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Integral Equations and Operator Theory

On a Class of Operators of Finite Type

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Abstract. This paper studies some class of pure operators A with finite rank self-commutators satisfying the condition that there is a finite dimensional subspace containing the image of the self-commutator and invariant with respect to A^* . Besides, in this class the spectrum of operator A is covered by the projection of a union of quadrature domains in some Riemann surfaces.

In this paper the analytic model, the mosaic and some kernel related to the eigenfunctions are introduced which are the analogue of those objects in the theory of subnormal operators.

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1. Introduction

Recently, several works [9], [10], [17], [19], [20], [23] have given the natural connection between operator theory and the theory of quadrature domains (cf. [1], [6], [13]). The goal of this paper is to give further connection between theory of operators of finite type and quadrature domains in Riemann surface (cf. [6], [19]).

In this paper, $\mathcal{H}$ is an infinite dimensional separable Hilbert space. Let A be an operator on $\mathcal{H}$. Let $[A^*, A] \stackrel{\text{def}}{=} A^*A - AA^*$ be the self-commutator of A. Let $M = M_A$ be the non-normal subspace of A, i. e. the closure of $[A^*, A]\mathcal{H}$. Let $K = K_A \stackrel{\text{def}}{=}$ closure of $\bigvee_m {A^*}^m M$. As in [19], if $\dim K_A < +\infty$, then A is said to be of finite type, or a finite type operator. This definition coincides with Yakubovich's (cf. [21], [22]) in the case of subnormal operators (cf. [4]), since in this case $K_A = M_A$ (cf. [14]).

For any operator A on $\mathcal{H}$, as in [17], [18], denote

$$C = C_A \stackrel{\text{def}}{=} [A^*, A]\big|_K \qquad \text{and} \qquad \Lambda = \Lambda_A \stackrel{\text{def}}{=} (A^*\big|_K)^*. \tag{1}$$

This pair $\{C_A,\ \Lambda_A\}$ is a complete unitary invariant for pure operator A (cf. [17], [19]). The following $\mathcal{L}(K_A)$-valued rational function is useful (cf. [18], [19]) for studying A:

$$R(z) = R_A(z) \stackrel{\text{def}}{=} C_A(zI - \Lambda_A)^{-1} + \Lambda_A^*, \qquad z \in \rho(\Lambda_A). \tag{2}$$

For an operator A of finite type, as in [17], [20], let

$$P(z,w) = P_A(z,w) \stackrel{\text{def}}{=} \det\big((wI - \Lambda_A^*)(zI - \Lambda_A) - C_A\big). \tag{3}$$

These Λ, C, $R(\cdot)$, $P(\cdot,\cdot)$ are basic tools for studying the pure operator A of finite type in this paper.

In §2, the quadrature domains on Riemann surface associated to a pure finite operator and some related concepts are studied. In §3, it introduces an $\mathcal{L}(K)$-valued function $\alpha(\cdot)$ on the domains in Riemann surface which satisfies the conditions $\alpha(\cdot) = \alpha(\cdot)^2$ and

$$\big(R(\Psi(\cdot)) - S(\cdot)I\big)\alpha(\cdot) = \alpha(\cdot)\big(R(\Psi(\cdot)) - S(\cdot)\big)$$

where $\Psi(\cdot)$ and $S(\cdot)$ are the projection to the complex plane and the Schwarz function on the quadrature domains on Riemann surface. Based on $\alpha(\cdot)$, an $\mathcal{L}(K)$-valued measure $e(\cdot)$ on the projection of the boundaries of the quadrature domains in Riemann surfaces is introduced which is an analogue of the $\mathcal{L}(M)$-valued measure $e(\cdot)$ for the pure subnormal operators. In §4, an analytic model for some pure operators of finite type is established, which is the extension of the case $\dim M = 1$ established in [20]. In §5, the mosaic of some pure operators of finite type is studied. In §6, the relation of two kernels $S(\cdot,\cdot)$ and $E(\cdot,\cdot)$ studied in [15] and [17] is also established for a class of operators of finite type. In §7, we study some special case and example.

2. Quadrature domains on a Riemann Surface

Let us briefly quote some in [16] and [19]. Let $\mathcal{D}$ be a finitely connected domain with boundary $\partial\mathcal{D}$ consisting of finite collection of piecewise smooth Jordan curves in a Riemann surface $\mathcal{R}$. If there are a bounded analytic function $\Psi(\cdot)$ and a meromorphic function $S(\cdot)$ on $\mathcal{D}$ which have continuous boundary values on $\partial\mathcal{D}$ satisfying

$$S(\zeta) = \overline{\Psi(\zeta)}, \qquad \zeta \in \partial\mathcal{D}, \tag{4}$$

then $\mathcal{D}$ is said to be a quadrature domain in the Riemann surface $\mathcal{R}$. The function $S(\cdot)$ is said to be the Schwarz function associated with $\mathcal{D}$ and the function $\Psi(\cdot)$ is said to be projection from $\mathcal{D}$ to the complex plane $\mathbb{C}$ (cf. [6], [16], [19]). If $\mathcal{R}$ is $\mathbb{C}$ and $\Psi(\zeta) \equiv \zeta$, then $\mathcal{D}$ is a quadrature domain in the complex plane (cf. [1], [6], [13]).

We select some quadrature domains associated with an operator A of finite type. Let $P(\cdot,\cdot)$ be the operator defined in the §1. There is a decomposition (cf. also [19], [21])

$$P(z,w) = P(z)\overline{P(\overline{w})}\prod_{j=1}^{l} P_j(z,w)^{k_j} \tag{5}$$

satisfying the following conditions that $P(\cdot)$ is a polynomial with the leading coefficient 1, $P_j(z,\overline{w}) = \overline{P_j(w,\overline{z})}$, $P_j(\cdot,\cdot)$ is an irreducible polynomial, the equation $P_j(z,w) = 0$ has no solution of type $w\equiv$constant, the leading term of $P_j(z,w)$ is $(zw)^{n_j}$ with $n_j > 0$ and $P_j(\cdot,\cdot) \neq P_{j'}(\cdot,\cdot)$ for $j \neq j'$. Denote $n_A = \sum n_j k_j$. Let $\mathcal{R}_j$ be the Riemann surface of the algebraic function $w = f_j(z)$ defined by $P_j(z, f_j(z)) = 0$. Let $\mathcal{R} = \cup\mathcal{R}_j$. Then there is an analytic function $\Psi(\cdot)$ and a meromophic function $S(\cdot)$ on $\mathcal{R}$ (although $\mathcal{R}$ may not be connected) satistying

$$P_j(\Psi(\zeta), S(\zeta)) = 0, \qquad \zeta \in \mathcal{R}_j. \tag{6}$$

The function $\Psi(\cdot)\big|_{\mathcal{R}_j}$ is a n_j to 1 mapping except at the branch points of the algebraic function $f_j(\cdot)$. Actually,

$$S((\Psi\big|_{\mathcal{R}_j})^{-1}(z)) = f_j(z).$$

The function $z = \Psi(\zeta)$ is said to be the projection from $\mathcal{R}$ to the complex plane and the function $S(\cdot)$ is related to some Schwarz function of quadrature domains. If $\mathcal{D}_{j,l}$ is a finitely connected domain in $\mathcal{R}_j$ with piecewise smooth boundary satisfying

$$S(\zeta) = \overline{\Psi(\zeta)}, \qquad \zeta \in \partial\mathcal{D}_{j,l} \tag{7}$$

and $\{\Psi(\zeta) : \zeta \in \mathcal{D}_{j,l}\}$ is bounded, then $\mathcal{D}_{j,l}$ is said to be a quadrature domain associated with the operator A. Let $\mathcal{D} = \mathcal{D}_A$ be the union of all quadrature domains $\mathcal{D}_{j,l}$ associated with the operator A. The union $\mathcal{D}$ is said to be complete, if in every component of $\rho(A)$ there is a point w in $\mathbb{C} \setminus \Psi(\mathcal{D} \cup \partial\mathcal{D})$ such that there exist n_A zeros (counting multiplicity) in $\mathcal{D}$ of the funciton $S(\cdot) - \overline{w}$. Remind that a zero ξ of $S(\cdot) - \overline{w}$ is said to be of multiplicity $l \geq 1$, if

$$S(\zeta) - \overline{w} = \sum_{i=l}^{\infty} a_i(\Psi(\zeta) - \Psi(\xi))^i,$$

and $a_l \neq 0$ for ζ in a neihgborhood of ξ.

