**Lagrangian Intersection Floer Homology** (sketch)

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Recall that a *symplectic* $2n$-manifold $(M,\omega)$ is a smooth manifold with a closed nondegenerate 2-form, i.e. $\omega(x,y) = -\omega(y,x)$ and $d\omega = 0$ and $\omega(x,y) = 0 \;\; \forall\, y \in T_pM \;\Rightarrow\; x = 0$.

An *isotropic* submanifold $N$ (with embedding $i : N \hookrightarrow M$) is one with $\omega|_N \equiv i^*\omega = 0$, i.e. it has an isotropic tangent space $TL \subseteq (TL)^\perp \subset TM$.

A *Lagrangian* submanifold $L$ is an isotropic submanifold of maximal dimension, i.e. $TL = (TL)^\perp$, i.e. $\omega|_L = 0$ and $\dim L = \frac{1}{2}\dim M = n$.

Two submanifolds $N_0$ and $N_1$ are *transverse* (or intersect *transversally*), written $N_0 \pitchfork N_1$, if $T_pM = T_p(N_0) + T_p(N_1)$ for all $p \in N_0 \cap N_1$.

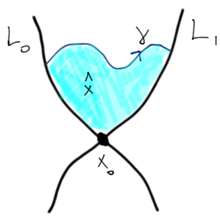
An *exact* symplectic manifold $(M,\omega)$ is one with $\omega = d\alpha$ for some 1-form. Then $\alpha|_L$ is closed on Lagrangian submanifolds $L$, since $0 = \omega|_L = (d\alpha)|_L = d(\alpha|_L)$. And an *exact* Lagrangian submanifold is one with $\alpha|_L$ exact, i.e. $\alpha|_L = df$ for some function $f : L \to \mathbb{R}$.

A *[time-dependent] almost complex structure* $J : [0,1] \times M \to \mathrm{End}(TM)$ is a smooth bilinear map with $J_t^2 = -\mathbb{1}$ (when considered as an endomorphism of $TM$). This structure is $\omega$-*compatible* if $g_J(\cdot,\cdot) \equiv \omega(\cdot, J\cdot)$ is a Riemannian metric (varying $t \in [0,1]$ implicitly), i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$ and $g_J(v,v) > 0$ ($g_J$ is *positive definite*).

*Motivation* (Morse Theory for a path-space action functional):

Consider a symplectic $(M,\omega)$ with transverse Lagrangian submanifolds $L_0, L_1$ and a time-independent $\omega$-compatible almost complex structure $J$. From $\mathcal{P}(L_0,L_1) = \{\gamma : [0,1] \to M \mid \gamma(0) \in L_0\,,\, \gamma(1) \in L_1\}$, the space of paths, pass to the universal cover $\tilde{\mathcal{P}}$ based at $x_0 \in L_0 \cap L_1$ (considered as a constant path), $\tilde{\mathcal{P}} = \{(\gamma, [\hat{x}]) \mid [\hat{x}] = \text{homotopy } \gamma \simeq x_0\}$. The *action functional* is $\mathcal{A} : \tilde{\mathcal{P}} \to \mathbb{R}$, $(\gamma,[\hat{x}]) \mapsto \int_0^1\int_0^1 \hat{x}^*\omega$. Noting that $T_\gamma\mathcal{P}$ is the set of sections $\xi$ of $\gamma^*(TM)$, i.e. the set of vector fields $\xi$ along $\gamma$, i.e. $T_\gamma\mathcal{P} = \{\xi(t) \in T_{\gamma(t)}M \;\;\forall\, t \in [0,1]\,,\, \xi(i) \in T_{\gamma(i)}L_i\}$, there is an $L_2$-inner product on $T\mathcal{P}$ given by $\langle \xi, \eta\rangle = \int_0^1 g_J(\xi(t), \eta(t))\,dt$. This gives $d\mathcal{A}_\gamma(\xi) = \int_0^1 \omega(\dot\gamma, \xi)\,dt = \int_0^1 g_J(J\dot\gamma, \xi)\,dt = \langle J\dot\gamma, \xi\rangle$. In other words, the critical points are the lifts to $\tilde{\mathcal{P}}$ of constant paths $\dot\gamma = 0$, and the gradient trajectories $v : \mathbb{R} \to \tilde{\mathcal{P}}$ of $\mathcal{A}$ are $J$-holomorphic strips $v = J\dot\gamma$. These are maps $u : \mathbb{R} \times [0,1] \to M$, $u(s,t) = v(s)(t)$, satisfying $\bar\partial_J u \equiv \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0$ and $u(s,i) \in L_i$.

