A point-source method for inverse acoustic and electromagnetic obstacle scattering problems.

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Abstract

We present the mathematical foundation for a *point-source method* to solve some inverse acoustic and electromagnetic obstacle scattering problems in three dimensions. We investigate the inverse acoustic scattering problem by a sound-soft and a sound-hard scatterer and the inverse *electromagnetic* scattering problem by a perfect conductor. Two independent approaches to the method are presented which reflect its strong relation to basic properties of obstacle scattering problems.

1 Introduction.

For numerous applications in medicine, geophysics and material science inverse acoustic and electromagnetic scattering problems are of great importance. Since many problems in these areas are not amenable to high or low frequency approximations one has to work with frequencies in the resonance region.

This paper is concerned with the development of a *point-source method* (PSM) for the solution of some inverse acoustic and electromagnetic obstacle scattering problems in three dimensions. Point-source methods have been recently introduced by Potthast [8],[9] and by Colton and Kirsch [1] to solve two-dimensional scalar inverse scattering problems. Here we investigate the inverse three-dimensional scattering problems for a sound-soft acoustic scatterer, a sound-hard acoustic scatterer and the electromagnetic scattering problem for a perfect conductor.

For scattering of an incident plane wave $u^i(.,d)$ with direction of incidence d by an obstacle D we assume the far field pattern $u^\infty(.,d)$ of the scattered field $u^s(.,d)$ to be given. The problem is to reconstruct the unknown obstacle D. To explain the main steps let us first investigate the inverse sound-soft acoustic scattering problem. We consider the far field pattern $w^\infty(\hat{x},z)$ of incident point-sources $w^i(.,z)$ with source

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point z. The vector $\hat{x}$ is the observation direction $\hat{x} \in \Omega$ where Ω denotes the unit sphere. The starting point for the PSM is the observation that there exists some explicitly given function $w_0^\infty(\hat{x}, z), \hat{x} \in \Omega, z \in I\!R^3$ such that

$$\delta(\hat{x}, z) := |w^\infty(\hat{x}, z) - w_0^\infty(\hat{x}, z)| \tag{1.1}$$

tends to zero for $z \to \partial D$. The PSM gives an explicit procedure how to compute from the measured far field pattern $u^\infty(., -\hat{x})$ an approximation $v^\infty(\hat{x}, .)$ for $w^\infty(\hat{x}, .)$ on some exposed set $\mathcal{G}$ with $\mathcal{G} \subset I\!R^3 \setminus \overline{D}$ depending on a parameter function M (see Sections 2.4.1, 3.4.1). Then

$$\alpha(\hat{x}, z) := |v^\infty(\hat{x}, z) - w_0^\infty(\hat{x}, z)| \tag{1.2}$$

approximates $\delta(\hat{x}, z)$ on $\mathcal{G}$. The PSM searches for parts of the unknown boundary ∂D as the surface where $\alpha(\hat{x}, z)$ is small.

A main problem is to choose an appropriate function δ for the different sound-soft, sound-hard or perfect conductor boundary condition such that $\delta(\hat{x}, z)$ tends to zero if z tends to the boundary of the unknown scatterer. For each scattering problem we give a function δ and present two different ways to prove $\delta(z) \to 0, z \to \partial D$. This reflects *two different approaches to the PSM* and shows that the method is strongly related to basic properties of obstacle scattering problems. The first approach uses *mixed reciprocity relations* (Sections 2.2 and 3.2) and the boundary condition. The second approach uses trivial explicit solutions for scattering of incident point-sources with $z \in D$ and *continuity properties of acoustic or electromagnetic surface potentials* (Sections 2.3 and 3.3). We need continuity properties or jump-relations, respectively, which extend the classical continuity properties to potentials with kernel functions instead of mere densities and to the L^p−spaces with $p \in (1, 2)$. This is worked out in Section 4.

2 Inverse acoustic scattering.

2.1 The direct and the inverse scattering problem.

We consider acoustic scattering from a *sound-soft* or a *sound-hard* impenetrable obstacle $D \subset I\!R^3$. We assume D to be bounded in $I\!R^3$. For simplicity we will restrict our representation to domains with boundary of class C^2. Let u^i be a solution to the Helmholtz equation

$$\triangle u + \kappa^2 u = 0 \tag{2.1}$$

with wave number $\kappa > 0$ on a domain containing D in its interior, representing an incident field. The direct scattering problem consist in looking for a scattered field u^s which solves the Helmholtz equation in $I\!R^3 \setminus \overline{D}$ and satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\Big(\frac{\partial u^s}{\partial r} - i\kappa u^s\Big) = 0, \quad r = |x|, \tag{2.2}$$

uniformly with respect to all directions such that the total field $u = u^i + u^s$ satisfies the *sound-soft* boundary condition $u^i + u^s = 0$ on ∂D, or the *sound-hard* boundary condition $\frac{\partial}{\partial \nu}\Big(u^i + u^s\Big) = 0$ on ∂D. Because of the Sommerfeld radiation condition (2.2) the scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|}\Big\{u^\infty(\hat{x}) + O\Big(\frac{1}{|x|}\Big)\Big\}, \quad |x| \to \infty, \tag{2.3}$$

where $\hat{x} := x/|x| \in \Omega := \{x \in I\!R^3, |x| = 1\}$. The function u^∞ is called the *far field pattern* of the scattered wave. We denote the far field pattern corresponding to the incident plane wave $u^i(x,d) := e^{i\kappa x\cdot d}$, $x \in I\!R^3$ with direction $d \in \Omega$ by $u^\infty(\hat{x}, d)$, $\hat{x} \in \Omega$.

We are now prepared to formulate the corresponding *inverse* scattering problems.

The inverse scattering problem. Given the far field pattern for scattering of one or a number of incident plane waves by a sound-soft or a sound-hard scatterer, find the unknown obstacle D.

Later we will use integral representations for the scattered fields. By

$$\Phi(x,y) := \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, \quad x \neq y,$$

we denote the fundamental solution to the Helmholtz equation. For a domain $D \subset I\!R^3$ we use the single-layer operator

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)ds(y), \quad x \in \partial D, \tag{2.4}$$

the double-layer operator

$$(K\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds(y), \quad x \in \partial D, \tag{2.5}$$

the adjoint operator

$$(K'\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)ds(y), \quad x \in \partial D, \tag{2.6}$$

and the normal derivative operator

$$(T\varphi)(x) := 2\frac{\partial}{\partial \nu(x)}\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds(y), \quad x \in \partial D. \tag{2.7}$$

The combined acoustic double- and single-layer potential

$$(P_D\varphi)(x) := \int_{\partial D} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\Phi(x,y) \right\} \varphi(y) ds(y), \quad x \in I\!R^M \setminus \partial D \tag{2.8}$$

with density $\varphi \in C(\partial D)$ solves the scattering problem for a sound-soft obstacle if and only if the density φ is a solution of the boundary integral equation

$$(I + K - iS)\varphi = -2u^i|_{\partial D}. \tag{2.9}$$

Analogously the modified single- and double-layer potential

$$(P_N\varphi)(x) := \int_{\partial D} \left\{ \Phi(x,y)\varphi(y) + i\frac{\partial \Phi(x,y)}{\partial \nu(y)}(S_0^2\varphi)(y) \right\} ds(y), \quad x \in I\!R^M \setminus \partial D, \tag{2.10}$$

where S_0 denotes S in the case $\kappa = 0$, solves the scattering problem from a sound-hard obstacle if and only if the density φ is a solution of the integral equation

$$(I - K' - iTS_0^2)\varphi = -2\frac{\partial u^i}{\partial \nu} \tag{2.11}$$

Colton and Kress show in [3] that the inverse operators $(I + K - iS)^{-1}$ and $(I - K' - iTS_0^2)^{-1}$ exist and are bounded in $C(\partial D)$. Thus we obtain the representation

$$u^s = -2P_D(I + K - iS)^{-1}u^i|_{\partial D} \tag{2.12}$$

for the scattered field from the *sound-soft* obstacle D and

$$u^s = 2P_N(I - K' - iTS_0^2)^{-1}\frac{\partial u^i}{\partial \nu} \tag{2.13}$$

for the scattered field from the *sound-hard* obstacle D. The far field pattern of the potentials P_D and P_N is given by the functions

$$(P_D^\infty\varphi)(\hat{x}) := \frac{1}{4\pi}\int_{\partial D} \left\{ \frac{\partial e^{-i\kappa\hat{x}\cdot y}}{\partial \nu(y)} - ie^{-i\kappa\hat{x}\cdot y} \right\} \varphi(y) ds(y), \quad \hat{x} \in \Omega \tag{2.14}$$

and

$$(P_N^\infty\varphi)(\hat{x}) := \frac{1}{4\pi}\int_{\partial D} \left\{ e^{-i\kappa\hat{x}\cdot y}\varphi(y) + i\frac{\partial e^{-i\kappa\hat{x}\cdot y}}{\partial \nu(y)}(S_0^2\varphi)(y) \right\} ds(y), \quad \hat{x} \in \Omega. \tag{2.15}$$

The mapping which maps the boundary values onto the far field pattern of the scattered field is called *scattering map* $\mathcal{S}$.