As it has been mentioned in p.127 of [19]. If $\mathcal{D}_A$ is complete then in every component of $\rho(A)$, there is a non-empty open set O such that for every $w \in O$, there are n_A zeros of $S(\cdot) - \overline{w}$ in $\mathcal{D}_A$.

Let $\mathcal{L} = \mathcal{L}_A \stackrel{\text{def}}{=} \partial\mathcal{D}_A, D = D_A \stackrel{\text{def}}{=} \Psi(\mathcal{D}_A)$ and $L = \Psi(\mathcal{L})$. we choose the orientation of $\mathcal{L}$ as counter clockwise with respect to $\mathcal{D}$ and pass the orientation from $\mathcal{L}$ to L by Ψ. It is easy to see that

$$L \subset \{z : P(z, \overline{z}) = 0\}\ . \tag{8}$$

An operator B on $\mathcal{K}$ is called *diagonalizable* at an $\lambda \in \sigma(B)$ if $(B - \lambda I)^i\eta = 0$ for some integer $i > 0$ implies that $\eta = 0$, i. e. $\lim_{z\to\lambda}(z - \lambda)(zI - B)^{-1}$ is finite.

Let $\mathcal{F}$ be the family of all pure operators A of finite type satisfying the condition that $\mathcal{D}_A$ is complete and $R_A(z)$ is diagonalized at $\overline{z}$ for almost all $z \in L$. In the present paper, we always assume that $A \in \mathcal{F}$. It is evident that pure subnormal operator of finite type is in $\mathcal{F}$. A pure hyponormal operator of finite type satisfying $\dim M_A = 1$ is in $\mathcal{F}$ (see §7). There are a lot of examples of pure non-hyponormal operator of finite type are in $\mathcal{F}$.

From (4) in [18], $D_A \subset \sigma(A)$ for any finite operator A. From [19], it is easy to see that if $\mathcal{D}$ is complete, then the closure of D_A must contain $\sigma(A)$ as a subset. Therefore for $A \in \mathcal{F}$, $\sigma(A) =$ closure of D_A.

3. Some lemmas

Let A be a pure operator of finite type on a separable Hilbert space $\mathcal{H}$. We adopt the notations in the second section such as $\mathcal{D} = \mathcal{D}_A$, etc.

For $\zeta \in \mathcal{D} \cup \mathcal{L}$, if ζ is not a pole of $S(\cdot)$ and $\Psi(\zeta) \in \rho(\Lambda)$, let

$$\alpha(\zeta) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma_\zeta} (zI - R(\Psi(\zeta))^{-1} dz \tag{9}$$

where γ_ζ is a counter clockwise contour $\{z : |z - S(\zeta)| = \epsilon\}$ where $\epsilon > 0$ and satisfies

$$\sigma(R(\Psi(\zeta))) \cap \{z : 0 < |z - S(\zeta)| \leq \epsilon\} = \emptyset.$$

From

$$\det(wI - R(z)) = P(z, w) \det(zI - \Lambda)^{-1}, \tag{10}$$

it is easy to see that $S(\zeta) \in \sigma(R(\Psi(\zeta)))$, $\alpha(\zeta) = \alpha(\zeta)^2 \neq 0$, and

$$\alpha(\zeta)K = \{\eta \in K : (R(\Psi(\zeta)) - S(\zeta)I)^i \eta = 0 \ \text{ for some } \ i \in \mathbb{N}\}.$$

The $\mathcal{L}(K)-$valued function $\alpha(\cdot)$ is meromorphic on a neighborhood of $\mathcal{D} \cup \mathcal{L}$.

If $R(\Psi(\zeta))$ is diagonalizable at $S(\zeta)$, then $\alpha(\zeta)$ is the parallel projection from K onto the eigenspace $\{\eta \in K : R(\Psi(\zeta))\eta = S(\zeta)\eta\}$ of $R(\Psi(\zeta))$ corresponding to the eigenvalue $S(\zeta)$ and

$$(R(\Psi(\zeta)) - S(\zeta)I)\alpha(\zeta) = \alpha(\zeta)\big(R(\Psi(\zeta)) - S(\zeta)I\big) = 0. \tag{11}$$

If $R(\Psi(\zeta))$ is diagonalizable at $S(\zeta)$ for ζ in a set with limiting point at which $S(\cdot)$ and $R(\Psi(\cdot))$ is analytic, then (11) holds for all ζ, for $A \in \mathcal{F}$.

Similarly, if ζ is not a pole of $S(\zeta)$ and $S(\zeta) \notin \sigma(\Lambda^*)$, let

$$\beta(\zeta) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{l_\zeta} \big(zI - R(\overline{S(\zeta)})^*\big)^{-1} dz,$$

where l_ζ is a counter clockwise contour $\{z : |z - \Psi(\zeta)| = \epsilon\}$, and $\epsilon > 0$ satisfying

$$\sigma\big(R(\overline{S(\zeta)})^*\big) \cap \{z : 0 < |z - S(\zeta)| \leq \epsilon\} = \emptyset.$$

From (10) and $P(\overline{S(\zeta)}, \overline{\Psi(\zeta)}) = \overline{P(\Psi(\zeta), S(\zeta))} = 0$, it is evident that $\Psi(\zeta) \in \sigma\big(R(\overline{S(\zeta)})^*\big)$, $\beta(\zeta) = \beta(\zeta)^2$ and

$$\beta(\zeta)K = \{\eta : \big(R(\overline{S(\zeta)})^* - \Psi(\zeta)I\big)^i \eta = 0, \ \ \text{for some } \ i \in \mathbb{N}\}.$$

If $R(\overline{S(\zeta)})^*$ is diagonalizable at $\Psi(\zeta)$, then

$$\big(R(\overline{S(\zeta)})^* - \Psi(\zeta)I\big)\beta(\zeta) = \beta(\zeta)\big(R(\overline{S(\zeta)})^* - \Psi(\zeta)I\big) = 0, \tag{12}$$

and $\beta(\zeta)K$ is the eigenspace of $R(\overline{S(\zeta)})^*$ corresponding to $\Psi(\zeta)$. The function $\beta(\cdot)$ is meromorphic on a neighborhood of $\mathcal{D} \cup \mathcal{L}$.