This is an attempt at an $\infty$-dimensional Morse Theory for $\mathcal{A}$, but the associated differential complex is not always defined (i.e. $\partial^2 \neq 0$).



Let $(M, \omega)$ be symplectic, with compact transverse Lagrangian submanifolds $L_0, L_1$, and fix a time-dependent $\omega$-compatible almost complex structure $J$.

*Remark*: It is very difficult to get rid of the compactness hypothesis on $L_0, L_1$ to define Floer homology $HF(L_0, L_1)$, but not so much for the transversality hypothesis.

For a Principal Ideal Domain $R$ (usually $\mathbb{Q}, \mathbb{C}, \mathbb{Z}_2, \mathbb{Z}$) we can form a ring of infinite sums with a formal variable, called the *Novikov ring* $\Lambda = \{\sum a_i T^{\lambda_i} \mid a_i \in R \,, \lambda_i \in \mathbb{R}_{\geq 0} \,, \varinjlim \lambda_i = \infty\}$. The *Floer complex* $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ is the free $\Lambda$-module generated by $L_0 \cap L_1$. Note that there are only finitely many generators since $L_0, L_1$ are compact and transverse.

$J$-holomorphic strips are maps $u : \mathbb{R} \times [0,1] \to M$ satisfying

(1) $\bar{\partial}_J u \equiv \frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = 0$

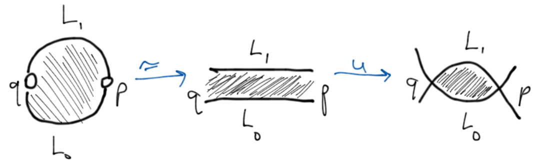
(2) $u(s, i) \in L_i \mid i = 0, 1$

(3) $\lim_{s \to \infty} u(s,t) = p \,, \lim_{s \to -\infty} u(s,t) = q \mid p, q \in L_0 \cap L_1$.

Note that we can consider punctured disks via the biholomorphism $\mathbb{R} \times [0,1] \simeq \mathbb{D}^2 - \{\pm 1\}$ $(s + it \in \mathbb{C})$; these are called *$J$-holomorphic Whitney disks*.

The asymptotic condition (3) is equivalent to $u$ having finite *energy* $E(u) \equiv \frac{1}{2} \int_{\mathbb{R} \times [0,1]} |du|^2_{g_J} = \int \int |\frac{\partial u}{\partial s}|^2 = \int u^* \omega < \infty$, which is equivalent to *symplectic area* $\int u^* \omega \equiv \omega(u)$.

The *Moduli space* $\mathcal{M}_J(p, q, [u])$ is the set of solutions $\{u\}$ for fixed $p, q \in L_0 \cap L_1$ and fixed $J$, with the same homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$.



*Aside*: The transversality hypothesis $L_0 \pitchfork L_1$ can be dropped by introducing a time-dependent Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ with *Hamiltonian* vector field $X_t$ on $M$ ($i_{X_t} \omega = dH_t$ for all $t \in [0,1]$) such that $\phi_1(L_0) \pitchfork L_1$, where $\phi_t$ is its *flow* (family of symplectomorphisms generated by $X_t$). In this case there are perturbed Cauchy-Riemann equations with Lagrangian boundary conditions, $\frac{\partial u}{\partial s} + J_t(u)[\frac{\partial u}{\partial t} - X_t(u)] = 0$, and the intersection points of $L_0 \cap L_1$ are replaced with paths $\gamma \in \mathcal{P}(L_0, L_1)$ such that $\dot{\gamma}(t) = X_t(\gamma(t))$.

It is intended for the Floer differential $\partial$ to "count" solutions $u \in \mathcal{M}$, or actually $u \in \mathcal{M}/\mathbb{R}$ since there is a free $\mathbb{R}$-action (translation in the $s$-coordinate).

