16), to obtain

$$\frac{h^2}{6} Z_{n-1} = -\mu_n + \mu_{n-1}. \tag{24}$$

In order to obtain the value of μ_n, we replace μ_i and μ_{i+1} in (18) by their values obtained from (19) and substitute $i = n - 1$ in (18). After that, we substitute the values of μ_n and μ_{n-1} from (19) into (24) to obtain

$$Z_{n-3} + 27Z_{n-2} + 93Z_{n-1} = \frac{120}{h^4}(-u_n + 3u_{n-1} + u_{n-3}). \tag{25}$$

Equations (20), (23) and (25) form a system of $n - 1$ equations with $n - 1$ unknowns. These unknowns can be solved using the *MATHEMATICA* software. Next, we describe the present fourth order quintic spline method. Hence, to construct an algorithm for the proposed method, we can use the following steps:

Step 1: Divide the interval$[a,b]$ into $n - 1$ subinterval by taking $x_i = a + ih$, where $h = 1/n$ and $i = 0, 1, \ldots, n$.

Step 2: Apply shooting method to problem (1), to obtain the approximate solution u_i at the grid points.

Step 3: Use (20), (23) and (25) to form a system of linear equations and then solve for the values of A_i, B_i, C_i and D_i, for $i = 0, 1, \ldots, n - 1$.

Step 4: Use the values of A_i, B_i, C_i, D_i, Z_i and u_i obtained from Steps 2 and 3 to construct the quintic spline solution $s_i(x)$ in (4), to approximate the solution for problem (1).

CONVERGENCE ANALYSIS

Let $s_i(x)$ given by (4), denotes the quintic spline using the exact values u_i, μ_i, η_i and Z_i and let $\tilde{s}_i(x)$ denotes the quintic spline constructed using the values $\tilde{u}_i, \tilde{\mu}_i, \tilde{\eta}_i$ and $\tilde{Z}_i$, where $\tilde{u}_i$ is the approximate solution of problem (1) which obtained by the shooting method, while $\tilde{\mu}_i, \tilde{\eta}_i$ and $\tilde{Z}_i$ were the second, third and fourth derivatives of the function$\tilde{s}_i(x)$ at the point $(x_i, \tilde{u}_i)$ respectively. Then, $\tilde{s}_i(x)$ was given by

$$\tilde{s}_i(x) = \tilde{Z}_{i+1}\frac{(x-x_i)^5}{120h} + \tilde{Z}_i\frac{(x_{i+1}-x)^5}{120h} + \tilde{A}_i(x-x_i)^3 + \tilde{B}_i(x_{i+1}-x)^2 + \tilde{C}_i(x-x_i) + \tilde{D}_i(x_{i+1}-x), \tag{26}$$

where

$$\tilde{A}_i = \frac{h}{12}\tilde{Z}_i + \frac{\tilde{\eta}_i}{6},$$

$$\tilde{B}_i = -\frac{h^2}{12}\tilde{Z}_i + \frac{\tilde{\mu}_i}{2},$$

$$\tilde{C}_i = \frac{\tilde{u}_{i+1}}{h} - \frac{h^3}{12}\tilde{Z}_i - \frac{h^3}{120}\tilde{Z}_{i+1} - \frac{h^2}{6}\tilde{\eta}_i, \text{ and}$$

$$\tilde{D}_i = \frac{\tilde{u}_i}{h} + \frac{9h^3}{120}\tilde{Z}_i - \frac{h}{2}\tilde{\mu}_i,$$

Assume that $e(x)$ defines the error between the exact solution $u(x)$ and the spline function $\tilde{s}_i(x)$ for problem (1), i.e.

$$e(x) = u(x) - \tilde{S}(x), x \in [a,b]. \tag{27}$$

It was easy to verify that we can rewrite the error function $e(x)$ as follows

$$e(x) = [u(x) - S(x)] + [S(x) - \tilde{S}(x)] = e_i(x) + e_D(x), \tag{28}$$

where $e_i(x)$ is the error caused by spline interpolation and $e_D(x)$ is the error caused by discretization of problem (1). Now, to estimate $e(x)$ we have to estimate $e_i(x)$ and $e_D(x)$ separately.