Lemma 3.1 *Suppose $\zeta \in \mathcal{L}$, $\Psi(\zeta) \in \rho(\Lambda)$, ζ is not a pole of $S(\cdot)$ and $R(\Psi(\zeta))$ is diagonalizable at $S(\zeta)$. Then*

$$\alpha(\zeta) = \beta(\zeta)^*. \tag{13}$$

Proof. It is obvious that $R(\Psi(\zeta))^* = R(\overline{S(\zeta)})^*$ is diagonalizable, since $\overline{S(\zeta)} = \Psi(\zeta)$ for $\zeta \in \mathcal{L}$ (see(7)). Therefore (12) implies that

$$\big(R(\Psi(\zeta)) - S(\zeta)I\big)\beta(\zeta)^* = 0.$$

Thus $\beta(\zeta)^*K \subset \alpha(\zeta)K$ and

$$\alpha(\zeta)\beta(\zeta)^* = \beta(\zeta)^* \tag{14}$$

On the other hand, from (11) we have

$$\big(R(\overline{S(\zeta)})^* - \Psi(\zeta)I\big)\alpha(\zeta)^* = 0.$$

Therefore, by the same argument, we have

$$\beta(\zeta)\alpha(\zeta)^* = \alpha(\zeta)^* \tag{15}$$

Thus (13) follows from (14) and (15). □

Let $u \in \mathcal{D} \cup \mathcal{L}$ and $f(\cdot)$ be a meromorphic function with a pole at $u \in \mathcal{D}$. Then

$$\operatorname{Res}(f(\cdot), u) \overset{\text{def}}{=} a_{-1}$$

where a_{-1} is the coefficient of the Laurant expansion

$$f(\zeta) = \sum_{n=-k}^{\infty} a_n \big(\Psi(\zeta) - \Psi(u)\big)^n$$

Define an $\mathcal{L}(K)$-valued measure $e_1(\cdot)$ on $\mathcal{L}$ as

$$e_1(d\zeta) = \frac{i}{2\pi}\beta(\zeta)\big(S(\zeta)I - \Lambda^*\big)^{-1} dS(\zeta) \tag{16}$$

Let $f(\cdot)$ be a meromorphic function on a neighborhood of $D \cup L$ with possible poles only in

$$D \setminus L \cup \sigma(\Lambda) \cup \Psi\big((\text{set of singularities of } \beta(\cdot)) \cup (\text{set of poles of } S(\cdot))\big).$$

Assume that the function $f(\cdot)$ also satisfies

$$\int_{\mathcal{L}} |f(\Psi(\zeta)| \|e_1(d\zeta)\| < +\infty. \tag{17}$$

Let us assume that w belongs to the non-empty open set O of a component of $\rho(A)$ where $\overline{w} - S(\zeta)$ has n_A zero. Also let

$$w \notin \{\overline{S(\zeta)} : \zeta \in \mathcal{D} \cup \mathcal{L},\ dS(\zeta)/d\Psi(\zeta) = 0\}. \tag{18}$$

By the calculus of residues, it is easy to see the following

$$\int_{\mathcal{L}} \frac{f(\Psi(\zeta))e_1(d\zeta)}{\overline{w} - S(\zeta)} = I_1(w) + I_2(w) + I_3(w). \tag{19}$$

where

$$\begin{aligned}
&I_j(w) = \sum_{u \in F_j} \operatorname{Res}(G(\cdot), u),\\
&G(\cdot) \overset{\text{def}}{=} f(\Psi(\cdot))\beta(\cdot)\big(S(\cdot) - \Lambda^*\big)^{-1}\big(S(\cdot) - \overline{w}\big)^{-1} dS(\cdot)/d\Psi(\cdot),\\
&F_1 = \{\text{poles of the function } \beta(\cdot)\big(S(\cdot)I - \Lambda^*\big)^{-1} dS(\cdot)/d\Psi(\cdot) \text{ in } \mathcal{D}\},\\
&F_2 = \{\text{poles of } f(\Psi(\cdot)) \text{ in } \mathcal{D}\}
\end{aligned}$$

and F_3 depends on w, i. e. ,

$$F_3 = \{\zeta \in \mathcal{D} : S(\zeta) = \overline{w}\} = \{\zeta_1(w), \ldots, \zeta_p(w)\}.$$

It is easy to calculate that

$$I_3(w) = \sum_{j=1}^{p} f(\Psi(\zeta_j(w)))\beta(\zeta_j(w))(\overline{w}I - \Lambda^*)^{-1} \tag{20}$$

Let us study $I_3(w)$. Let

$$P(z) = \prod_{j=1}^{q} (z - z_j)^{\nu_j} \tag{21}$$

in (5), where $z_j \neq z_{j'}$ for $j \neq j'$. Let

$$\hat{P}(z, w) \overset{\text{def}}{=} \prod_{j=1}^{l} P_j(z, w)^{k_j} \tag{22}$$

in (5). If $w \in O$, then

$$\{z : P(z, \overline{w}) = 0\} = \{z_i\} \cup \{z : \hat{P}(z, \overline{w}) = 0\}$$

But there are n_A zeros $\overline{w} - S(\zeta)$ in $\mathcal{D}$, those must be $\zeta_i(w)$, with multiplicity k_i. Thus

$$\sigma(R(w)^*) = \{z : P(z, \overline{w}) = 0\} = \{\zeta_j(w)\} \cup \{z_i\}$$

$R(w)^*$ is diagonalizable at $\Psi(\zeta_j(w))$ for each j, since $A \in \mathcal{F}$. Therefore by Jordan decomposition, we have

$$\big(zI - R(w)^*\big)^{-1} = \sum_j \frac{1}{z - \Psi(\zeta_j(w))} \beta(\zeta_j(w)) + \sum_{i=1}^{q} \sum_{j=0}^{\nu_i - 1} \frac{j! Q_{ji}(w)}{(z - z_i)^{j+1}}$$

for some $Q_{ji}(w)$. Thus

$$f(R(w)^*) = \sum_j f(\Psi_j(w))\beta(\zeta_j(w)) + \sum_{i=1}^{q}\sum_{j=0}^{\nu_i-1} f^{(j)}(z_i)Q_{ji}(w).$$

But, by (4) of [18]

$$f(R(w)^*)(\overline{w}I - \Lambda^*)^{-1} = P_K(\overline{w}I - A^*)^{-1}f(A)\big|_K.$$

Hence

$$I_3(w) = P_k(\overline{w}I - A^*)^{-1}f(A)\big|_K - \sum_{i=1}^{q}\sum_{j=0}^{\nu_i-1} f^{(j)}(z_j)\hat{Q}_{ji}(w) \tag{23}$$

where $\widehat{Q}_{ji}(w) = Q_{ji}(w)(\overline{w}I - \Lambda^*)^{-1}$.

It is easy to see that (19)-(23) can be analytically extended to all $w \in \rho(A)$ except a finite set.

In the proof of next lemma, we only consider the case that f is analytic on a neighborhood of $D \cup L$, i. e. $F_2 = \emptyset$ and hence $I_2(w) = 0$. However in the proof of Theorem 5.3, we have to consider the more general case that $f(\cdot)$ may be meromorphic.

Suppose $g(\cdot)$ is analytic on a neighborhood U of $D \cup L$. Choose a contour ℓ in $\mathbb{C} \setminus (D \cup L)$ such that

$$\frac{1}{2\pi i}\int_\ell \frac{g(w)dw}{w - z} = g(z), \qquad z \in D \cup L$$

Multiplying (19) through by $-\overline{g(w)dw}/2\pi i$ and then integrating on ℓ, we have

$$\int_{\mathcal{L}} f(\Psi(\zeta))\overline{g(\Psi(\zeta))}e_1(d\zeta) = I_1' + I_2' + I_3'. \tag{24}$$

where

$$I_j' = \sum_{u \in F_j'} \operatorname{Res}(G(\cdot)\overline{g(S(\cdot))}, u), \quad j = 1, 2, \tag{25}$$

where

$$F_1' = \{\text{poles } u \text{ of } G(\cdot) \text{ satisfying } \overline{S(u)} \in D \cup L\}$$

and

$$F_2' = \{\text{poles } u \text{ of } f(\cdot) \text{ satisfying } \overline{S(\zeta)} \in D \cup L\}.$$

Besides,

$$I_3' = P_K\big(g(A)^*f(A)\big)\big|_K + \sum f^{(j)}(z_i)Q_{jl}(g) \tag{26}$$

where

$$Q_{jl}(g) = \frac{1}{2\pi i}\int_L \overline{g(w)}\widehat{Q}_{jl}(w)\overline{dw}. \tag{27}$$

By Hermitian property of I_3', it is easy to see that $Q_{jl}(g) = \sum Q_{ijkl}\overline{g^{(l)}(z_k)}$, where $Q_{ijkl} \in \mathcal{L}(K)$.