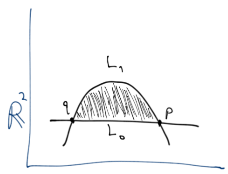
The expected dimension of $\mathcal{M}$ is $\text{ind}([u])$, the *Maslov index*, which arises from $\pi_1(\Lambda_{\text{Gr}}) \cong \mathbb{Z}$ for the Lagrangian Grassmannian):

Let $L_0, \{L_1(t)\}_{t \in [0,1]} \subset \mathbb{R}^{2n}$ be linear subspaces such that $L_1(0), L_1(1) \pitchfork L_0$. Then
$\text{ind}(L_1(t); L_0) \equiv$ number of times $L_1(t)$ is not transverse to $L_0$ (counted with signs and multiplicities).

*Example:* $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$
$L_1(t) = (e^{i\theta_1(t)}\mathbb{R}) \times \cdots \times (e^{i\theta_n(t)}\mathbb{R})$ with all $\theta_i$ sweeping through $0^o$
$\Rightarrow$ $\text{ind} = n$

For a $J$-holomorphic strip $u$, $u^*(TM)$ can be trivialized to give two paths $u|^*_{\mathbb{R} \times \{0\}}(TL_0)$ and $u|^*_{\mathbb{R} \times \{1\}}(TL_1)$ of Lagrangian submanifolds, and this trivialization can be chosen so that $TL_0$ remains constant. Then $\text{ind}(u) \equiv \text{ind}(TL_1(t); TL_0)$ where $t$ dictates movement from $p$ to $q$.

*Example*: $L_0 = \{(x, 0)\} \subset \mathbb{R}^2$
$L_1 = \{(x, x^2 - x)\} \subset \mathbb{R}^2$
$\Rightarrow$ $\text{ind}([u]) = 1$ (slope of $L_1$ equals slope of $L_0$ at $x = \frac{1}{2}$)

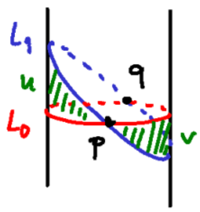


Define $\partial(p) = \sum_{q,[u]} \#[\mathcal{M}_J(p,q,[u])/\mathbb{R}]\, T^{\omega([u])} \cdot q$ , for $q \in L_0 \cap L_1$, $[u] \in \pi_2(M, L_0 \cup L_1)$ $\mid \mathrm{ind}([u]) = 1$.
We need to be clear on what $\#[\ \ ]$ actually means.

*Theorem* (Floer): If $[\omega] \cdot \pi_2 M = 0$ and $[\omega] \cdot \pi_2(M, L_i) = 0$ then $\partial$ is well-defined, $\partial^2 = 0$, and
$HF^*(L_0, L_1) \equiv H^*(CF, \partial)$ is independent of the chosen $J$
and is invariant under Hamiltonian isotopies of $L_i$.

This gives a special case of *Arnold's Conjecture*:
If $[\omega] \cdot \pi_2(M, L) = 0$ and $\psi$ is a Hamiltonian diffeomorphism such that $\psi(L) \pitchfork L$,
then $|\psi(L) \cap L| \geq \sum b_i(L)$.
The proof uses the result $HF^*(L, L) \cong H^*(L; \Lambda)$, where $HF^*(L, L) \equiv HF^*(L, \psi(L))$.

*Example*: $M = T^*(S^1) \approx \mathbb{R} \times S^1$ with $L_0 = \{(0, \theta)\}$ and $L_1 = \{(\sin\theta + \frac{1}{\sqrt{2}}, \theta)\}$
$L_0 \cap L_1 = \{p, q\} = \{(0, \frac{5\pi}{4}), (0, \frac{7\pi}{4})\}$ and region between them decomposes into disks $u, v$.
$CF(L_0, L_1) = \Lambda p \oplus \Lambda q$ and $\partial(p) = (T^{\omega(u)} - T^{\omega(v)})q$ and $\partial(q) = 0$
*Remark*: $\#[\ \ ]$ counts $\mathcal{M}$ with signs (assuming an orientation can be placed on $\mathcal{M}$).
*Remark*: Here we can assign a $\mathbb{Z}$-grading on $CF$, with $\deg(p) = 0$ and $\deg(q) = 1$,
because the Maslov index is independent of $[u]$.
If $u, v$ have equal symplectic areas then $\partial = 0$ and $HF^*(L_0, L_1) \cong H^*(S^1; \Lambda)$.
Otherwise, $HF^*(L_0, L_1) = 0$.