Since our spline was polynomial of degree four, then we can write $e_i(x)$ over the interval $[x_i, x_{i+1}]$ as

$$u(x) - s_i(x) = \frac{u^{(6)}(\xi_i)}{6!}(x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2})(x - x_{i+3}), \tag{29}$$

for some $\xi_i \in [x_i, x_{i+1}]$. Recall that every subinterval has length h and if we let $t = x - x_i$, then (29) can be rewritten as

$$u(x) - s_i(x) = \frac{u^{(6)}(\xi_i)}{6!}(2h + t)(h + t)(t)(h - t)(2h - t)(3h - t). \tag{30}$$

Calculation on the expression $(2h + t)(h + t)(t)(h - t)(2h - t)(3h - t)$ in (30) shows that it has maximum value at $t = \frac{h\left(3-\sqrt{3}\sqrt{35+8\sqrt{7}}\right)}{6}$ and it was equal to $\frac{16\left(10+7\sqrt{7}\right)}{27}$ h^6. Then, $\|u(x)-s_i(x)\|_\infty$ was bounded by

$$\|u(x)-s_i(x)\|_\infty \le \frac{10+7\sqrt{7}}{1215}h^6\|u^{(6)}(\xi_i)\|_\infty. \tag{31}$$

Let $W^6 = \max_{x\in[a,b]}\|u^{(6)}(x)\|_\infty$. Therefore, it was easy to conclude that

$$\|e_I(x)\|_\infty \le 0.02347W^6h^6. \tag{32}$$

In order to estimate the error function $e_D(x)$, we can subtract (26) from (4), to obtain

$$s_i(x) - \tilde{s}_i(x) = (Z_{i+1} - \tilde{Z}_{i+1})\frac{(x-x_i)^5}{120h} + (Z_i - \tilde{Z}_i)\frac{(x_{i+1}-x)^5}{120h} + (A_i - \tilde{A})(x - x_i)^3 + (B_i - \tilde{B}_i)(x - x_{i+1})^2 + (C_i - \tilde{C}_i)(x - x_i) + (D_i - \tilde{D}_i)(x_{i+1} - x),$$

$$\text{where } x \in [x_i, x_{i+1}]. \tag{33}$$

Let $X = (x_0, \ldots, x_n)^t$, $U = (u_0, \ldots, u_n)^t$, $\tilde{U} = (\tilde{u}_0, \ldots, \tilde{u}_n)^t$, $\mu = (\mu_0, \ldots, \mu_n)^t$, $\tilde{\mu} = (\tilde{\mu}_0, \ldots, \tilde{\mu}_n)^t$, $\eta = (\eta_0, \ldots, \eta_n)^t$, $\tilde{\eta} = (\tilde{\eta}_0, \ldots, \tilde{\eta}_n)^t$, $Z = (Z_0, \ldots, Z_n)^t$, and $\tilde{Z} = (\tilde{Z}_0, \ldots, \tilde{Z}_n)^t$. From (33), it was easy to see that

$$\left\|e_D(x)\right\|_\infty \le \left\|U-\tilde{U}\right\|_\infty + h^2\left\|\mu-\tilde{\mu}\right\|_\infty + \frac{h^3}{3}\left\|\eta-\tilde{\eta}\right\|_\infty + \frac{h^4}{6}\left\|Z-\tilde{Z}\right\|_\infty. \tag{34}$$

Next, we will estimate $\|\eta-\tilde{\eta}\|_\infty$. We use (16) to obtain

$$\eta-\tilde{\eta}_i = -\frac{4h}{12}\left(Z_i-\tilde{Z}_i\right)-\frac{2h}{12}\left(Z_{i+1}-\tilde{Z}_{i+1}\right) -\frac{1}{h}\left(\mu_i-\tilde{\mu}_i\right)+\frac{1}{h}\left(\mu_{i+1}-\tilde{\mu}_{i+1}\right). \tag{35}$$

Therefore, from (35), we obtain

$$\|\eta-\tilde{\eta}\|_\infty \le \frac{1}{h}\|\mu-\tilde{\mu}\|_\infty + \frac{h}{3}\left\|Z-\tilde{Z}\right\|_\infty. \tag{36}$$

On substituting (36) into (34), we obtain

$$\left\|e_D(x)\right\|_\infty \le \left\|U-\tilde{U}\right\|_\infty + \frac{4h^2}{3}\|\mu-\tilde{\mu}\|_\infty + \frac{5h^4}{18}\left\|Z-\tilde{Z}\right\|_\infty. \tag{37}$$

Next, we will estimate $\|\mu-\tilde{\mu}\|_\infty$. We use (19) to obtain

$$\mu_i-\tilde{\mu}_i = \frac{1}{h^2}\left(u_{i-1}-\tilde{u}_{i-1}\right)-\frac{2}{h^2}\left(u_i-\tilde{u}_i\right)+\frac{1}{h^2}\left(u_{i+1}-\tilde{u}_{i+1}\right)- \frac{h^2}{120}\left(Z_{i-1}-\tilde{Z}_{i-1}\right)-\frac{8h^2}{120}\left(Z_i-\tilde{Z}_i\right)-\frac{h^2}{120}\left(Z_{i+1}-\tilde{Z}_{i+1}\right),$$

where $i = 1, \ldots, n-1$. (38)