For $u \in L\backslash$ a finite set, there is only one $\zeta \in \mathcal{L}$ such that $\Psi(\zeta) = u$ and $S(\zeta) = \overline{u}$. Therefore define an $\mathcal{L}(K)$-valued measure $e(du)$ on L satisfying $e(du) = e_1(d\zeta)$ where $u = \Psi(\zeta)$, i. e.

$$e(du) \stackrel{\text{def}}{=} \frac{i}{2\pi}\beta(\zeta)\big(\overline{u}I - \Lambda^*\big)^{-1}d\overline{u}$$

where $\Psi(\zeta) = u$ and $S(\zeta) = \overline{u}$.

Let $p_A(\cdot)$ be a polynomial with leading coefficient 1 and the minimal degree such that F_1' and $\{z_i\}$ are zeros of $p(\cdot)$ with sufficient multiplicities such that $I_1(w) = 0$ and $\sum\limits_{j=0}^{\nu_i-1} f^{(j)}(z_i)Q_{ji}(w) = 0$, i. e. $I_1' = I_3' = 0$ for any analytic function $f(\cdot) \in \mathcal{F}_{p_A}$ where $\mathcal{F}_{p_A}$ is the family of analytic function $f(\cdot)$ on a neighborhood of $D \cup L$ satisfying the condition that $f(\cdot)/p_A(\cdot)$ is analytic.

Lemma 3.2 *For $A \in \mathcal{F}$, the measure*

$$e(du) \geq 0 \qquad \text{on } L. \tag{28}$$

Proof. For $f \in \mathcal{F}_{p_A}$, we have

$$0 \leq P_K f(A)^* f(A)\big|_K = \int_L |f(u)|^2 e(du). \tag{29}$$

which proves (28). □

Corollary 3.3 *For $A \in \mathcal{F}$,*

$$e(du) = \frac{1}{2\pi i}(uI - \Lambda)^{-1}\alpha(\zeta)du \qquad \text{on } L \tag{30}$$

where $\Psi(\zeta) = u$ and $S(\zeta) = \overline{u}$, and

$$e_1(d\zeta) = \frac{1}{2\pi i}\big(\Psi(\zeta)I - \Lambda\big)^{-1}\alpha(\zeta)d\Psi(\zeta), \qquad \zeta \in \mathcal{L}.$$

Besides,

$$\alpha(\zeta) = -\big(\Psi(\zeta)I - \Lambda\big)\beta(\zeta)\big(S(\zeta)I - \Lambda^*\big)^{-1}\frac{dS(\zeta)}{d\Psi(\zeta)}.$$

For pure subnormal operator S of finite type, if N is the m.n.e.(minimal normal extension, cf. [4]) of S, then $\sigma(N) = L_S \cup Q$, where Q is a finite set. The measure $e(\cdot)$ defined in (30) coincides with the measure $e(\cdot)$ defined in [14] and [16] on $\sigma(N) \setminus Q$, (see also the begining of §5 of the present paper.).

4. Analytic model

In this section, we generalize, to some extend, the analytic model in [20] of a pure hyponormal operator of finite type with rank one self-commutator to the operators in $\mathcal{F}$.

For $A \in \mathcal{F}$, let $q(u) = q_A(u)$ be the polynomial with minimal degree and with leading term $u^p (p = p_A)$ satisfying

$$\int_L |q(u)|^2 \|e(du)\| < +\infty.$$

Let $\mathcal{H}_A \stackrel{\text{def}}{=}$ the closure of $q_A(A)\mathcal{H}$ and $G = G_A \stackrel{\text{def}}{=} \mathcal{H} \ominus \mathcal{H}_A$. Then it is easy to see that $\dim G = p_A$, $A^*G \subset G$ and $q(A)^*\big|_G = 0$. The function model of G and $A^*\big|_G$ will be discussed in §6.

Let T be a subspace of $\mathcal{H}$. T is said to be a cyclic subspace of the pure operator A, if the set $\mathcal{H}_{A,T} \stackrel{\text{def}}{=} \text{span}\{r(A)x : x \in T, r(\cdot)$ is a rational function with poles in $\rho(A)\}$ is dense in $\mathcal{H}$. As an example K_A is a cyclic subspace of a pure operator A, since

$$A^*(\lambda I - A)^{-1}x = (\lambda I - A)^{-1} R_A(\lambda)x, \qquad x \in K_A, \lambda \in \rho(A)$$

(cf. Lemma 2 of [18]). Therefore the closure of $\mathcal{H}_{A,K}$ reduces A and hence it is $\mathcal{H}$.

Now, let us introduce an analytic model of $A\big|_{\mathcal{H}_A}$. Suppose $M_A \subset T \subset K_A$, and T is a cyclic subspace of A. Define an $\mathcal{L}(T)$-valued measure on L

$$\Theta(du) \stackrel{\text{def}}{=} P_T e(du)\big|_T. \tag{31}$$

Let $H^2(D, T, \Theta)$ be the Hilbert space completion of all T-valued rational functions with possible poles in $\rho(A)$ with respect to the following inner product:

$$(f, g)_{H^2(D,T,\Theta)} \stackrel{\text{def}}{=} \int_L (\Theta(du) f(u), g(u))_T.$$

Let $F_A = \{\zeta \in D : \zeta$ is a pole of $\beta(\cdot)\big(S(\cdot)I - \Lambda^*\big)^{-1} dS(\cdot)/d\Psi(\cdot)$ satisfying $S(\zeta) \in D\}$,

$$B(f, g) \stackrel{\text{def}}{=} \sum_{\zeta \in F_A} \text{Res}\big(((\Psi(\cdot)I - \Lambda)^{-1} \alpha(\cdot) f(\Psi(\cdot)), g(\overline{S(\cdot)})^*), \zeta\big), \tag{32}$$

$$Q(f, g) \stackrel{\text{def}}{=} \sum (Q_{ijkl} f^{(j)}(z_i), g^{(l)}(z_k)), \tag{33}$$

and

$$< f, g >_A \stackrel{\text{def}}{=} B(f, g) + Q(f, g).$$

Theorem 4.1 *Let $A \in \mathcal{F}$. Then $A\big|_{\mathcal{H}_A}$ is unitarily equivalent to the multiplication operator, denoted still by A,*

$$(Af)(u) = uf(u), \qquad u \in D,$$

on the Hilbert space $H^2(D, T, \Theta)$, endowed with the inner product

$$(f, g) \stackrel{\text{def}}{=} (f, g)_{H^2(D,T,\Theta)} - < f, g >_A . \tag{34}$$

Proof. From (24), it is easy to see that for any pair of rational functions $f(\cdot)$ and $g(\cdot)$ with poles in $\rho(A)$ satisfying the condition that $f(\cdot)/q(\cdot)$ and $g(\cdot)/q(\cdot)$ are analytic at zeros of $q(\cdot)$, we have

$$\int_{\mathcal{L}} f(u)\overline{g(u)}\big(e(du)x, y\big) = (f(A)x, g(A)y) + < fx, gy >_A, \qquad (35)$$

for $x, y \in K$. Thus (34) follows from (35), which proves the theorem. □

Corollary 4.2 *If $A \in \mathcal{F}$, then the restriction of A at the closure of $p_{_A}(A)\mathcal{H}$ is a subnormal operator of finite type.*

Remark Lemma 6 of [20] still holds for the present case. Therefore there is a more concrete form of $B(f, g)$ as in Lemma 6 of [20].

5. Mosaics

In [14], for a pure subnormal operator S, the author introduced a sort of musaic inspired by [5] and [11], but different from there, as

$$\mu(z) = \int_{\sigma(N)} \frac{uI - \Lambda_S}{u - z} e(du), \qquad z \in \mathbb{C} \setminus \sigma(N), \qquad (36)$$

where N is the m.n.e. of S, and $e(du)$ is a positive $\mathcal{L}(M)$-valued measure defined as

$$e(F) = P_{M_S} E(F)\big|_{M_S}$$

for any Borel set $F \subset \sigma(N)$, where $E(\cdot)$ is the spectral measure of N and Λ_S and M_S are defined in §1. This $\mu(\cdot)$ satisfies $\mu(\cdot)^2 = \mu(\cdot)$. Recently Gleason and Rosentrater proved the following:

Theorem 5.1 [7]. *Let S be a pure subnormal operator with trace class self-commutator on a Hilbert space $\mathcal{H}$ with mininal normal extension N, then*

$$[S^{*k}S^l, S]\big|_{M_S} = \frac{1}{\pi} \iint_{\mathbb{C}} k\bar{z}^{k-1} z^l \mu(z) dm_2(z),$$

where $m_2(\cdot)$ is the planar Lebesgue measure.