2.2 Mixed reciprocity relations.

Here we present a mixed reciprocity relation on which our inversion method may be based. In contrast to the usual *far field reciprocity relation*, where incident plane waves and the far field pattern of the scattered waves are involved, or to the *near field reciprocity relation* for incident point sources and measured data on a surface surrounding the obstacle we call this type of reciprocity a *mixed reciprocity relation.*

In the following D is either a sound-soft or a sound-hard scatterer. Let us denote by $u^s(., d)$ the scattered field for scattering of an incident plane wave with direction d. If the incident field is given by a point-source $\Phi(., z)$ with source point z we will use $w^s(., z)$ for the scattered field and $w^\infty(., z)$ for its far field pattern. We use the constant $\gamma := \frac{1}{4\pi}$ for the three dimensional case. The following theorem with a different constant γ also holds in two dimensions.

THEOREM 1 **(Mixed reciprocity relation.)** *For scattering from a sound-soft and a sound hard obstacle we have the relation*

$$w^\infty(\hat{x}, z) = \gamma\ u^s(z, -\hat{x}), \quad z \in I\!R^M \setminus \overline{D}, \hat{x} \in \Omega, \tag{2.16}$$

i.e. in the observation direction $\hat{x}$ the far field pattern for scattering of a point-source with source point z is the same than the scattered wave of an incident plane wave with direction $-\hat{x}$ measured in the observation point z.

Proof. The proof for the sound-soft scatterer is due to Kress [6]. By Green's theorem we have that

$$\int_{\partial D} \Big(w^s(y, z) \frac{\partial u^s(y, d)}{\partial \nu(y)} - \frac{\partial w^s(y, z)}{\partial \nu(y)} u^s(y, d) \Big) ds(y) = 0, \quad z \in I\!R^M \setminus \overline{D}, d \in \Omega \tag{2.17}$$

where $\nu(y)$ is the unit outer normal vector to ∂D in the point $y \in \partial D$. Also from Green's theorem we have the representation

$$w^\infty(\hat{x}, z) = \gamma \int_{\partial D} \Big(w^s(y, z) \frac{\partial e^{-i\kappa \hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial w^s}{\partial \nu}(y, z) e^{-i\kappa \hat{x}\cdot y} \Big) ds(y), \quad \hat{x} \in \Omega. \tag{2.18}$$

Adding γ times (2.17) with d replaced by $-\hat{x}$ to equation (2.18) we obtain

$$w^\infty(\hat{x}, z) = \gamma \int_{\partial D} w^s(y, z) \frac{\partial u(y, -\hat{x})}{\partial \nu(y)} ds(y), \quad z \in I\!R^M \setminus \overline{D}, \quad d \in \Omega, \tag{2.19}$$

for the sound-soft scatterer and

$$w^\infty(\hat{x}, z) = -\gamma \int_{\partial D} \frac{\partial w^s(y, z)}{\partial \nu(y)} u(y, -\hat{x}) ds(y), \quad z \in I\!R^M \setminus \overline{D}, \quad d \in \Omega, \tag{2.20}$$

for the sound-hard scatterer where $u(.,d)$ denotes the total field for the sound-soft or the sound-hard scattering problem with incident plane wave of direction d. Again from Green's theorem have the representation formula

$$u^s(x,d) = -\int_{\partial D} \Phi(x,y) \frac{\partial u(y,d)}{\partial \nu} ds(y), \quad x \in I\!R^M \setminus \overline{D} \tag{2.21}$$

for the sound-soft boundary condition and

$$u^s(x,d) = \int_{\partial D} \frac{\partial \Phi}{\partial \nu}(x,y) u(y,d) ds(y), \quad x \in I\!R^M \setminus \overline{D} \tag{2.22}$$

for the sound-hard boundary condition. Now from (2.19), (2.21) and (2.20), (2.22) using the boundary condition for w^s we obtain $w^\infty(\hat{x}, z) = \gamma \; u^s(z, -\hat{x})$, $\hat{x} \in \Omega$, $z \in I\!R^M \setminus \overline{D}$, both for the sound-soft and sound-hard boundary condition. □

2.3 Incident point-sources near the boundary.

Now we prepare the second approach to the point-source method. We investigate the behaviour of the scattered fields for incident point-sources and dipoles when the source point tends to the boundary of the scatterer. This behaviour is strongly related to the jump-relations or continuity properties, respectively, for single- and double-layer potentials. More precisely, we will need to investigate the behaviour of single- and double-layer potentials where the density $\varphi(y)$ is replaced by kernel-functions $k(x,y)$ with $k(x,.) \in L^p(\partial D)$ for $p \in (1,2)$. The proofs are worked out in Section 4 of the paper.

2.3.1 The sound-soft boundary condition.

Let us consider scattering of the incident point-source $w^i(x,z) = \Phi(.,z)$ with source point $z \in I\!R^3$ by a *sound-soft* scatterer. We denote the corresponding far field pattern by $w^\infty(\hat{x}, z), \hat{x} \in \Omega$. The far field pattern of the function $-\Phi(.,z)$ is given by the function $w_0^\infty(\hat{x}, z) = -\gamma e^{-i\kappa \hat{x}\cdot z}$.

THEOREM 2 *Let D be a sound-soft scatterer. For $z \in D$ we have*

$$w^\infty(.,z) = w_0^\infty(.,z), \tag{2.23}$$

and for $z \notin \overline{D}$

$$w^\infty(.,z) \neq w_0^\infty(.,z). \tag{2.24}$$

The far field pattern $w^\infty(.,z) \in C(\Omega)$ depends continuously on z in $I\!R^3$.

Remark. The *Green's function* for the sound-soft scattering problem is given by $w^s(x,z)+w^i(x,z)=w^s(x,z)-w_0^i(x,z)$ with $w_0^i(x,z):=-\Phi(x,z)$, i.e. the theorem states that the far field pattern of the Green's function tends to zero for $z\to\partial D$.

Proof. The first statement of the theorem is almost trivial. Incident point sources $w^i(.,z)$ satisfy both the Helmholtz equation in $I\!R^3\setminus\{z\}$ and the Sommerfeld radiation condition. Thus for $z\in D$ the solution of the scattering problem with incident wave $w^i(.,z)$ is given by $-w^i(.,z)$ and the far field pattern $w^\infty(.,z)$ is $w_0^\infty(.,z)$. If $z\notin\overline{D}$ the singular function $-w^i(.,z)$ with far field pattern $w_0^\infty(.,z)$ cannot be a scattered field. By the Rellich Lemma ([3]) the far field pattern uniquely determines the scattered field. Thus $w_0^\infty(.,z)$ cannot be a far field pattern of a scattered field and we obtain (2.24). From Section 2.1 we obtain for the far field pattern w^∞ the representation

$$\begin{aligned} w^\infty(.,z) &= -2P_D^\infty(I+K-iS)^{-1}w^i(.,z) \\ &= -2P_D^\infty w^i(.,z) \;+\; 2P_D^\infty(I+K-iS)^{-1}(K-iS)w^i(.,z). \end{aligned} \tag{2.25}$$

The kernel $k^\infty(\hat{x},y)$ of P_D^∞ is continuous for $\hat{x}\in\Omega$ and $y\in\partial D$. From the proofs of the classical continuity results for the single-layer potential (see [2]) it can easily be seen that

$$f^\infty(\hat{x},z):=\int_{\partial D}k^\infty(\hat{x},y)\Phi(y,z)ds(y) \tag{2.26}$$

depends continuously on $z\in I\!R^3$ and $\hat{x}\in\Omega$. Let us now consider the operators K and S. Their kernels $k(x,y)$ are continuous for $x\neq y$ and weakly singular, but not continuous for all x and $y\in\partial D$. Thus we cannot use the classical continuity properties of the single-layer potential to treat the terms $Kw^i(.,z)$ and $Sw^i(.,z)$. The function $k(x,.)$ is in $L^p(\partial D)$ for $p\in(1,2)$, but not in $L^2(\partial D)$. Also note that we need to know the behaviour of

$$f(.,z):=\int_{\partial D}k(.,y)\Phi(y,z)ds(y) \tag{2.27}$$

as a function $f(.,z)\in L^p(\partial D)$, for $z\to z_0\in\partial D$. In contrast to this the theory of potentials with L^p-densities investigates $f(x,.\pm h\nu(.))\in L^p(\partial D)$ for $h\to 0$ where x is fixed. We also need to investigate the mapping properties of the integral operator $(I+K-iS)^{-1}$ in $L^p(\partial D)$. This is worked out in the Section 4 of the paper. As a result we first obtain L^p-continuity statements for the single- and double-layer potential operators, i.e. the functions $(K-iS)w^i(.,z)\in L^p(\partial D),\quad z\in I\!R^3$, depend continuously on $z\in I\!R^3$ for $p\in(1,2)$. Second the operator $(I+K-iS)$ is continuously invertible in $L^p(\partial D)$. The remark that P_D^∞ maps $L^p(\partial D)$ boundedly into $C(\Omega)$ and the use of (2.25) now completes the proof of the continuity statement of the theorem. $\square$