From (22), we also obtain

$$\mu_0-\tilde{\mu}_0 = \frac{65h^2}{120}\left(Z_1-\tilde{Z}_1\right)+\frac{26h^2}{120}\left(Z_2-\tilde{Z}_2\right)+\frac{h^2}{120}\left(Z_3-\tilde{Z}_3\right)- \frac{5}{h^2}\left(u_1-\tilde{u}_1\right)+\frac{5}{h^2}\left(u_2-\tilde{u}_2\right)-\frac{1}{h^2}\left(u_3-\tilde{u}_3\right). \tag{39}$$

Therefore, from (38) and (39), it is easy to verify that

$$\|\mu-\tilde{\mu}\|_\infty \le \frac{5}{h^2}\left\|U-\tilde{U}\right\|_\infty + \frac{65h^2}{120}\left\|Z-\tilde{Z}\right\|_\infty. \tag{40}$$

On substituting (40) into (37), we obtain

$$\left\|e_D(x)\right\|_\infty \le \frac{23}{3}\left\|U-\tilde{U}\right\|_\infty + h^4\left\|Z-\tilde{Z}\right\|_\infty. \tag{41}$$

Next, we will estimate $\left\|Z-\tilde{Z}\right\|_\infty$. To estimate $\left\|Z-\tilde{Z}\right\|_\infty$, we let $Q = (q_{i,j})$ denotes a matrix with

$q_{1,1} = 93,$	$q_{n-1,n-1} = 93,$
$q_{1,2} = 27,$	$q_{n-1,n-2} = 27,$
$q_{1,3} = 1,$	$q_{n-1,n-3} = 1,$
$q_{i,j} = 66,$	$i = 2, \ldots, n-1,$
$q_{i,j+1} = q_{i,j-1} = 26,$	$i = 2, \ldots, n-2,$ and
$q_{i,j+2} = q_{i,j-2} = 1,$	$i = 2, \ldots, n-2.$

We also let $J = (j_{m,l})$ denotes a matrix with

$j_{1,1} = 3,$	$j_{n-1,n-1} = 3,$
$j_{1,2} = -3,$	$j_{n-1,n-2} = -3,$
$j_{1,3} = 1,$	$j_{n-1,n-3} = 1,$
$j_{m,m} = 6,$	$m = 2, \ldots, n-1,$
$j_{m,m+1} = j_{m,m-1} = -4,$	$m = 2, \ldots, n-2,$ and
$j_{m,m+2} = j_{m,m-2} = 1,$	$m = 2, \ldots, n-2.$

Let $\psi = \frac{120}{h^4}(-u_0, u_0, 0, \ldots, u_n, -u_n)^t$, then our system which consists of (20), (23) and (25) can be rewritten in a matrix form as

$$QZ = \frac{120}{h^4}JU + \psi, \tag{42}$$

From (42), we obtain

$$Q\tilde{Z} = \frac{24}{h^3}J\tilde{U} + \psi + \tau(h), \tag{43}$$

where $\tau(h) = (\tau_0(h), \tau_1(h), \ldots, \tau n(h))^t$ is the error in the fourth derivative due to the discretization. On subtracting (43) from (42), we obtain

$$Q(Z-\tilde{Z}) = \frac{24}{h^3}J(U-\tilde{U}) - \tau(h). \tag{44}$$

Since $u_0 = \tilde{u}_0$ and $u_n = \tilde{u}_n$, then it is not difficult to show that

$$\tau_0(h) = \tau_n(h) = 0, \tag{45}$$

and

$$\tau_i(h) = \frac{-h^2}{6}u^{(4)}(\xi_i), \xi_i \in (x_i, x_{i+1}). \tag{46}$$

From (45) and (46), it follows that

$$\|\tau(h)\|_\infty \le c_1 h, \tag{47}$$

where $c_1 = \max_{a\le x\le b}\left\|u^{(4)}(x)\right\|_\infty$.