For any polynomial $P(w, z) = \sum p_{mn} w^m z^n$, let us adopt the Weyl ordering

$$P(A^*, A) \stackrel{\text{def}}{=} \sum p_{mn} A^{*m} A^n$$

for any operator A. Denote $P_w(w, z)$ as $\frac{\partial}{\partial w} P_w(w, z)$. Inspired by Theorem 5.1, let us introduce the following:

Definition 5.1 Let A be an operator of finite type. If there is a $\mathcal{L}(K_A)$-valued function $\mu(\cdot)$ on $\sigma(A) \setminus L$, where L is a union of a finite collection of piecewise smooth curves in $\sigma(A)$, satisfying the condition that (i) $\mu(z) = \mu(z)^2,\ z \in \sigma(A)\backslash L_A$ and (ii) there is a non-zero polynomial $p(\cdot)$ satisfying

$$\iint\limits_{\sigma(A)} |p(z)|\|\mu(z)\|dm_2(z) < +\infty \tag{37}$$

and (iii) for every polynomial $p(w,z)$ satisfying

$$\iint\limits_{\sigma(A)} p_w(\overline{z},z)\|\mu(z)\|dm_2(z) < +\infty, \tag{38}$$

the identity

$$[p(A^*,A),A]\big|_{K_A} = \frac{1}{\pi}\iint\limits_{\sigma(A)} p_w(\overline{z},z)\mu(z)dm_2(z) \tag{39}$$

holds good. Then $\mu(\cdot)$ is said to be the **mosaic** of A.

It is obvious that if mosaic exists, then it is unique (up to equivalence in the sense of almost everywhere). By Theorem 5.1 the mosaic defined in [15] (or (36)) of a subnormal of finite type satisfies the definition 5.1. In this case

$$\iint\limits_{\sigma(A)} \|\mu(z)\|dm_2(z) < +\infty.$$

For $A \in \mathcal{F}$, let $\mu(z) = \mu_A(z) = 0$ for $z \in \rho(A)$ and

$$\mu(z) = \mu_A(z) \stackrel{\text{def}}{=} \sum_{\Psi(\zeta)=z} \alpha(\zeta), \quad z \in D \setminus L. \tag{40}$$

Theorem 5.2 *For $A \in \mathcal{F}$, $\mu(\cdot)$ is defined in* (40) *is the mosaic of A satisfying Definition* 5.1.

Proof. First, we have $\alpha(\zeta)^2 = \alpha(\zeta)$ as in §3. For different ζ and ζ' satisfying

$$\Psi(\zeta) = \Psi(\zeta') = z,$$

we have $S(\zeta) \neq S(\zeta')$. Therefore $\alpha(\zeta)$ and $\alpha(\zeta')$ as the projections to the eigenspaces of $R(z)$ corresponding to different eigenvalues $S(\zeta)$ and $S(\zeta')$, we have

$$\alpha(\zeta)\alpha(\zeta') = \alpha(\zeta')\alpha(\zeta) = 0.$$

Thus $\mu(z)^2 = \mu(z)$.

Let $p(\cdot)$ be a polynomial such that $p(\Psi(\zeta))\alpha(\zeta)$ is analytic on $\mathcal{D} \cup \mathcal{L}$. Then $p(\cdot)\mu(\cdot)$ is a bounded function. Hence condition (ii) is satisfied.

In order to prove (39), we only have to prove that for any polynomial $f(\cdot) \in \mathcal{F}_p$ as described before Lemma 3.2, the following equality

$$[(\overline{w}I - A^*)^{-1}f(A),A]\big|_K = -\frac{1}{\pi}\iint\limits_{\sigma(A)} \frac{f(z)\mu(z)dm_2(z)}{(\overline{w}-\overline{z})^2}, \quad w \in f(A) \tag{41}$$

holds good. By (4) of [18], the left hand side of (41) equals to

$$(\overline{w}I-\Lambda^*)^{-1}Cf(R(w)^*)(\overline{w}I-\Lambda^*)^{-1}. \tag{42}$$

The right hand side of (41) equals to

$$\begin{aligned}
&-\frac{1}{2\pi i}\iint\limits_D d\left(\frac{f(z)\mu(z)dz}{\overline{w}-\overline{z}}\right) = -\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\frac{f(\Psi(\zeta))\alpha(\zeta)d\Psi(\zeta)}{\overline{w}-S(\zeta)}\\
&=\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\frac{f(\Psi(\zeta))(\Psi(\zeta)I-\Lambda)e_1(d\zeta)}{S(\zeta)-\overline{w}}\\
&=\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\frac{f(\Psi(\zeta))\Psi(\zeta)e_1(d\zeta)}{S(\zeta)-\overline{w}}-\frac{\Lambda}{2\pi i}\int\limits_{\mathcal{L}}\frac{f(\zeta)e_1(d\zeta)}{S(\zeta)-\overline{w}}.
\end{aligned} \tag{43}$$

□

By (19) and the calculations followed (19), (43) equals to

$$R(w)^*f(R(w)^*)\big(\overline{w}I-\Lambda^*\big)^{-1}-\Lambda f(R(w)^*)\big(\overline{w}I-\Lambda^*\big)^{-1}$$

which equals (42) and proves (41).

The following is an integral representation of $\mu(\cdot)$, which is an analogue to the integral representation (36).

Theorem 5.3 *Suppose $A\in\mathcal{F}$. Let $f(\cdot)$ be any analytic function on a neighborhood on $\sigma(A)$ satisfying the condition that $f(\cdot)/p_A(\cdot)$ is analytic. Then*

$$\mu(z)=\frac{1}{f(z)}\int\limits_L\frac{f(u)(uI-\Lambda)e(du)}{u-z},\qquad z\in D\setminus L. \tag{44}$$

Proof. It is easy to see that

$$\begin{aligned}
&\int\limits_L\frac{f(u)(uI-\Lambda)e(du)}{u-z}=\int\limits_{\mathcal{L}}\frac{f(\Psi(\zeta))(\Psi(\zeta)I-\Lambda)e_1(d\zeta)}{\Psi(\zeta)-z}\\
&=\frac{1}{2\pi i}\int\frac{f(\Psi(\zeta))\alpha(\zeta)d\Psi(\zeta)}{\Psi(\zeta)-z}
\end{aligned} \tag{45}$$

Since the factor $p(\Psi(\zeta))$ cancells the poles of $\alpha(\cdot)$, the right hand side of (45) equals to

$$\sum_{\Psi(\zeta)=z}\mathrm{Res}\left(\frac{f(\Psi(\zeta))\alpha(\zeta)}{\Psi(\zeta)-z},\zeta\right)=f(z)\sum_{\Psi(\zeta)=z}\alpha(\zeta)$$

which proves the theorem. □

Similarly, we have

$$\int\limits_L\frac{f(u)(uI-\Lambda)e(du)}{u-z}=0,\qquad z\in\rho(A),$$

where $f(\cdot)/p_A(\cdot)$ is analytic on a neighborhood of $\sigma(A)$. Besides, just like in [15], for $A \in \mathcal{F}$

$$R_A(z)\mu(z) = \mu(z)R_A(z).$$

From (24), if $f(\cdot) \in \mathcal{F}_{p_A}$, then

$$\int_L \frac{f(u)e(du)}{(u-z)(\overline{u}-\overline{w})} = f(R(w)^*)(R(w)^* - z)^{-1}(\overline{w}I - \Lambda^*)^{-1} + f(z)((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}\mu(z)$$

for $z, w \in \rho(A)$. If $w \in D_A$, by Plemelj's formula and (30), we may prove that

$$\int_L \frac{f(u)e(du)}{(u-z)(\overline{u}-\overline{w})} = (I - \mu(w)^*)f(R(w)^*)(R(w)^* - zI)^{-1}(\overline{w}I - \Lambda^*)^{-1} + f(z)((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}\mu(z)$$

which is a generalization of Lemma 6 in [14] in some sense. This formula may be useful for future study.