2.3.2 The sound-hard boundary condition.

We now investigate the *sound-hard* boundary condition. Let $a \in I\!R^3$ be a constant vector with $||a|| = 1$. Let us consider scattering of the *dipole*

$$\psi^i(x, z, a) := a \cdot \text{grad}_z \Phi(x, z), \quad x \in I\!R^3 \setminus \{z\} \tag{2.28}$$

by a sound-hard scatterer. We denote the far field pattern of the scattered wave by $\psi^\infty(\hat{x}, z, a)$, $\hat{x} \in \Omega$. The far field pattern of the function $-\psi^i(., z, a)$ is given by the function $\psi_0^\infty(\hat{x}, z, a) := i\kappa\gamma \; a \cdot \hat{x} \; e^{-i\kappa\hat{x}\cdot z}$. Recall that for an incident point-source, its scattered field and its far field pattern we use the notation $w^i(., z)$, $w^s(., z)$ and $w^\infty(., z)$.

THEOREM 3 *Let D be a sound-hard scatterer. For $z \in D$ we have*

$$w^\infty(., z) = w_0^\infty(., z) \quad \text{and} \quad \psi^\infty(., z, a) = \psi_0^\infty(., z, a) \tag{2.29}$$

and for $z \notin \overline{D}$

$$w^\infty(., z) \neq w_0^\infty(., z) \quad \text{and} \quad \psi^\infty(., z, a) \neq \psi_0^\infty(., z, a). \tag{2.30}$$

Let $z_0 \in \partial D$ be a point with $\nu(z_0) = a$. Then we have

$$\Big|\Big|\psi^\infty(., z_0 + h\nu(z_0), a) - \psi^\infty(., z_0 - h\nu(z_0), a)\Big|\Big|_{C(\Omega)} \to 0, \quad h \to 0. \tag{2.31}$$

The convergence (2.31) is uniform for $z_0 \in \partial D$.

Proof. The statement of (2.29) and (2.30) can be obtained with the same arguments as in Theorem 2. To investigate the limit in (2.31) we look at the representation

$$\begin{aligned} \psi^\infty(\hat{x}, z, a) \;\; = \;\; & -2P_N^\infty(I - K' - iTS_0^2)^{-1}\frac{\partial\psi^i(., z, a)}{\partial\nu} \; - \; 2P_N^\infty\frac{\partial\psi^i(., z, a)}{\partial\nu} \\ & -2P_N^\infty(I - K' - iTS_0^2)^{-1}(K' + iTS_0^2)\frac{\partial\psi^i(., z, a)}{\partial\nu}. \end{aligned} \tag{2.32}$$

The kernel $k^\infty(\hat{x}, y)$ of P_N^∞ is continuous for $\hat{x} \in \Omega$ and $y \in \partial D$. From the proofs of the classical results for the normal derivative of the double-layer potential (see [2], Theorem 2.21) we obtain that for the function

$$\begin{aligned} f(\hat{x}, z) \;\; := & \;\; \int_{\partial D} k^\infty(\hat{x}, y)\frac{\partial}{\partial\nu(y)}\Big(\nu(z_0) \cdot \text{grad}_z \Phi(y, z)\Big) \, ds(y). \\ = & \;\; \nu(z_0) \cdot \text{grad}_z \int_{\partial D} k^\infty(\hat{x}, y)\frac{\partial\Phi(z, y)}{\partial\nu(y)}ds(y). \end{aligned} \tag{2.33}$$

we have

$$f(\hat{x}, z_0 + h\nu(z_0)) - f(\hat{x}, z_0 - h\nu(z_0)) \to 0, \quad h \to 0 \tag{2.34}$$

uniformly for $\hat{x} \in \Omega$. Note that this is different from the uniform convergence with respect to $z_0 \in \partial D$ as given in the statement of the quoted theorem, but can be seen in the same way from the proof! We obtain the convergence (2.31) for the term (2.33). Let us now consider the operators K' and S_0. Their kernel $k(x,y)$ is continuous for $x \neq y$ and weakly singular. For fixed $x \in \partial D$ we have $k(x,.) \in L^p(\partial D)$ for $p \in (1,2)$, but $k(x,.) \notin L^2(\partial D)$. So as in the proof of Theorem 2 we have to consider special continuity properties for integral operators with L^p-kernels and the mapping properties of the operator $(I - K' - iTS_0^2)^{-1}$ in $L^p(\partial D)$. The proofs for this can be found in section 4. We obtain for the kernels $k(x,y)$ of K' and S_0

$$\Big|\Big|\nu(z_0) \cdot \text{grad}_z\Big(\int_{\partial D} k(.,y) \frac{\partial \Phi(z,y)}{\partial \nu(y)} ds(y)\Big)\Big|_{z=z_0+h\nu(z_0)} - \nu(z_0) \cdot \text{grad}_z\Big(\int_{\partial D} k(.,y) \frac{\partial \Phi(z,y)}{\partial \nu(y)} ds(y)\Big)\Big|_{z=z_0-h\nu(z_0)}\Big|\Big|_{L^p(\partial D)} \to 0, \quad h \to 0 \tag{2.35}$$

and using the boundedness of TS_0^2 in $L^p(\partial D)$

$$\Big|\Big|\Big(K' + iTS_0^2\Big)\frac{\partial \psi^i(.,z_0 + h\nu(z_0),a)}{\partial \nu} - \Big(K' + iTS_0^2\Big)\frac{\partial \psi^i(.,z_0 - h\nu(z_0),a)}{\partial \nu}\Big|\Big|_{L^p(\partial D)} \to 0, \quad h \to 0 \tag{2.36}$$

Since the operator $(I - K' - iTS_0^2)^{-1}$ is continuously invertible in $L^p(\partial D)$ and P_N^∞ maps $L^p(\partial D)$ boundedly into $C(\Omega)$ using (2.32) the proof of (2.31) is complete and for $a = \nu(z_0)$ the convergence is uniform for $z_0 \in \partial D$. □

2.4 A point-source method in inverse scattering.

In this section we describe a *point-source method* for the solution of the inverse scattering problem based either on Section 2.2 or on Section 2.3. Independently of these sections we first describe a procedure how to compute from the far field pattern for an incident plane wave with direction $d \in \Omega$ an approximation for the far field pattern for incident point-sources or incident dipoles in the observation point $-d$. Then we develop the PSM for the inverse sound-soft and the inverse sound-hard scattering problem.

2.4.1 An approximation procedure.

Let us consider an incident field Ψ_0^i which solves the Helmholtz equation on $\mathbb{R}^3 \setminus \{0\}$ and the rotated and translated field $\Psi^i(x, M, z) := \Psi_0^i(M^{-1}(x - z))$, where z is the translation vector and M is an orthogonal rotation matrix. We denote the far field pattern for scattering of $\Psi^i(.,M,z)$ by a scatterer D by $\Psi^\infty(.,M,z)$. In this section we develop a procedure to compute on some special subset $\mathcal{G}$ of $I\!R^3$ an approximation

$$v^\infty\Big(\hat{x}, g(.,M,z)\Big) = \int_\Omega u^\infty(d,-\hat{x})g(-d,M,z)ds(d), \quad z \in \mathcal{G}, \tag{2.37}$$

for the far field pattern $\Psi^\infty(\hat{x},M,z)$ with $g(.,M,z) \in L^2(\Omega)$. By $B_R(x)$ we denote the ball with radius R and center x. We assume about the unknown domain D the apriori information $D \subset B_{R_i}(x_0) \subset B_{R_e}(0)$ for some known $R_e > R_i > 0$ and some unknown $x_0 \in B_{R_e}(0)$, i.e. we know an apriori bound on the *size* of the object, but its *shape* and *location* is unknown. For known $\beta > 0$ we assume that an *exterior β-cone condition* is valid, more precisely that for all $x \in I\!R^3 \setminus D$ there is a cone $\mathcal{C}_{x,p,\beta} := \{y \in I\!R^3, \langle y-x,p\rangle \geq \cos(\beta)\}$ with vertex x, direction $p \in \Omega$ and opening angle β in the exterior $I\!R^3 \setminus D$ of D, i.e. $\mathcal{C}_{x,p,\beta} \subset I\!R^3 \setminus D$.