*Issues*:

(1) Transversality of $\bar{\partial}_J$-operator
- Places $\mathcal{M}$ into a function space framework, equips a manifold structure
- Requires the linearization $D_u \bar{\partial}_J$ to be surjective for all $u \in \mathcal{M}$

$\Rightarrow$ Solution: techniques with $J_t$ and $H_t$ (and "virtual cycles" and "Kuranishi structures")

(2) Orientation of $\mathcal{M}$
- Used for defining $\#[\ \ ]$ with signs
- Requires auxiliary data (topological conditions on $M, L_0, L_1$)

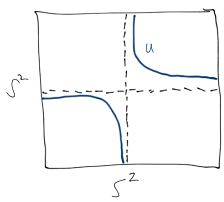
$\Rightarrow$ Solution: (1) assign $L_0$ and $L_1$ a *relatively spin structure*
i.e. $\exists\, c \in H^2(M; \mathbb{Z}_2)$ such that $c|_{L_0} = w_2(TL_0)$ and $c|_{L_1} = w_2(TL_1)$
or (2) ignore orientation by replacing $\Lambda$ with $\mathbb{Z}_2$ (and $\#[\mathcal{M}] = |\mathcal{M}| \bmod 2$)

(3) *Bubbling* phenomena
- Limiting behavior of sequences with uniformly bounded energy
- Requires some compactness condition on $\mathcal{M}$

$\Rightarrow$ Solution: explained below

Given a sequence of $J$-holomorphic strips $\{u_i\}$ with uniformly bounded energy, if $\sup ||du_i||_\infty < \infty$ then $\{u_i\}$ has a uniformly convergent subsequence (with derivatives on compact sets). If the sequence has unbounded derivatives, the resulting limit is called a *bubble*.

*Example*: By the Removable Singularity Theorem, every $u : \mathbb{C} \to M$ extends to $S^2 = \mathbb{C}P^1$.
Consider $u_i : \mathbb{CP}^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$, $u_i(z) = (z, \frac{1}{iz})$. Then $\underrightarrow{\lim}\, u_i = (z, 0)$ if $z \neq 0$,
but at $z = 0$ the derivative blows up, giving "bubbling."



There are three types of phenomena:

(1) *Sphere bubbling*, which occurs when $|du_i| \to \infty$ at an interior point of the strip.

This bubble is a holomorphic sphere which has a transverse intersection to the rest of the strip.

(2) *Disk bubbling*, which occurs when $|du_i| \to \infty$ at a boundary point of the strip.

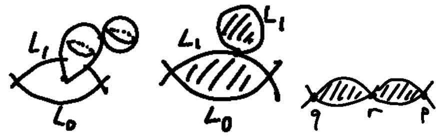
(3) *Breaking of strips*, which occurs when $|du_i| \to \infty$ at the endpoints $p, q$.

The resulting limit is a sequence of broken $J$-holomorphic strips (i.e. a strip with nodes).

Considering the strip as the Whitney disk, "breaking of strips" is bubbling at the endpoints.

We must hope there is no disk bubbling, since it prevents $\partial^2 = 0$ (an obstruction to this Floer theory). A condition of *Gromov compactness* makes $\mathcal{M}$ precompact and gives it a natural compactification, which allows bubbling and breaking of strips (the compactification $\overline{\mathcal{M}}$ contains broken strips).

*If the ambient manifold and Lagrangian submanifolds are exact (with associated 0-forms $f_i$ on $L_i$), then the first two phenomena (bubbling) do not occur! Indeed, the energy $E(u) = \int_{\mathbb{R}\times[0,1]} u^*\omega = \int_{L_1(q\to p)} \alpha|_{L_1} - \int_{L_0(q\to p)} \alpha|_{L_0} = \int_q^p df_1 - \int_q^p df_0 = f_1(p) - f_1(q) - f_0(p) - f_0(q)$ is constant (via Stokes' theorem). For disk bubbling, the energy of the disk $D \approx \mathbb{D}^2$ (with boundary in $L_1$, intersecting the limiting curve at some $r \in L_1$) is $\int_D \omega = \int_{L_1(r\to r)} \alpha = \int_r^r df_1 = 0$. For sphere bubbling, the energy of the bubble $S \approx S^2$ is $\int_S \omega = \int_{\partial S} \alpha = \int_\varnothing \alpha = 0$ (where $\alpha$ is the associated 1-form on $M$).