6. An identity of two kernels $S(\cdot,\cdot)$ and $E(\cdot,\cdot)$

For an operator A, let $J = J_A$ be the self-adjoint operator on M_A defined by

$$J_A = |C_A|^{-\frac{1}{2}} C_A |C_A|^{-\frac{1}{2}} \tag{46}$$

where $|C| = |C_A|$ is the restriction of the operator $(C_A^* C_A)^{\frac{1}{2}}$ on M_A. If A is hyponormal, then $J_A = I$. Let $S(z,w) = S_A(z,w)$ be the Brodzkii-Lifshitz kernel,

$$S_A(z,w) \overset{\text{def}}{=} J_A|C_A|^{\frac{1}{2}} P_{M_A}(\overline{w}I - A^*)^{-1}(zI - A)^{-1}|C_A|^{\frac{1}{2}}, \quad \text{for } z, w \in \rho(A)$$

(cf. [2]), where P_{M_A} is the projection to M_A.

If A is a pure hyponormal operator with rank one self-commutator, then this $S_A(\cdot,\cdot)$ coincides with the definition in [15], [12] and [20].

Lemma 6.1 *For $z \in \rho(A)$ and $w \in \rho(A^*)$*

$$S_A(z,w) = \left(I_M - J_A|C_A|^{\frac{1}{2}} P_{M_A}((zI_K - \Lambda_A)^{-1}(wI_K - \Lambda_A^*)^{-1}|C_A|^{\frac{1}{2}}\right)^{-1} - I_M \tag{47}$$

where I_M and I_K are the identity operators on M_A and K_A respectively.

Proof. From the proof in [17], for any operator A, we have

$$P_K(wI - A^*)^{-1}(zI - A)^{-1}\big|_K = ((wI - \Lambda^*)(zI - \Lambda) - C)^{-1} \tag{48}$$

From (46) and (48) we may prove that $XY = YX = I_M$, where

$$X = I_M - J|C|^{\frac{1}{2}} P_{M_A}(zI_K - \Lambda)^{-1}(wI_K - \Lambda^*)^{-1}|C|^{\frac{1}{2}}$$

and

$$Y = I_M + J|C|^{\frac{1}{2}} P_{M_A}((wI_K - \Lambda^*)(zI_K - \Lambda) - C)^{-1}|C|^{\frac{1}{2}}$$

which proves (47). □

Corollary 6.2 [2]. *$S(\cdot,\cdot)$ is a complete unitary invariant for pure operator A.*

Proof. It is obvious that $S_A(\cdot,\cdot)$ is a complete unitary invariant. Let us give a proof of the completeness by (47), which is different from the proof in [2]. Firstly, it is easy to see that $S_A(\cdot,\cdot)$ determines $J_A|C_A|$ and hence C_A. Secondly, it determines ${\Lambda^*}^n C, n = 1, 2, \ldots$ from (47). Therefore $S_A(\cdot,\cdot)$ determines $\{C_A, \Lambda_A\}$ up to a unitary equivalence. But $\{C_A, \Lambda_A\}$ is a complete unitary invariant, which proves the completeness of $S_A(\cdot,\cdot)$. □

Corollary 6.3 *There is an analytic continuation of the analytic function $S_A(z,w)$ from $\rho(A)\times\rho(A^*)$ to $\rho(\Lambda)\times\rho(\Lambda^*)$ as the function defined in the right-hand side of* (47).

We always make this analytic continuation of $S_A(z,\overline{w})$. Therefore $S_A(z,w)$ is defined as analytic function of z and $\overline{w}$ for $(z,w)\in\rho(\Lambda)\times\rho(\Lambda^*)$.

Now let us introduce another kernel $E(\cdot,\cdot)$ related to the eigenfunction of the adjoint operator of a subnormal operator.

Let A be a pure subnormal operator with m.n.e. N. Let us adopt the analytic model of A as in [14]. From [14], for $z\in\sigma(A)\cap\rho(N)$, a function $f(\cdot)$ is an eigenfunction of the operator A^* corresponding to $\overline{z}$, iff there is a vector $x\in M_A = K_A$ such that

$$f_{z,x}(u) = \frac{\overline{u}I-\Lambda^*}{\overline{u}-\overline{z}}\mu(z)^*x.$$

The both sides of above identity are equal as vectors in the Hilbert space. Define a $\mathcal{L}(M)$-valued kernel

$$\widehat{E}_A(z,w) \overset{\text{def}}{=} \mu(z)\int\frac{(uI-\Lambda)e(du)(\overline{u}I-\Lambda^*)}{(u-z)(\overline{u}-\overline{w})}\mu(w)^*, \qquad z,w\in\sigma(A)\cap\rho(N)$$

Then

$$(f_{w,x}, f_{z,y}) = (\widehat{E}_A(z,w)x, y).$$

Suppose $\nu_j, j = 1,2$ are piecewise smooth curves in $\partial\sigma(A)$. Then from Plemej's formula the boundary value of $\mu(\cdot)$ from $\sigma(A)\setminus\sigma(N)$ is

$$\mu(z) = 2\pi i(zI-\Lambda)e(dz)/dz \qquad \text{for } a.\ e. \text{ in } \nu_j.$$

From (40) in [14], we have

$$((\overline{z}I-\Lambda^*)(zI-\Lambda)-C)e(dz) = e(dz)((\overline{z}I-\Lambda^*)(zI-\Lambda)-C) = 0, \qquad z\in\nu.$$

Therefore

$$R(z)\mu(z) = \mu(z)R(z) = \overline{z}\mu(z), \qquad z\in\nu_j. \tag{49}$$

Let $r(z)$ be the closure of the range $C^{-\frac{1}{2}}\mu(z)M$ and $P(z)$ be the projection from M to $r(z)$. Let $r^*(z)$ be the closure of the range $C^{-\frac{1}{2}}\mu(z)^*M$. Let us normalize $\widehat{E}_A(\cdot,\cdot)$ as

$$\widetilde{E}_A(z,w) = C_A^{-\frac{1}{2}}\widehat{E}_A(z,w)C_A^{-\frac{1}{2}}.$$

Theorem 6.4 *Suppose $\nu_j, j = 1, 2$ are piecewise smooth curves in $\partial\sigma(A)$. Then*

$$\widetilde{E}_A(z, w)|z - w|^2 = P(z)(I + S_A(z, w))^{-1}\big|_{r^*(w)} \tag{50}$$

for $z \in \nu_1$ and $w \in \nu_2$.