We now come to the first step of the approximation procedure. Let us choose a small parameter $\rho > 0$ and use the notation $e_1 := (1,0,0)$. On the *domain of approximation*

$$G_0 := B_{2R_i}(-\rho e_1) \setminus \mathcal{C}_{-\rho e_1, e_1, \beta} \tag{2.38}$$

we want to approximate the source $\Psi_0^i(.)$ by a *Herglotz wave function*

$$v^i(x, g_0) := \int_\Omega e^{i\kappa x\cdot d} g_0(d)ds(d), \quad x \in I\!R^3 \tag{2.39}$$

with density $g_0 \in L^2(\Omega)$. In the following lemma we prove that this is possible.

LEMMA 4 *Take $\mu \in I\!N$. For $\epsilon > 0$ there exists $g_0 \in L^2(\Omega)$ with*

$$||v^i(.,g_0) - \Psi_0^i(.)||_{C^\mu(G_0)} < \epsilon. \tag{2.40}$$

Proof: For $\eta > 0$ define $G_{0,\eta} := \{x \in I\!R^3, \text{dist}\,(x, G_0) < \eta\}$. Note that though G_0 has corners, the boundary of $G_{0,\eta}$ is of class C^1. If η is sufficiently small we have $0 \notin G_{0,\eta}$. Let us choose η sufficiently small and such that the homogeneous interior Dirichlet problem for $G_{0,\eta}$ has only the trivial solution. In this case the operator

$$(Vg)(x) := \int_\Omega e^{i\kappa x\cdot d} g(d)ds(d), \quad x \in \partial G_{0,\eta}, \tag{2.41}$$

has dense range in $L^2(\partial G_{0,\eta})$ since according to Theorem 5.13 of [3] its adjoint operator is injective. Now the statement of the lemma is a consequence of the continuous dependence of solutions to the interior Dirichlet problem for the domain $G_{0,\eta}$ with respect to the L^2-norm of the boundary values on $\partial G_{0,\eta}$. □

We now translate and rotate both the domain of approximation G_0 and the Herglotz wave function $v^i(., g_0)$. The translated and rotated domain of approximation is $G(M, z) := MG_0 + z$. The translated and rotated Herglotz wave function is given by

$$\begin{aligned} v^i(M^{-1}(x-z), g_0) &= \int_\Omega e^{i\kappa M^{-1}(x-z)\cdot d} g_0(d)\, ds(d) \\ &= \int_\Omega e^{i\kappa x\cdot Md}\ \ e^{-i\kappa z\cdot Md} g_0(d)\ \ ds(d) \\ &= \int_\Omega e^{i\kappa x\cdot \tilde{d}}\ \left\{ e^{i\kappa z\cdot \tilde{d}} g_0(M^{-1}\tilde{d}) \right\}\ ds(\tilde{d}) \\ &= v^i(x, g(., M, z)), \end{aligned} \tag{2.42}$$

where $g(., M, z)$ is given by

$$g(d, M, z) := e^{-i\kappa z\cdot d} g_0(M^{-1}d). \tag{2.43}$$

From Lemma 4 we obtain that the Herglotz wave function $v^i(., g(., M, z))$ approximates the function $\Psi^i(., M, z)$ on the domain $G(M, z)$, i.e. we have

$$||\Psi^i(., M, z) - v^i(., g(., M, z))||_{C^\mu(G(M,z))} \leq \epsilon. \tag{2.44}$$

Let us consider a matrix M and a point $z \in I\!R^3$ such that the scatterer D is a subset of the domain of approximation $G(M, z)$. By continuity of the scattering problem there is a constant c (depending only on the domain D) such that from (2.44) we obtain

$$||\Psi^\infty(., M, z) - v^\infty(., g(M, z))||_{C(\Omega)} \leq \ \ c\epsilon \tag{2.45}$$

for the far field pattern Ψ^∞ for scattering of the incident field Ψ^i and the far field pattern $v^\infty(., M, z)$ for scattering of $v^i(., g(M, z))$. From the continuity of the scattering map which maps the boundary data of the incident field onto the far field pattern, the linearity of this mapping and the continuity of the mapping $L^2(\Omega) \to C(G(M, z)), g \mapsto v^i(., g)$ we obtain the representation

$$v^\infty(\hat{x}, g(., M, z)) \ = \ \int_\Omega u^\infty(\hat{x}, d)\ g(d, M, z)\ ds(d) \tag{2.46}$$

for the far field pattern v^∞ of the scattered field produced by the incident field v^i. (For the representation (2.46) see also Theorem 3.16 of [3], where the proof is done by integral equation methods.) We use the reciprocity relation for the far field pattern of incident plane waves to transform this into

$$\begin{aligned} v^\infty(\hat{x}, g(., M, z)) &= \int_\Omega u^\infty(-d, -\hat{x})\ g(d, M, z)\ ds(d) \\ &= \int_\Omega u^\infty(d, -\hat{x})\ g(-d, M, z)\ ds(d), \end{aligned} \tag{2.47}$$

i.e. the form (2.37). Given the function $z \mapsto M(z)$, we denote by $\mathcal{G}$ the set of points z in $I\!R^3$ such that the scatterer D is a subset of the domain of approximation $G(M(z), z)$ and call $\mathcal{G}$ the set of points *exposed* by M. The approximation (2.45) is valid uniformly on $\mathcal{G}$.

2.4.2 The PSM for the inverse sound-soft scattering problem.

From either Theorem 1 (Section 2.2) and the boundary condition or from Theorem 2 (Section 2.3) we conclude that for the sound-soft scatterer the function

$$\delta^w(\hat{x}, z) := |w^\infty(\hat{x}, z) - w_0^\infty(\hat{x}, z)| \tag{2.48}$$

tends to zero if z tends to the boundary ∂D of the unknown scatterer. To obtain an approximation for d we consider the point-source $\Psi_0^i(x) := w^i(x, 0)$. Following the approximation procedure of Section 2.4.1 we obtain an approximation $v^\infty(., g(., M(z), z))$ for the far field pattern $w^\infty(., z)$ on the set $\mathcal{G}$ exposed by $M(z)$. This yields on $\mathcal{G}$ an approximation

$$\alpha^w(\hat{x}, M(z), z) := |v^\infty(\hat{x}, g(., M(z), z)) - w_0^\infty(\hat{x}, z)| \tag{2.49}$$

for the function $\delta^w(\hat{x}, z)$. Given the far field pattern u^∞ and a matrix function M the approximation α^w can be computed using (2.43) to (2.47).

The point-source method searches parts Λ of the boundary ∂D of the unknown sound-soft domain D as the set of points where the function $\alpha^w(\hat{x}, M(z), z)$ is close to zero. The diskussion of search strategies for this minimization problem and their effective implementation is beyond the range of this paper.

2.4.3 The PSM for the inverse sound-hard scattering problem.

First of all let us state that for scattering by a sound-hard obstacle D the function δ^w defined by (2.48) is not continuous at the boundary. Therefore it does not necessarily tend to zero if z tends to the boundary ∂D and we cannot use it to reconstruct the scatterer. For the inverse sound-hard scatterering problem we use the function

$$\delta^\psi(\hat{x}, z, a) := |\psi^\infty(\hat{x}, z, a) - \psi_0^\infty(\hat{x}, z, a)| \tag{2.50}$$

for $z \in I\!R^3$ and $a \in \Omega$ with ψ^∞ defined by (2.28). From

$$\psi^\infty(., z, a) = \mathcal{S}\Big(a \cdot \text{grad}\,_z w^i(., z)\Big) = a \cdot \text{grad}\,_z \mathcal{S} w^i(., z) = a \cdot \text{grad}\,_z w^\infty(., z) \tag{2.51}$$

and Theorem 1 we observe that $\psi^\infty(\hat{x}, z, a) - \psi_0^\infty(\hat{x}, z, a) = a \cdot \text{grad}\,_z u(z, -\hat{x})$ with the total field u for scattering by the sound-hard scatterer. Thus for $a = \nu(z_0)$, $z_0 \in \partial D$, from the boundary condition we obtain that $\delta^\psi(\hat{x}, z, a)$ tends to zero for

$z \to z_0 \in \partial D$. Following the second approach we obtain the same statement as a consequence of Theorem 3.

Principally we could quit at this point and use the approximation procedure of Section 2.4.1 applied to the function $\Psi_0^i(x) := \psi^i(x, 0, a_0)$ to obtain on the exposed set $\mathcal{G}$ an approximation $\alpha^\psi(\hat{x}, z, M(z)a_0)$ for $\delta^\psi(\hat{x}, z, M(z)a_0)$. But we would like to have the freedom to adapt the direction a independently from the rotation $M(z)$. Using the following *trick* we can avoid to compute initial approximations for all directions $a_0 \in \Omega$.