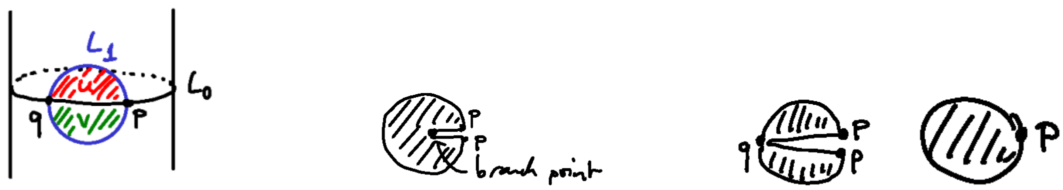



*Example*: $M = T^*(S^1)$, with $L_0, L_1$ sketched below

View $M$ as $\mathbb{R}^2$ in some neighborhood of $L_1$, to make $L_1 = S^1$ and $[-1,1] \subset L_0$.

Then $CF(L_0, L_1) = \Lambda p \oplus \Lambda q$ and $\partial(p) = \pm T^{\omega(u)} q$ and $\partial(q) = \pm T^{\omega(v)} p$ and hence $\partial^2 \neq 0$.

Look at the moduli space of index-2 strips from $p$ to $p$, which consists of holomorphic maps whose image is the unit disk minus the segment from $p$ to some $\alpha \in (-1,1)$ along the axis. Reparameterizing the strip (i.e. Whitney disk) to consider the upper half of the unit disk as our domain, there are $u_\alpha(z) = \frac{z^2+\alpha}{1+\alpha z^2}$ for $\alpha \in (-1,1)$, so $\mathcal{M}_{\mathrm{ind}(u)=2}(p,p,[u]) \approx (-1,1)$. There are two endpoints, $\alpha \to -1$ in which there is a broken strip $p \to q \to p$, and $\alpha \to 1$ in which there is a constant strip at $p$ and a disk bubble with boundary in $L_1$. This disk bubble prevents $\partial^2 = 0$ !

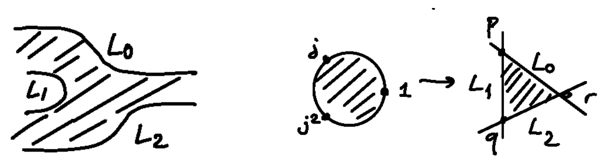



Now that the issues have been somewhat addressed, the differential $\partial$ needs to be verified, i.e. to prove that $\partial^2 = 0$. For simplicity, assume there is no bubbling phenomenon. Consider $\mathcal{M}_J(p,q,[u])/\mathbb{R}$ for $\mathrm{ind}(u) = 2$, which is expected to be a 1-manifold (since $\mathcal{M}$ is of dimension $\mathrm{ind}(u)$). This is compactified by adding once-broken strips $\bigsqcup_{r\in L_0\cap L_1, u_1\# u_2=u} (\mathcal{M}_J(p,r,[u_1])/\mathbb{R}) \times (\mathcal{M}_J(r,q,[u_2])/\mathbb{R})$. By a gluing theorem the resulting $\overline{\mathcal{M}/\mathbb{R}}$ is a manifold with boundary. Now $\partial^2$ counts all broken strips $p \to r \to q$ (the "ends" of index-2 moduli spaces), which are the contributions to the coefficient of $T^{\omega(u)} q$ in $\partial^2(p)$. But the number (including signs) of "ends" of an oriented compact 1-manifold is zero (even number of boundary components); this is the same if orientation is ignored and counting is done modulo 2.

In particular, broken strips $p \to r \to q$ give two honest index-1 strips $p \to r$ and $r \to q$, so $\partial(p)$ contains an $r$ factor and $\partial(r)$ contains a $q$ factor. Normally, $|L_0 \cap L_1| > 2$, but points $q$ and $r$ can actually be the same point (and reached from $p$ by different strips).

In the case of exact Lagrangian submanifolds, the formal variable $T$ need not appear in the definition of $\partial$ because all symplectic areas are equal. We might as well write $\partial(p) = n(p,q) \cdot q$ where $n(p,q)$ is the number of solutions in $\mathcal{M}(p,q)/\mathbb{R}$ modulo 2, and work over $\mathbb{Z}_2$-coefficients instead of $\Lambda$. Since no bubbling can occur, we have $HF^*$ ready to go.