Proof. From Lemma 6 of [14],

$$\int \frac{e(du)}{(u - z)(\overline{u} - \overline{w})} = Q(z, w)(I - \mu(z)) - \mu(w)^* Q(z, w)$$

where $Q(z, w) = ((\overline{w}I - \Lambda^*)(zI - \Lambda) - C)^{-1}$. Thus

$$\begin{aligned}
&\int \frac{(uI - \Lambda)e(du)(\overline{u}I - \Lambda^*)}{(u - z)(\overline{u} - \overline{w})} \\
&= \mu(z) + \mu(w)^* - I + (zI - \Lambda)\int \frac{e(du)}{(u - z)(\overline{u} - \overline{w})}(\overline{w}I - \Lambda^*) \\
&= -I + \mu(z) + \mu(w)^* + (\overline{w}I - R(z))^{-1}(I - \mu(z))(\overline{w}I - \Lambda^*) \\
&\quad -(zI - \Lambda)\mu(w)^*(zI - R(w)^*)^{-1},
\end{aligned}$$

since $Q(z, w) = (zI - \Lambda)^{-1}(\overline{w} - R(z))^{-1} = (z - R(w)^*)^{-1}(\overline{w}I - \Lambda^*)^{-1}$. From Lemma 7 of [14]

$$R(z)\mu(z) = \mu(z)R(z), \qquad z \in \sigma(N) \cap \rho(S).$$

We have

$$\widehat{E}_A(z, w) = \mu(z)\mu(w)^* - \mu(z)(zI - \Lambda)(zI - R(w)^*)^{-1}\mu(w)^*$$

for $z, w \in \sigma(A) \cap \rho(N)$. The above formula may be extended continuously to $z \in \nu_1$ and $w \in \nu_2$. Therefore from (49), we have for $z \in \nu_1$ and $w \in \nu_2$,

$$\begin{aligned}
|z - w|^2 \widehat{E}_A(z, w) &= \mu(z)(\Lambda - R(w)^*)\mu(w)^*(\overline{z} - \overline{w}) \\
&= \mu(z)(\overline{w} - \overline{z})(\overline{w}I - \Lambda^*)^{-1}C\mu(w)^* \\
&= \mu(z)(\overline{w}I - R(z))(\overline{w}I - \Lambda^*)^{-1}C\mu(w)^* \\
&= \mu(z)(C - C(zI - \Lambda)^{-1}(\overline{w}I - \Lambda^*)^{-1}C)\mu(w)^*
\end{aligned}$$

which proves (50) by Lemma 6.1. □

Now, let us study the case that the operator A is a hyponormal operator of finite type. Let us redefine $E_{\zeta,x}$ in [18] as

$$E_{\zeta,x} = \mathcal{F}(\overline{S(\zeta)}, (\overline{S(\zeta)}I - \Lambda)\alpha(\zeta)^* x), \qquad \zeta \in \mathcal{D},\ x \in M$$

where $\mathcal{F}(\lambda, v)$ is the unique solution of $(\lambda I - A)\mathcal{F}(\lambda, v) = v$ (cf. [18]). Then

$$A^* E_{\zeta,x} = \overline{\Psi(\zeta)} E_{\zeta,x}.$$

In the case of that A is a subnormal operator of finite type, if $w = \overline{S(\zeta)} \in \rho(A)$ and $z = \Psi(\zeta)$, then in the analytic model

$$E_{\zeta,x} = \frac{wI - \Lambda}{w - u}\alpha(\zeta)^* x.$$

By the way, we can show that as a vector in the Hilbert space

$$E_{\zeta,x} = f_{z,x} \tag{51}$$

for $x \in \alpha(\zeta)^* M$. Actrually, it is easy to see that for any $\lambda \in \rho(A), y \in M$

$$(E_{\zeta,x}, (\lambda - (\cdot))^{-1} y) = (((\overline{\lambda} I - \Lambda^*)(wI - \Lambda) - C)^{-1} \alpha(\zeta)^* x, y), \tag{52}$$

and

$$(f_{z,x}, (\lambda - (\cdot))^{-1} y) = \frac{(\mu(z)^* x, y)}{\overline{\lambda} - \overline{z}}. \tag{53}$$

But $(R(w) - \overline{z})(w - \Lambda)\alpha(\zeta)^* = 0$ and $\mu(z)^* x = \alpha(\zeta)^* x$. Therefore (51) and (52) are equal for every $\lambda \in \rho(A)$ and $y \in M$, which proves (51).

Let $r(\zeta),\ r^*(\zeta)$ be the range $C^{-\frac{1}{2}}\alpha(\zeta)M$ and $C^{-\frac{1}{2}}\alpha(\zeta)^* M$ respectively. Let $P(\zeta)$ be the projection from M to $\gamma(\zeta)$. If for $z \in D$, there is only one $\zeta \in \mathcal{D}$ such that $z = \Psi(\zeta)$, then $r(\zeta),\ r^*(\zeta)$ and $P(\zeta)$ defined here concide with $r(z),\ r^*(z)$ and $P(z)$ before respectively. Let $\widehat{E}(\zeta, \xi)$ be the operator on $\mathcal{L}(M)$ satisfying

$$(\widehat{E}(\zeta, \xi)x, y) = (E_{\xi,x}, E_{\zeta,y}).$$

Then from Lemma 5 of [18] we have

$$\widehat{E}_A(\zeta, \xi) = \frac{\alpha(\zeta)((S(\zeta)I - \Lambda^*)(\overline{S(\xi)}I - \Lambda) - C)\alpha(\xi)^*}{(\Psi(\zeta) - \overline{S(\xi)})(\overline{\Psi(\xi)} - S(\zeta))}. \tag{54}$$

Normalize $\widehat{E}_A(\zeta, \xi)$ as

$$E_A(\zeta, \xi) \stackrel{\text{def}}{=} C_A^{-\frac{1}{2}} E_A(\zeta, \xi) C_A^{-\frac{1}{2}}.$$

Theorem 6.5 *Let A be a pure hyponormal operator of finite type. For $\zeta,\ \xi \in \mathcal{D}$, if $\overline{S(\zeta)}$ and $\overline{S(\xi)} \in \rho(A)$, then*

$$E_A(\zeta, \xi)(\Psi(\zeta) - \overline{S(\xi)})(S(\zeta) - \overline{\Psi(\xi)}) = P(\zeta)(I + S_A(\Psi(\zeta), \Psi(\xi)))^{-1}\big|_{r(\xi)^*}. \tag{55}$$

Proof. It is easy to see that

$$\alpha(\zeta)(S(\zeta)I - \Lambda^*) = \alpha(\zeta)(R(\Psi(\zeta)) - \Lambda^*) = \alpha(\zeta)C(\Psi(\zeta)I - \Lambda)^{-1}. \tag{56}$$

Thus (55) follows from (47), (54) and (56).

In the case of that A is a pure hyponormal operator of finite type with rank one self-commutator, the kernel $E_A(\zeta, \xi)$ defined here coincides with that defined in [12], [15], [17] and (54) becomes

$$E_A(z, w)(z - \overline{S(w)})(S(z) - \overline{w})(1 + S_A(z, w)) = 1$$

which has been proved in [17] (cf. [15], [12]). □

Now, let us discuss the subspace G_A in §4. Suppose all zeros of $q_A(\cdot)$ are in the $\partial\sigma(A)$, with multiplicities $l_j, j = 1, 2, \ldots, s$. The following is a function model of G_A. The subspace G_A is the span of the vectors

$$g_{v_j,l,x} = \frac{\partial^l}{\partial \overline{\zeta}^l} \frac{(\overline{S(\zeta)}I - \Lambda)}{\overline{S(\zeta)} - (\cdot)} \alpha(\zeta)^* x\big|_{\zeta = \Psi^{-1}(v_j)}, \qquad x \in T, l = 0, 1, 2, \ldots, l_{j-1}$$

where Λ is regarded as an operator from T-valued functions to T-valued functions. The inner product is

$$(g_{v_j,l,x}, g_{v_{j'},l',x'}) = (\frac{\partial^{l+l'}}{\partial\overline{\zeta}^l \partial\xi^{l'}} \widehat{E}_A(\xi,\zeta)x, x')\big|_{\zeta=\Psi^{-1}(v_j),\xi=\Psi^{-1}(v_{j'})}.$$

The operator A acts as $Aq_{v_j,l,x} = (\cdot)g_{v_j,l,x}$.

7. Special cases

We give some examples of operators in $\mathcal{F}$. Let $\mathcal{F}_1$ be the family of all pure hyponormal operators of finite type with rank one self-commutators.