We apply for $\mu = 1$ the approximation procedure to the function $w^i(., z)$. Note that we have $\psi^i(x, z, a) = a \cdot \text{grad}_z w^i(x, z) = -a \cdot \text{grad}_x w^i(x, z)$. Since for the approximation (2.40) we use the norm of $C^1(G)$, the function

$$a \cdot \text{grad}_z v^i(x, g(., M, z)) = -a \cdot \text{grad}_x v^i(x, g(., M, z)) \tag{2.52}$$

approximates $\psi^i(., z, a)$ on $G(z, M)$ in the norm of $C(G(z, M))$. For the following considerations assume that $D \subset G(z, M)$. The functions $w^i(., z)$, $v^i(., g(., M, z))$, $w^\infty(., z)$ and $v^\infty(., g(., M, z))$ depend in $C(D)$ or $C(\Omega)$, respectively, continuously differentiable on the parameter z. As already used in (2.51) the differentiation with respect to z and the scattering operator $\mathcal{S}$ can be exchanged, i.e. we have

$$\mathcal{S}\Big(a \cdot \text{grad}_z v^i(., g(., M, z))\Big) \;=\; a \cdot \text{grad}_z v^\infty(., g(., M, z)). \tag{2.53}$$

Thus the function

$$\begin{aligned} \sigma(\hat{x}, z, M, a) \;&:=\; \int_\Omega u^\infty(d, -\hat{x}) \;\; (-i\kappa a \cdot d)\; g(d, M, z)\; ds(d) \\ &=\; a \cdot \text{grad}_z v^\infty(\hat{x}, g(., M, z)), \quad \hat{x} \in \Omega, \end{aligned} \tag{2.54}$$

approximates $\psi^\infty(., z, a)$. For a matrix function $z \mapsto M(z)$ the approximation of $\psi^\infty(., z, a)$ by $\sigma(., z, M(z), a)$ is uniform on the set $\mathcal{G}$ exposed by $M(z)$ and it is also uniform for $a \in \Omega$. We may choose a in dependence of the point z, i.e. $a = a(z)$. On the exposed set $\mathcal{G}$ we obtain an approximation

$$\alpha^\psi(\hat{x}, z, M(z), a(z)) := |\sigma(\hat{x}, z, M(z), a(z)) - \psi_0^\infty(\hat{x}, z, a(z))| \tag{2.55}$$

for the function $\delta^\psi(\hat{x}, z, a(z))$.

The point-source method for the solution of the inverse sound-hard acoustic scattering problem searches for parts Λ of the unknown sound-hard boundary ∂D as the set of points where the function $\alpha^\psi(\hat{x}, z, M(z), a(z))$ is close to zero and the normal vector $\nu(z)$ of Λ is close to $a(z)$. The discussion of an effective implementation of this minimization problem is not an aim of this work.

3 Inverse electromagnetic scattering.

3.1 The direct and the inverse scattering problem.

We consider the scattering of electromagnetic waves from a perfect conductor $D \subset I\!R^3$. We assume D to be bounded in $I\!R^3$ and the boundary ∂D to be of class C^2. An incident time-harmonic electromagnetic wave is modelled by two fields E^i and H^i which satisfy the *reduced Maxwell equations* $\text{curl}\ E - i\kappa H = 0, \quad \text{curl}\ H + i\kappa E = 0$ with the wave number $\kappa > 0$ on a domain containing D in its interior. The direct scattering problem is to find a scattered electromagnetic field E^s, H^s which solves the reduced Maxwell equations in $I\!R^3 \setminus \overline{D}$ and satisfies the *Silver Müller radiation condition*

$$\lim_{r\to\infty} \Big(H \times x - rE\Big) = 0 \tag{3.1}$$

where $r = |x|$ and where the limit is assumed to hold uniformly in all directions $x/|x|$ and the *boundary condition* $\nu \times (E^i + E^s) = 0$ on ∂D where ν is the unit outward normal to ∂D. From the radiation condition (3.1) it can be shown that the scattered field has the asymptotic behaviour

$$E^s(x) = \frac{e^{i\kappa|x|}}{|x|}\left\{E^\infty(\hat{x}) + O\Big(\frac{1}{|x|}\Big)\right\}, \quad |x| \to \infty, \tag{3.2}$$

uniformly for all directions $\hat{x} = x/|x|$. The vector field E^∞ is defined on the unit sphere Ω and is known as the electric *far field pattern* of E^s. We denote the scattered field for an incident plane wave

$$E^i_{pl}(x,d,q) = i\kappa(d\times q)\times d\ e^{i\kappa x\dot{d}}, \quad H^i_{pl}(x,d,q) = i\kappa d \times q\ e^{i\kappa x\times d} \tag{3.3}$$

with direction d and polarization q by $E^s_{pl}(.,d,q)$, the corresponding far field pattern by $E^\infty_{pl}(.,d,q)$.

Inverse scattering problem. *Given the electric far field pattern $E^\infty_{pl}(.,d,q)$ for scattering of one or a number of incident plane waves $E^i_{pl}(x,d,q), H^i_{pl}(x,d,q)$ with direction d and polarization q by a perfect conductor D, find the unknown obstacle.*

Let us consider an integral representation for the scattered electromagnetic field. With the fundamental solution $\Phi(x,y)$ as introduced in Section 2.1 we use the *magnetic dipole operator*

$$(Ma)(x) := 2\int_{\partial D} \nu(x) \times \text{curl}_x \left\{a(y)\Phi(x,y)\right\}\ ds(y), \quad x \in \partial D \tag{3.4}$$

and the *electric dipole operator*

$$(Nb)(x) := 2\nu(x) \times \text{curl curl} \int_{\partial D} \nu(y) \times b(y)\ \Phi(x,y)\ ds(y), \quad x \in \partial D. \tag{3.5}$$

Further we define the *projection operator* P by

$$(Pb)(x) := (\nu(x) \times b(x)) \times \nu(x), \quad x \in \partial D. \tag{3.6}$$

The *combined magnetic and electric dipole potential*

$$\begin{aligned}(P_E a)(x) &= \text{curl} \int_{\partial D} a(y)\Phi(x,y)ds(y) \\ &\quad + i \text{ curl curl} \int_{\partial D} \nu(y) \times (S_0^2 a)(y)\Phi(x,y)ds(y), \end{aligned} \tag{3.7}$$

$$(P_H a)(x) = \frac{1}{i\kappa}\text{curl } E^s(x) \tag{3.8}$$

with density $a \in T_d^{0,\alpha}(\partial D) := \{a \in C^{0,\alpha}(\partial D), \text{Div } a \in C^{0,\alpha}(\partial D), \nu \cdot a = 0\}$ and S_0 given by (2.4) in the case $\kappa = 0$ solves the Maxwell scattering problem provided the density a solves the integral equation $a + Ma + \ i\, NPS_0^2 a = -2\nu \times E^i$. Colton and Kress show in [3] that the inverse operator $(I + M + \ iNPS_0^2)^{-1}$ exists and is bounded in $T_d^{0,\alpha}(\partial D)$. Thus we obtain the representation

$$E^s = -2\ P_E\ (I + M + \ iNPS_0^2)^{-1}\nu \times E^i \tag{3.9}$$

for the scattered field from a perfect conductor. The far field pattern of the potential $P_E a$ is given by the function

$$\begin{aligned}(P_E^\infty a)(\hat{x}) &:= \frac{i\kappa}{4\pi}\ \hat{x} \times \int_{\partial D} a(y)e^{-i\kappa\hat{x}\cdot y}\ ds(y) \\ &\quad + i\ \frac{\kappa^2}{4\pi}\ \hat{x} \times \int_{\partial D} \Big(\nu(y) \times (S_0^2 a)(y)\Big) \times \hat{x}\ e^{-i\kappa\hat{x}\cdot y}\ ds(y), \hat{x} \in \Omega.\end{aligned} \tag{3.10}$$

3.2 A mixed reciprocity relation.

In the following D is a perfect conductor. Let $p \in I\!R^3$ be a constant vector. The electromagnetic field of an *electric dipole* with polarization p is given by

$$E^i_{edp}(x,z,p) := \frac{-1}{i\kappa}\text{ curl }_y\text{curl }_y\Big(p\Phi(x,z)\Big), \quad H^i_{edp}(x,z,p) := \text{curl }_y\Big(p\Phi(x,z)\Big) \tag{3.11}$$

for $x \neq z$. We denote the scattered electromagnetic field for an incident electric dipole with source point z and polarization p by $E^s_{edp}(.,z,p), H^s_{edp}(.,z,p)$, the corresponding far field pattern by $E^\infty_{edp}(.,z,p), H^\infty_{edp}(.,z,p)$. For the total field, i.e. the sum of incident and scattered field, we use $E_{edp}(.,z,p)$, $H_{edp}(.,z,p)$. The total field for plane waves is denoted by $E_{pl}(.,d,q), H_{pl}(.,d,q)$.