Since Q is strictly diagonally dominant matrix, then Q^{-1} exist, $\left\|Q^{-1}\right\|_\infty \le \frac{1}{12}$, $\|Q\|_\infty = 121$, and $\|J\|_\infty = 16$. Together with (44) and (47), we obtain

$$\left\|Z-\tilde{Z}\right\|_\infty \le \frac{160}{h^4}\left\|U-\tilde{U}\right\|_\infty + \frac{1}{12}c_1 h^2. \tag{48}$$

From equations (41) and (48), we obtain

$$\left\|e_D(x)\right\|_\infty \le \frac{3680}{3}\left\|U-\tilde{U}\right\|_\infty + \frac{1}{12}c_1 h^6. \tag{49}$$

Theorem 1. (Chawla & Subramanian 1988)

Assume that $u(x)$ is sufficiently smooth. Then there exist a constant c independent of h such that

$$\left\|U-\tilde{U}\right\|_\infty \le ch^4.$$

From (49) together with Theorem 1, we obtain

$$\left\|e_D(x)\right\|_\infty \le c_2 h^4, \tag{50}$$

where $c_2 = \frac{3680}{3}c + \frac{(b-a)^2 c_1}{300}$. Finally, from (28) and (50) together with (32), we obtain

$$\left\|e(x)\right\|_\infty \le \left\|e_i(x)\right\|_\infty + \left\|e_D(x)\right\|_\infty \le c_3 h^4, \tag{51}$$

where $c_3 = \frac{0.02347(b-a)^2 W^6}{25} + c_2$.

Theorem 2. With the assumptions of Theorem 1, if is the quintic spline method (4) that used to approximate the solution of problem (1), $u(x)$ then

$$\left\|u(x)-\tilde{S}(x)\right\|_\infty \le c_3 h^4,$$

where $c_3 = \frac{0.02347(b-a)^2 W^6}{25} + c_2$.

NUMERICAL EXPERIMENTS

In this section, we implement the proposed method on three test problems of the second order BVPs and the maximum absolute errors between the nodal points were tabulated in Table 1 for step sizes equal to 0.1 and 0.01. To show the performance of the proposed method, we compared our results with those obtained by the cubic spline method derived in Chawla and Subramanian (1988).

Problem 1 (Burden & Faires 2001) Consider the linear second order BVPs:

$$u'' = \frac{-2}{x}u' + \frac{2}{x^2}u + \frac{\sin(\ln(x))}{x^2}, \qquad 1 \le x \le 2,$$

$$u(1) = 1,\ u(2) = 2.$$

The exact solution for *Problem 1* is given by

$$u(x) = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln(x)) - \frac{1}{10}\cos(\ln(x)), \text{ where}$$

$$c_2 = \frac{1}{70}\left(80 - 12\sin(\ln 2) - 4\cos(\ln 2)\right), \text{ and}$$

$$c_1 = \frac{11}{10} - c_2.$$

Problem 2 (Burden & Faires 2001) Consider the nonlinear second order BVPs:

$$u'' = 2u^3 - 1 \le x \le 0,$$

$$u(-1) = \frac{1}{2},\ u(0) = \frac{1}{3}.$$

The exact solution for *Problem 2* is given by

$$u(x) = \frac{1}{x+3}.$$

Problem 3 (Khuri 2004) Consider the Bratu type equation:

$$u'' + 2e^{u} = 0.$$

$$u(0) = u(1) = 0.$$

The exact solution for *Problem 3* is given by

$$u(x) = -2\ln\left(\frac{\cosh(1.178787(x-0.5))}{\cosh(0.589388)}\right).$$

From Table 1, we noticed that the proposed quintic spline method and the existing cubic spline method by Chawla and Subramanian (1988) were found to have comparable accuracy in solving linear BVPs such as *Problem 1*. However, the proposed quintic spline method is more accurate than the existing cubic spline method in solving nonlinear BVPs such as *Problems 2* and *3*.

TABLE 1. Maximum absolute errors for *Problems 1, 2* and *3*

Problem	Method	Step size	Maximum absolute errors
1	Chawla and Subramanian (1988)	0.1	9.82365×10^{-6}
		0.01	7.82189×10^{-9}
	Our method	0.1	5.12238×10^{-6}
		0.01	8.32690×10^{-9}
2	Chawla and Subramanian (1988)	0.1	1.68860×10^{-5}
		0.01	1.69450×10^{-7}
	Our method	0.1	8.82381×10^{-6}
		0.01	9.50081×10^{-9}
3	Chawla and Subramanian (1988)	0.1	6.26403×10^{-6}
		0.01	6.24981×10^{-6}
	Our method	0.1	6.22187×10^{-6}
		0.01	6.39223×10^{-6}

Conclusion

In this article, a new quintic spline method for the numerical solutions of second order BVPs (1) has been presented. An algorithm to apply the new method is presented as well. Convergence analysis showed that the order of convergence of the new method is order 4. Three test problems have been chosen for comparison purposes. Numerical results seemed to indicate that the new quintic spline method is more accurate than the existing cubic spline method for the case of nonlinear BVPs but vice versa for the case of linear BVPs.

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