Proposition 7.1 *$\mathcal{F}_1 \subset \mathcal{F}$.*

Proof. It is easy to show that $\mathcal{D} = D$ is complete. Now we have to show that $R(z)$ is diagonalizable at $S(z)$. We only have to show that $P_w(z, S(z)) \not\equiv 0$, where $P_w(z,w) = \frac{\partial}{\partial w}P(z,w)$. Actually, in [20] we already applied that in defining $k(\cdot)$, but we did not write the proof. We adopt all the notations in [20].

Suppose on contrary that $P_w(z, S(z)) = 0$. Then $P_z(z, S(z)) = 0$, where $P_z(z,w) = \frac{\partial}{\partial z}P(z,w)$, since

$$0 = \frac{d}{dz}P(z, S(z)) = P_z(z, S(z)) + S'(z)P_w(z, S(z)).$$

By the Corollary 1 of [20],

$$C - C(zI-\Lambda)^{-1}(\overline{w}I-\Lambda^*)^{-1}C = P(z,\overline{w})CQ(z)^{-1}\overline{Q(w)}^{-1},$$

we have

$$\begin{aligned}&C(zI-\Lambda)^{-2}(\overline{w}I-\Lambda^*)^{-1}C\\ =&P_z(z,\overline{w})C(Q(z)\overline{Q(w)})^{-1} - P(z,\overline{w})Q'(z)Q(z)^{-2}\overline{Q(w)}^{-1}C\end{aligned}$$

Therefore

$$C(zI-\Lambda)^{-2}(S(z)I-\Lambda^*)^{-1} = 0,$$

since $P_z(z, S(z)) = P(z, S(z)) = 0$. Hence

$$((S(z)I-\Lambda^*)^{-1}1, (\overline{z}I-\Lambda^*)^{-1}1) = 0 \tag{57}$$

From (24) of [20], there is an analytic function $h(\cdot,u)$ of $(\cdot)$ and $\overline{u}$ such that

$$(\overline{z}I-\Lambda^*)^{-1}1 = E(\cdot,z)\overline{S(z)} + h(\cdot,z).$$

Therefore

$$(\overline{z}I-\Lambda^*)^{-2}1 = -E(\cdot,z)\overline{S'(z)} - \overline{S(z)}\partial_{\overline{z}}E(\cdot,z) + \partial_{\overline{z}}h(\cdot,z).$$

By (6) of [20], z_k is a pole of $S(z)$ of order n_k. Thus

$$\lim_{z\to z_k}\frac{(\overline{z}-\overline{z_k})}{\overline{S(z)}}(\overline{z}I-\Lambda^*)^{-2}1 = n_kE(\cdot,z_k).$$

Multiplying $z - z_k$ to the both sides of (57) and letting $z \to z_k$, (57) leads to a contradiction $(1, E(\cdot, z_k)) = 0$, since $E(\cdot, z)$ is normalized as $(E(\cdot, z), 1) = 1$. Thus $A \in \mathcal{F}$. □

By means of the technique in §4 of [20], we may also prove that if $A \in \mathcal{F}$ and $\mathcal{G}$ is any invariant subspace of A with finite codimension, then $A\big|_{\mathcal{G}} \in \mathcal{F}$.

As in [17], we proved that for a hyponormal operator H, if $\dim K_K = 1$ (in this case $\dim M_H$ must be 1), then H must be subnormal. Actually, if it is also pure then it must be a linear combination of the identity operator and a unilateral shift of multiplicity one. The question is the following. For a hyponormal H, does the condition $\dim M_H = \dim K_H$ (> 1) implies that H must be subnormal. We still don't know the answer when $\dim M_H = \dim K_H = 2$. But the following example show that there is a pure non-subnormal hyponormal operator H satisfying $\dim M_H = \dim K_H = 3$. From that example it is also easy to construct for every $n > 3$, a pure non-subnormal hyponormal operator satisfying $\dim M_H = \dim K_H = n$.

Example 7.1 Let $\mathcal{H}$ be the Hardy space $H^2(\mathbb{T})$ endowed with the inner product

$$(f, g) \stackrel{\text{def}}{=} (f, g)_{H^2(\mathbb{T})} + a_0 f(0)\overline{g(0)} + a_1 f'(0)\overline{g'(0)}$$

where $1 + a_0 > 0$ and $1 + a_1 > 0$. Let A be the multiplication operator

$$(Af)(z) \stackrel{\text{def}}{=} zf(z), \qquad f \in \mathcal{H}$$

Then

$$(A^*f)(z) = (f(z) - f(0))/z + (a_1 - a_0)f(0)/(1 + a_0) - a_1 f''(0)z/2(1 + a_1),$$

and

$$([A^*, A]f)(z) = \alpha_0 f(0) + \alpha_1 z f'(0) + \alpha_2 z f''(0)/2,$$

where

$$\alpha_0 = \frac{1 + a_1}{1 + a_0}, \quad \alpha_1 = \frac{1 + a_0 - (1 + a_1)^2}{(1 + a_0)(1 + a_1)}, \quad \alpha_2 = \frac{a_1}{1 + a_1}.$$

Thus there is an orthonormal basis $\{\eta_1, \eta_2, \eta_3\}$ of the subspace $M_A = K_A$, where

$$\eta_1 = z^2, \quad \eta_2 = (1 + a_1)^{-\frac{1}{2}} z, \quad \eta_3 = (1 + a_0)^{-\frac{1}{2}},$$

and z is the complex variable. Let us rewrite the vectors η_i as column vectors (δ_{nj}) where δ_{nj} is the Kronecker δ. Then

$$C_A = \begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_0 \end{pmatrix}, \ \Lambda_A = \begin{pmatrix} 0 & \sqrt{1 - \alpha_2} & 0 \\ 0 & 0 & \sqrt{\alpha_0} \\ 0 & 0 & 0 \end{pmatrix}.$$

The mosaic of this operator H is

$$\mu(z) = \begin{pmatrix} \alpha_2\sqrt{1+a_1}/z \\ 1-\alpha_0 \\ \sqrt{\alpha_0}z \end{pmatrix} \begin{pmatrix} z(1+a_1)^{-\frac{1}{2}} & 1-\alpha_2 & \sqrt{\alpha_0}(1+a_1)^{-1}/z \end{pmatrix}$$

It is easy to see that $S(z) = \frac{1}{z}$, D is the unit disc and $L = \mathbb{T}$. Then the $\mathcal{L}(K_A)$-valued $e(\cdot)$ on $\mathbb{T}$ is

$$e(de^{i\theta}) = \frac{d\theta}{2\pi(1+a_1)} \begin{pmatrix} \sqrt{1+a_1}e^{-i\theta} \\ 1 \\ \sqrt{\alpha_0}e^{i\theta} \end{pmatrix} \begin{pmatrix} \sqrt{1+a_1}e^{-i\theta} & 1 & \sqrt{\alpha_0}e^{-i\theta} \end{pmatrix}$$

Therefore, for $z \in D$,

$$\mu(z) - \int_{\mathbb{T}} \frac{(uI-\Lambda)e(du)}{u-z} = \begin{pmatrix} 0 & \alpha_2(1-\alpha_0)/z & \alpha_2(1+\alpha_0)^{-\frac{1}{2}}/z^2 \\ 0 & 0 & (1-\alpha_0)\sqrt{\alpha_0}(1+a_1)^{-1}/z \\ 0 & 0 & 0 \end{pmatrix} \quad (58)$$

Thus $e(\cdot)$ can not be a measure at $\{0\}$, since 0 is a pole of order 2 of the function defined by (58). Therefore A can not be subnormal by (36).

This $A \in \mathcal{F}$ and it is hyponormal iff $a_1 \geq 0$ and $(1+a_1)^2 \leq 1+a_0$. This example shows that in general for $A \in \mathcal{F}$, $e(\cdot)$ can not be a set function at the poles of $\mu(\cdot)$, i. e. (36) can not be extended to the operator $A \in \mathcal{F}$, even if we release the requirement that $e(\{a\}) \geq 0$ for the pole a of $\mu(\cdot)$. This example also shows some non-hyponormal operators in $\mathcal{F}$.

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