Theorem 5 **(Mixed electromagnetic reciprocity.)** *For scattering by a perfect conductor we have*

$$q \cdot E^\infty_{edp}(\hat{x},z,p) = \gamma\ p \cdot E^s_{pl}(z,-\hat{x},q) \tag{3.12}$$

for $\hat{x} \in \Omega,\ z \in I\!R^M \setminus \overline{D}$ *and* $p,q \in \Omega,$ *where* $\gamma = \frac{1}{4\pi}$.

Proof. Using Green's Vector Theorem for electromagnetic plane waves we obtain the equation

$$\begin{aligned} 0 &= \operatorname{curl} \int_{\partial D} \nu(y) \times E^i_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y) \\ &\quad - \frac{1}{i\kappa} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H^i_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y), \quad z \in I\!R^3 \setminus \overline{D}. \end{aligned} \tag{3.13}$$

We add (3.13) to the representation formula (Stratton-Chu formula)

$$\begin{aligned} E^s_{pl}(z, -\hat{x}, q) &= \operatorname{curl} \int_{\partial D} \nu(y) \times E^s_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y) \\ &\quad - \frac{1}{i\kappa} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H^s_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y), \quad z \in I\!R^3 \setminus \overline{D} \end{aligned} \tag{3.14}$$

to obtain

$$\begin{aligned} E^s_{pl}(z, -\hat{x}, q) &= \operatorname{curl} \int_{\partial D} \nu(y) \times E_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y) \\ &\quad - \frac{1}{i\kappa} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y) \\ &= - \frac{1}{i\kappa} \operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times H_{pl}(y, -\hat{x}, q)\Phi(z, y) ds(y), \quad z \in I\!R^3 \setminus \overline{D} \end{aligned} \tag{3.15}$$

where for the last equality we used the boundary condition for a perfect conductor. By elementary calculations we may verify the equality

$$p \cdot \operatorname{curl}_z \operatorname{curl}_z (a(y)\Phi(y, z)) = a(y) \cdot \operatorname{curl}_z \operatorname{curl}_z (p\Phi(y, z)). \tag{3.16}$$

Using (3.16) we obtain from (3.15)

$$\begin{aligned} p \cdot E^s_{pl}(z, -\hat{x}, q) &= \int_{\partial D} \nu(y) \times H_{pl}(y, -\hat{x}, q) \cdot E^i_{edp}(y, z, p)\, ds(y), \\ &= -\int_{\partial D} \nu(y) \times E^i_{edp}(y, z, p) \cdot H_{pl}(y, -\hat{x}, q)\, ds(y) \end{aligned} \tag{3.17}$$

for $z \in I\!R^3 \setminus \overline{D}$ and $p, q \in \Omega$.

From the Stratton-Chu formula for the far field patterns and the definition of the electromagnetic plane waves we obtain

$$\begin{aligned} q \cdot E^\infty_{edp}(\hat{x}, z, p) &= \gamma \int_{\partial D} \Big\{ \nu(y) \times E^s_{edp}(y, z, p) \cdot H^i_{pl}(y, -\hat{x}, q) \\ &\quad + \nu(y) \times H^s_{edp}(y, z, p) \cdot E^i_{pl}(y, -\hat{x}, q) \Big\}\, ds(y) \end{aligned} \tag{3.18}$$

for $\hat{x} \in \Omega$ and $z \in I\!R^3 \setminus \overline{D}$. Analogously to (3.13) we obtain the formula

$$\begin{aligned} 0 &= i\kappa \int_{\partial D} \Big\{ \nu(y) \times E^s_{edp}(y, z, p) \cdot H^s_{pl}(y, -\hat{x}, q) \\ &\quad + \nu(y) \times H^s_{edp}(y, z, p) \cdot E^s_{pl}(y, -\hat{x}, q) \Big\}\, ds(y) \end{aligned} \tag{3.19}$$

from Green's vector formula and the Maxwell equations applied to the scattered electromagnetic fields. We multiply (3.19) by $\gamma/i\kappa$ and add it to (3.18) to get

$$\begin{aligned} q \cdot E^{\infty}_{edp}(\hat{x}, z, p) &= \gamma \int_{\partial D} \Big\{ \nu(y) \times E^{s}_{edp}(y, z, p) \cdot H_{pl}(y, -\hat{x}, q) \\ &\quad + \nu(y) \times H^{s}_{edp}(y, z, p) \cdot E_{pl}(y, -\hat{x}, q) \Big\} \, ds(y) \\ &= \gamma \int_{\partial D} \nu(y) \times E^{s}_{edp}(y, z, p) \cdot H_{pl}(y, -\hat{x}, q) ds(y) \qquad (3.20) \end{aligned}$$

for $z \in I\!R^3 \setminus \overline{D}$ and $p, q \in \Omega$ where we used the boundary condition for $E_{pl}(., -\hat{x}, q)$. Now from (3.17) and (3.20) using the boundary condition for $E_{edp}(., z, p)$ we obtain the statement of the theorem. □

3.3 Incident dipoles near the boundary.

We will use this section to investigate the electromagnetic field for scattering of incident electric dipoles by a perfectly conducting obstacle D when the source point z approaches the boundary ∂D of the obstacle. We recall that the far field pattern for scattering of the dipole $E^{i}_{edp}(., z, p)$ is denoted by $E^{\infty}_{edp}(., z.p)$. The far field pattern of the function $-E^{i}_{edp}(., z, p)$ is given by the function $E^{\infty}_{0}(\hat{x}, z, p) := -i\kappa \hat{x} \times (p \times \hat{x})\ e^{-i\kappa \hat{x} \cdot z}$, $\hat{x} \in \Omega$.

THEOREM 6 *Let $p \in \Omega$ be a fixed polarization. For $z \in D$ we have*

$$E^{\infty}_{edp}(., z, p) = E^{\infty}_{0}(., z, p) \qquad (3.21)$$

and for $z \notin \overline{D}$

$$E^{\infty}_{edp}(., z, p) \neq E^{\infty}_{0}(., z, p) \qquad (3.22)$$

for all $p \in \Omega$. Let further $z_0 \in \partial D$ and $\nu(z_0) \cdot p = 0$, i.e. the polarization of the electric dipole $E^{i}_{edp}(., z, p)$ is in the tangential plane of ∂D in z_0. Then we have

$$\Big|\Big| E^{\infty}_{edp}(., z_0 + h\nu(z_0), p) - E^{\infty}_{edp}(., z_0 - h\nu(z_0), p) \Big|\Big|_{C(\Omega)} \to 0, h \to 0. \qquad (3.23)$$

If $p = p(z_0)$ is a continuous tangential function on ∂D, the limit (3.23) is uniform for $z_0 \in \partial D$.

Proof. We will only present the proof of (3.23), the proofs of the other parts of the theorem are parallel to the proofs of Theorem 2 and Theorem 3. From Section 3.1 we obtain for the far field pattern $E^{\infty}_{edp}(., z, p)$ the representation

$$\begin{aligned} E^{\infty}_{edp}(., z, p) &= -\, 2\ P^{\infty}_{E}\ (I + M +\ iNPS^{2}_{0})^{-1}\ \ \nu \times E^{i}_{edp}(., z, p) \\ &= -2\ P^{\infty}_{E}\ \nu \times E^{i}_{edp}(., z, p) \qquad (3.24) \\ &\quad + 2P^{\infty}_{E}\ (I + M +\ iNPS^{2}_{0})^{-1}(M +\ iNPS^{2}_{0})\ \ \nu \times E^{i}_{edp}(., z, p). \end{aligned}$$

We have to investigate the behaviour of $f(.,z+h\nu(z)) - f(.,z-h\nu(z))$ for the function

$$f(.,z) := \int_{\partial D} k(.,y)\, \nu(y) \times \text{curl}\,_y \text{curl}\,_y \Big(p\Phi(y,z)\Big)\, ds(y), \tag{3.25}$$

where $k(.,y)$ is a part of the kernels of P_E^∞, M or S_0. This is done in Section 4.2. It is shown that in $L^p(\partial D)$ for $p \in (1,2)$ we have

$$f(.,z_0+h\nu(z_0)) - f(.,z_0-h\nu(z_0)) \to 0,\ h \to 0$$

and that for a continuous tangential field $p = p(z)$ this is uniform for $z_0 \in \partial D$. Lemma 11 of Section 4.2 also states that the operators M, S_0, P and N are bounded and that $(I + M + iNPS_0^2)$ is continuously invertible in $L^p(\partial D)$. Clearly P_E^∞ maps $L^p(\partial D)$ continuously into $C(\Omega)$. Together this yields the convergence (3.23) and ends the proof. □

3.4 The point-source method for inverse electromagnetic scattering from a perfect conductor.

In this section we describe a *point-source method* to solve the inverse electromagnetic scattering problem from a perfect conductor.