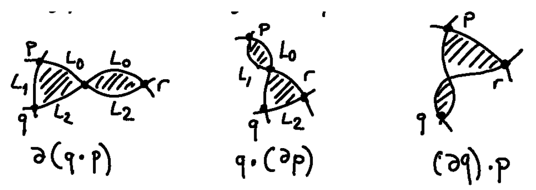
*Product Structure*: Want to define a map $\mu^2 : CF^*(L_0, L_1) \otimes CF^*(L_1, L_2) \to CF^*(L_0, L_2)$. Consider $u : \mathbb{D}^2 \to M$ where $\mathbb{D}^2$ has three marked points on its boundary, whose image is a "triangle" between $L_0, L_1, L_2$. Biholomorphically, $\mathbb{D}^2$ minus those marked points is a Riemann surface with boundary, with three strip-like ends (of finite energy). Since $\mathrm{Aut}(\mathbb{D}^2)$ acts transitively on cyclically-ordered triples of boundary points, the choice of marked points is arbitrary. Let $\mathcal{M}_J(p, q, r, [u])$ be the moduli space of such maps, and assume sufficient conditions (such as transversality and orientability).




Define $\mu^2(q, p) = q \circ p \equiv \sum_{r \in L_0 \cap L_2, \mathrm{ind}(u)=0} \#[\mathcal{M}_J(p, q, r, [u])]\ T^{\omega(u)} \cdot r$
where $\mathrm{ind}(u) = \deg(r) - [\deg(p) + \deg(q)]$ (expected dimension of $\mathcal{M}$) will be discussed later.

*Proposition*: If $[\omega] \cdot \pi_2(M, L_i) = 0$ then the aforementioned product structure satisfies the Leibniz rule with respect to $\partial$, and hence induces a product on $HF^*$. Moreover, this product is associative.

*proof*: Consider index-1 moduli spaces (triangles $L_0, L_1, L_2$ as above), and compactify it by adding limit configurations (assuming no bubbling), i.e. adding broken strips. These strips give contributions $\partial(q \circ p)$, $q \circ \partial(p)$, and $\partial(q) \circ p$. By a gluing theorem this compactification is a 1-manifold with boundary, and since the number of "ends" is zero (same argument as before) we have the Leibniz rule $\partial(q \circ p) = \pm\partial(q) \circ p \pm q \circ \partial(p)$. If $p, q$ are closed then $\partial(q \circ p) = 0$ and hence $q \circ p$ is closed. If in addition $p = \partial p'$ is exact, then $q \circ p = \pm\partial(q \circ p')$ and hence $q \circ p$ is exact. And this gives a well-defined product on $HF^*$.




*Higher-Order Operations*: There exist maps $\mu^k : CF^*(L_0, L_1) \otimes \cdots \otimes CF^*(L_{k-1}, L_k) \to CF^*(L_0, L_k)$ with a grading shift of $2 - k$ (explanation of grading will be discussed later). We set $\mu^1 = \partial$ and have $\mu^2$ being the above product (this presentation is leading to an $\mathcal{A}_\infty$-category, called the *Fukaya category*). To obtain these maps, look at $J$-holomorphic maps $u$ from disks $\mathbb{D}^2$ with $k+1$ marked points $z_0, \ldots, z_k$ (cyclically ordered and distinct) on the boundary, such that the image under $u$ is a disk between $L_0, \ldots, L_k$ with $u(z_0) = q \in L_0 \cap L_k$ and $u(z_i) = p_i \in L_{i-1} \cap L_i$. Repeating the above procedure, we obtain a moduli space $\mathcal{M}_J(p_1, \ldots, p_k, q, [u])$.

There are two ways to view this moduli space in terms of the domain $\mathcal{M}_{0,k+1}$ (the moduli space of disks with $k+1$ boundary marked points):

1) All marked points have been chosen/fixed. But then we have to quotient by the disk automorphisms and only count $\mathcal{M}/\mathrm{Aut}(\mathbb{D}^2)$.

2) Only 3 marked "input" points have been chosen, letting the other marked "output" points vary (say between the first and third input points). Then $\mathcal{M}_{0,k+1}$ is this infinite collection of disks with fixed marked points, indexed over $k-2$ of those marked points. Note that we can uniquely fix at most three marked points because $\mathrm{Aut}(\mathbb{D}^2) \cong PSL_2(\mathbb{R})$ is three-dimensional.