3.4.1 An approximation procedure.

In the present section we show how to obtain on some exposed set $\mathcal{G}$ an approximation for the function $q \cdot E_{edp}^\infty(\hat{x},z,p),\ z \in \mathcal{G},\ p \in \Omega$ from the far field pattern $E_{pl}^\infty(.,-\hat{x},q)$ for an incident plane wave with direction $-\hat{x}$ and polarization q. Analogously to Section 2.4.1 we could try to approximate the function $\Psi_0^i(x) := E_{edp}^i(x,0,p_0)$ by a superposition of plane waves $E_{pl}^i(x,\tilde{d},p_0), \tilde{d} \in \Omega$ to obtain an approximation for the corresponding far field pattern of the scattered field. But then the approximation density would be linked to the polarization p_0 and we would have to compute an approximation density for each $p_0 \in \Omega$. The following proceeding shows how this can be avoided.

Assume that D satisfies the conditions described in Section 2.4.1 and let G_0 be defined by (2.38). For $\mu = 2$ we use Lemma 4 to approximate $\Psi_0^i(x) := \Phi(x,0)$ by a Herglotz wave function (2.39). Then by (2.42) we obtain on the set $G(M,z)$ an approximation of

$$E_{edp}^i(x,z,p) \;=\; \text{curl}\,_x \text{curl}\,_x \Big(p\Phi(x,z)\Big) \;=\; (p\cdot\nabla_x)\nabla_x \Phi(x,z) \tag{3.26}$$

by the function

$$\begin{aligned}(p\cdot\nabla_x)\nabla_x \int_\Omega e^{i\kappa x\cdot d}\, g(d,M,z)\, ds(d) \;&=\; \int_\Omega (p\cdot\nabla_x)\nabla_x e^{i\kappa x\cdot d}\, g(d,M,z)\, ds(d)\\ &=\; \int_\Omega E_{pl}^i(x,d,p)\, g(d,M,z)\, ds(d) \qquad (3.27)\end{aligned}$$

with g given by (2.43). This yields by the same arguments as presented in Section 2.4.1 on the set $\mathcal{G}$ exposed by M an approximation

$$q \cdot \int_{\Omega} E_{pl}^{\infty}(\hat{x}, d, p)\ g(d, M, z)\ ds(d) \tag{3.28}$$

for $q \cdot E_{edp}^{\infty}(\hat{x}, z, p)$. Using the reciprocity relation for scattering of incident electromagnetic plane waves we transform (3.28) into

$$\int_{\Omega} p \cdot E_{pl}^{\infty}(d, -\hat{x}, q)\ g(-d, M, z)\ ds(d). \tag{3.29}$$

3.4.2 The PSM for the inverse perfect-conductor scattering problem.

From either the mixed reciprocity-relation (3.12) of Theorem 5 and the boundary condition or from the continuity property (3.23) of Theorem 6 we obtain that, if for $p \in \Omega$ we have $p \cdot \nu(z_0) = 0$, i.e. p is in the plane tangential to ∂D in the point z_0, the function

$$\delta^E(\hat{x}, z) := |q \cdot E_{edp}^{\infty}(., z, p) - q \cdot E_0^{\infty}(., z, p)| \tag{3.30}$$

tends to zero if z tends to $z_0 \in \partial D$ for all $q \in \Omega$. From the approximation procedure of Section 3.4.1 for $D \subset G(z, M)$ we obtain an approximation

$$V^{\infty}(\hat{x}, M, z, p, q) := \int_{\Omega} p \cdot E_{pl}^{\infty}(d, -\hat{x}, q)\ g(-d, M, z)\ ds(d) \tag{3.31}$$

for $q \cdot E_{edp}^{\infty}(\hat{x}, z, p)$, thus the function

$$\alpha^E(z, M, p, q) := |V^{\infty}(\hat{x}, M, z, p, q) - q \cdot E_0^{\infty}(., z, p)| \tag{3.32}$$

approximates $\delta^E(\hat{x}, z)$ on the set $\mathcal{G}$ exposed by M.

The PSM searches an approximation Λ for a part Δ of the unknown boundary ∂D as a set of points where the function $\alpha^E(z, M(z), p(z), q)$ is close to zero and where $p(z) \cdot \nu_\Lambda(z), z \in \Lambda$ is small. Here ν_Λ denotes a unit normal vector of the surface Λ. The investigation of effective implementations of the search is beyond the range of this work.

4 Properties of potentials with L^p-kernels.

Regularity properties of acoustic and electromagnetic potentials on surfaces have been investigated and used extensively in direct and inverse scattering theory. Here we investigate jump-relations or continuity properties, respectively, of potentials with *L^p-kernels*. In contrast to the classical jump-relations instead of a potential with density $\varphi(y)$ we we have a potential with a density-kernel $k(x, y)$ and instead of the behaviour of the complex number $f(z) := \int_{\partial D} \varphi(y)\Psi(z,y)ds(y)$ for $z \to \partial D$ we investigate the behaviour of the function $f(x,z) := \int_{\partial D} k(x,y)\Psi(z,y)ds(y)$, $x \in \partial D$, for $z \to \partial D$ where $f(.,z)$ is element of an appropriate function space. Since in the sections 2.3 and 3.3 we need $k(x,y)$ to be *weakly singular* we will have to work with L^p-kernels.

4.1 Acoustic potentials.

Let X_p for $p \in (1,2)$ be the completion of the space of continuous functions $k : \partial D \times \partial D \to \mathbb{C}$ with respect to the norm

$$||k||_{X_p}^p := \sup_{y \in \partial D} \int_{\partial D} |k(x,y)|^p ds(x) \quad < \quad \infty. \tag{4.1}$$

The set of continuous kernels k on $\partial D \times \partial D$ is dense in X_p. We consider integral operators $S_h : X_p \to L^p(\partial D)$ defined by

$$(S_h k)(x) := \int_{\partial D} k(x,y)\Phi(y,z_h)ds(y), \quad x \in \partial D, \tag{4.2}$$

where $z_h := z_0 + h\nu(z_0)$ and $h \in [0,h_0)$ for some $h_0 > 0$.

THEOREM 7 *For h_0 sufficiently small the operators $S_h : X_p \to L^p(\partial D)$ are well defined for $h \in [0,h_0)$. For fixed $k \in X$ we have the convergence*

$$S_h k \to A_0 k, \quad h \to 0 \tag{4.3}$$

in $L^p(\partial D)$. The convergence is uniform for $z_0 \in \partial D$.

Proof. For continuous kernels k clearly the mapping is well defined and the convergence (4.3) can be seen from the classical proofs of the jump relations as worked out in [2], Theorem 2.12 and Theorem 2.7.

We compute using the Hölder inequality for p and q with $1/p + 1/q = 1$ for continuous kernels k the estimate

$$\begin{aligned} |\Big(S_h k\Big)(x)|^p &= |\int_{\partial D} k(x,y)\Phi(y,z_h)ds(y)|^p \\ &= \Big(\int_{\partial D} |k(x,y)||\Phi(y,z_h)|^{1/p}|\Phi(y,z_h)|^{1/q}ds(y)\Big)^p \\ &\leq \Big(\int_{\partial D} |k(x,y)|^p|\Phi(y,z_h)|ds(y)\Big)\Big(\int_{\partial D}|\Phi(y,z_h)|\Big)^{p/q}. \end{aligned} \tag{4.4}$$

We integrate (4.4) with respect to $x \in \partial D$, exchange the integration with respect to x and y and use (4.1) to obtain the estimate

$$||S_h k||_{L^p(\partial D)} \leq c \, ||k||_{X_p} \tag{4.5}$$

with the constant $c := \sup_{h\in[0,h_0)} \int_{\partial D} |\Phi(y,z_h)|\ ds(y)$. Since X_p is the completion of the space of continuous kernels k, by standard arguments we now obtain that S_h is well defined on X_p. The convergence (4.3) is a consequence of (4.5) and the convergence for continuous kernels due to the Banach-Steinhaus Theorem. □

We now investigate the normal derivative of the double-layer potential. For $k \in X_p$ we define

$$(V_h k)(x) := \Big(\int_{\partial D} k(x,y) \nu(z_0) \cdot \operatorname{grad}_z \frac{\partial \Phi(z,y)}{\partial \nu(y)} ds(y) \Big) \Big|_{z=z_0+h\nu(z_0)},\ \ x \in \partial D \quad (4.6)$$

for $\pm h \in (0,h_0)$ with some positive constant h_0.

THEOREM 8 *For h_0 sufficiently small the operators $V_h : X_p \to L^p(\partial D)$ are well defined for $\pm h \in (0,h_0)$. For $k \in X_p$ we have the convergence*

$$||\Big(V_h - V_{-h}\Big)k||_{L^p(\partial D)} \to 0,\ \ h \to 0. \quad (4.7)$$

The convergence is uniform for $z_0 \in \partial D$.

Proof. For continuous kernels V_h is well defined and the convergence (4.7) can be obtained from the proofs of Theorem 2.21 and Theorem 2.13 of [2]. For fixed $\pm h \in (0,h_0)$ the function

$$\tilde{\Phi}(y,h) := \nu(z_0) \cdot \operatorname{grad}_z \frac{\partial \Phi(y,z)}{\partial \nu(y)} \Big|_{z=z_0+h\nu(z_0)} \quad (4.8)$$

is continuous in y. Thus using the same estimate as in (4.4) with Φ replaced by $\tilde{\Phi}$ we obtain the well-posedness of $V_h : X_p \to L^p(\partial D)$. To investigate the convergence (4.7) we proceed as in (4.4) with Φ replaced by

$$\tilde{\Phi}(y,z_0,h) := \nu(z_0) \cdot \operatorname{grad}_z \frac{\partial \Phi(y,z)}{\partial \nu(y)} \Big|_{z=z_0+h\nu(z_0)} - \nu(z_0) \cdot \operatorname{grad}_z \frac{\partial \Phi(y,z)}{\partial \nu(y)} \Big|_{z=z_0-h\nu(z_0)} \quad (4.9)$$

to obtain

$$||\Big(V_h - V_{-h}\Big)k||_{L^p(\partial D)} \leq\ c\ ||k||_{X_p} \quad (4.10)$$

with $c := \sup_{\pm h\in(0,h_0)} \int_{\partial D} |\tilde{\Phi}(y,z_0,h)|\ ds(y)$. The fact $c < \infty$ for sufficiently small h_0 can be obtained directly from the proofs of the quoted Theorems 2.21 and 2.13 of [2] where the kernel $\tilde{\Phi}$ is estimated. Now using the Banach-Steinhaus Theorem the convergence (4.7) follows from the convergence for continuous kernels and (4.10). □

LEMMA 9 *The operators K, S, S_0 and T are bounded in $L^p(\partial D)$. The operators $I + K - iS$ and $I - K' - iTS_0^2$ are continuously invertible in $L^p(\partial D)$.*

Proof. The boundedness of K, S and S_0 can be easily shown using (4.4). The boundedness of T in $L^p(\partial D)$ is a consequence of the Theorem of Calderon and Zygmund for singular integral operators (see [7]). By standard arguments the operators $K - iS$ and $K' + iTS_0^2$ can be seen to be the limit of a sequence of compact operators and thus compact in $L^p(\partial D)$. The injectivity of $I + K - iS$ and $I - K' - iTS_0^2$ in $L^p(\partial D)$ can be obtained with the help of the Fredholm Alternative in the same way than their injectivity in $L^2(\partial D)$ (see the proof of Theorem 3.20 of [3]). Hence by the Riesz-Fredholm theory for compact operators both operators are bijective and have bounded inverse. □

4.2 Electromagnetic potentials.

Now we investigate jump-relations for *electromagnetic* potentials with L^p-kernels $k \in X_p$. For $z_0 \in \partial D$ and $p \in I\!R^3$ we consider integral operators $A_h : X_p \to L^p(\partial D)$ defined by

$$(A_h k)(x) := \int_{\partial D} \nu(y) \times \text{curl}_z \text{curl}_z \Big(k(x,y) p \Phi(y,z) \Big)\Big|_{z = z_0 + h\nu(z_0)} ds(y) \tag{4.11}$$

with $\pm h \in (0, h_0)$ for sufficiently small h_0.

THEOREM 10 *Let $k(x,y)$ be a continuous kernel and $p \cdot \nu(z_0) = 0$. Then for the potential A_h we have*

$$||A_h(.) - A_{-h}(.)||_{C(\partial D)} \to 0, \quad h \to 0. \tag{4.12}$$

For $k \in X_p$ and $p \cdot \nu(z_0) = 0$ we have

$$||A_h(.) - A_{-h}(.)||_{L^p(\partial D)} \to 0, \quad h \to 0. \tag{4.13}$$

If $p = p(z_0)$ is a continuous function with $p(z_0) \cdot \nu(z_0) = 0$ for all $z_0 \in \partial D$, both the convergences (4.12) and (4.13) are uniform for $z_0 \in \partial D$.

Proof. As in [2], Theorem 2.21, or in Lemma 2.1 of [4] it is sufficient to carry out the proof for $\kappa = 0$. Let us use $z_h := z + h\nu(z)$. We calculate

$$(A_h k)(x) = \int_{\partial D} \left\{ 3 \frac{(z_h - y) \cdot k(x,y) p}{|z_h - y|^5} \, \nu(y) \times (z_h - y) - \frac{\nu(y) \times k(x,y) p}{|z_h - y|^3} \right\} ds(y). \tag{4.14}$$

By Gauss' theorem we obtain

$$\begin{aligned} 0 &= \int_{\partial D} \nu(y) \times \text{grad}_y \text{div}_y \Big(k(x,z_0) \Phi(z_h, y) \Big) ds(y) \qquad (4.15) \\ &= \int_{\partial D} \left\{ 3 \frac{(z_h - y) \cdot k(x,z_0)}{|z_h - y|^5} \, \nu(y) \times (z_h - y) - \frac{\nu(y) \times k(x,z_0) p}{|z_h - y|^3} \right\} ds(y) \end{aligned}$$

and thus

$$(A_h k)(x) = \int_{\partial D} \left\{ 3 \frac{(z_h - y)\cdot(k(x,y)-k(x,z_0))p}{|z_h - y|^5}\, \nu(y)\times(z_h - y) - \frac{\nu(y)\times(k(x,y)-k(x,z_0)p)}{|z_h - y|^3} \right\} ds(y). \quad (4.16)$$

We use the equality

$$\begin{aligned} &(z_h - y)\cdot(k(x,y)-k(x,z_0))p\,\nu(y)\times(z_h - y) \\ &\quad -(z_{-h} - y)\cdot(k(x,y)-k(x,z_0))p\,\nu(y)\times(z_{-h} - y) \quad (4.17) \\ &= h\Big[\nu(z_0)\cdot(k(x,y)-k(x,z_0))p\,\nu(y)\times(z_0 - y) \\ &\quad +(z_0 - y)\cdot(k(x,y)-k(x,z_0))p\,\nu(y)\times\nu(z_0)\Big] \quad (4.18) \\ &= h\,(k(x,y)-k(x,z_0))\,\Big[p\cdot(z_0 - y)\,(\nu(y)-\nu(z_0))\times\nu(z_0)\Big], \end{aligned}$$

where in the last step we used $\nu(z_0)\cdot p = 0$, and proceed as in the proof of Theorem 2.21 of [2] to obtain (4.12). For $p(z_0)\cdot\nu(z_0) = 0$ the convergence is uniform for $z_0 \in \partial D$. Let us now come to the proof of (4.13). For fixed $h \neq 0$ the kernel

$$\tilde{\Phi}(y, z_0, h) := \nu(y)\times \text{curl}_y \text{curl}_y \Big(p\Phi(y, z_0 + h\nu(z_0))\Big) \quad (4.19)$$

is continuous in y and thus the operators $A_h : X_p \to L^p(\partial D)$ are well defined by the arguments used in (4.4). We consider the kernel

$$\begin{aligned} \tilde{\Phi}(y, z_0, h) &:= \nu(y)\times \text{curl}_y \text{curl}_y \Big(p\Phi(y, z_0 + h\nu(z_0))\Big) \\ &\quad -\nu(y)\times \text{curl}_y \text{curl}_y \Big(p\Phi(y, z_0 - h\nu(z_0))\Big) \quad (4.20) \end{aligned}$$

of $A_h - A_{-h}$. In the case $\kappa = 0$ we compute

$$\begin{aligned} \tilde{\Phi}(y, z_0, h) &= 3\,(z_0 - y)\cdot p\,\left\{ \frac{\nu(y)\times(z_h - y)}{|z_h - y|^5} - \frac{\nu(y)\times(z_{-h} - y)}{|z_{-h} - y|^5} \right\} \\ &\quad -(\nu(y)-\nu(z_0))\times p\,\left\{ \frac{1}{|z_h - y|^3} - \frac{1}{|z_{-h} - y|^3} \right\} \quad (4.21) \end{aligned}$$

and we obtain $c := \sup_{\pm h \in (0,h_0)} \int_{\partial D} |\tilde{\Phi}(y, z_0, h)| ds(y) < \infty$ in the same way than in the acoustic cases. We can use (4.4) for $A_h - A_{-h}$ to estimate its norm $X_p \to L^p(\partial D)$ uniformly for $\pm h \in (0, h_0)$. Then from (4.12) with the help of the Banach-Steinhaus Theorem we obtain (4.13). □

In the same way than Lemma 9 we obtain

LEMMA 11 *The operators M and NPS_0 are bounded in $L^p(\partial D)$. The operator $I + M + iNPS_0^2$ is continuously invertible in L^p.* □

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