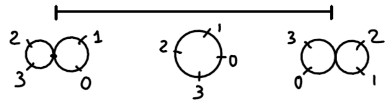
Define $\mu^k(p_k, \ldots, p_1) = \sum_{q \in L_0 \cap L_k, \mathrm{ind}(u)=0} \#[\mathcal{M}_J(p_1, \ldots, p_k, q, [u])]\ T^{\omega(u)} \cdot q$ using view (2).

*Proposition*: Assuming no bubbling, for all fixed $m \geq 1$
$\sum_{k,l \geq 1, k+l=m+1, 0 \leq j \leq l-1} (-1)^* \mu^l(p_m, \ldots, p_{j+k+1}, \mu^i(p_{j+k}, \ldots, p_{j+1}), p_j, \ldots, p_1) = 0.$
In particular, $\mu^2$ is associative up to homotopy given by $\mu^3$ (i.e. it is associative in $HF^*$).

To prove the above proposition, we consider $\mathcal{M}_{0,k+1}$, which is contractible of dimension $k-2$. It compactifies to $\overline{\mathcal{M}}_{0,k+1}$, the moduli space of "stable" genus 0 Riemann surfaces with one boundary component and $k+1$ boundary marked points (i.e. they are trees of disks attached at marked nodal points such that each component carries at least three marked points). In other words, the output points can move freely, and when one of them hits an input point, "bubbling" at the boundary occurs, inducing limit configurations (i.e. degeneration of the domain to $\partial\overline{\mathcal{M}}_{0,k+1}$). Here the domain breaks up into 2 disks (the new marked point is where the disks attach together).

*Example*: For $k=3$, $\overline{\mathcal{M}}_{0,4}$ contains the disks with a varying marked output point (between the first and third input points) and its boundary contains two limit configurations which corresponds to the output point hitting the first input point and the third input point. Thus $\overline{\mathcal{M}}_{0,4} \approx [0,1]$.

In general, $\overline{\mathcal{M}}_{0,k+1}$ is an *associahedron*.



Ignoring sphere/disk bubbling, Gromov compactness allows the ends of $\overline{\mathcal{M}}_{\mathrm{ind}(u)=1}$ to be either A) broken strips (energy buildup at one of the marked points) where the positions of the marked points do not hit each other, in which case the limit curve consists of a rigid index-1 strip plus a rigid disk with $k+1$ marked points (they attach at one of the marked points), or B) degeneration of domain to $\partial\overline{\mathcal{M}}_{0,k+1}$, in which case the limit curve consists of two disks with associated marked points.

The usual argument that the ends of 1-manifolds cancel in pairs now tells us that the sum of both types of broken configurations is zero. This is the $\mathcal{A}_\infty$-relation in the proposition. The type-A broken configurations correspond to terms involving $\mu^1$ and the type-B broken configurations correspond to terms involving only $\mu^{k\geq 2}$.

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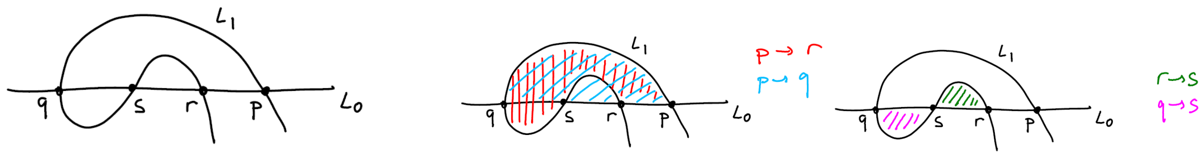
*Example*:

Consider two Lagrangian submanifolds of $\mathbb{R}^2$ that intersect transversally four times.
There are four isolated strips, $p \to r$ and $p \to q$ and $r \to s$ and $q \to s$.
The differentials are $\partial(p) = \pm q \pm r$ and $\partial(q) = \pm s$ and $\partial(r) = \pm s$.




The 1-dimensional family of index-2 strips $p \to s$ have a slit along either $\overline{qs} \in L_0$ or $\overline{sr} \in L_1$.
As the depth of the slit varies, different elements arise in $\mathcal{M}(p,s)$.
The ends are the broken strips $p \to q \to s$ and $p \to r \to s$.