Quarter Symmetric Metric Finsler Connections

on Kenmotsu and P-Kenmotsu Vector Bundles

K. L. Sai Prasad

GVP College of Engineering
Visakhapatnam, India
klsprasad@yahoo.com

Abstract

Quarter symmetric Finsler connections are defined on the total spaces of Kenmotsu and P-Kenmotsu vector bundles. In addition, torsion tensor properties of the connection are derived. It has been shown that the quarter symmetric Finsler connection on both Kenmotsu and P-Kenmotsu vector bundles is metrical Finsler connection.

Mathematics Subject Classification: 53C07

Keywords: manifolds, vector bundles, Finsler structures, Finsler connections, quarter symmetric metric connections, Kenmotsu structures, Para Kenmotsu structures

Introduction

Semi symmetric and quarter symmetric connections are known and well studied on a manifold [3,12]. Miron [6] introduced a sophisticated method for the study of Finsler Geometry of vector bundles. Using the techniques of Miron’s theory, Atanasiu [1,2] defined the semi symmetric and quarter symmetric Finsler connections on the total space of vector bundle and studied several properties. Sai Prasad [8] defined Kenmotsu and Para Kenmotsu Finsler structures on the total space of vector bundles and called the corresponding vector bundles as Kenmotsu and P-Kenmotsu vector bundles respectively.

In this paper, Quarter symmetric Finsler connections are defined on the total spaces of Kenmotsu and P-Kenmotsu vector bundles. In addition, torsion tensor properties of the connection are derived. It has been shown that the quarter symmetric Finsler connection on both Kenmotsu and P-Kenmotsu vector bundles is metric.
1. Preliminaries

Let \( V(M) = \{ VM, \pi, M \} \) be a vector bundle whose total space \( VM \) is a \((n + m)\) dimensional \( C^\infty \) manifold and base space \( M \) is an \( n \)-dimensional \( C^\infty \) manifold.

A Finsler connection \( \nabla \) on the total space \( VM \) of a vector bundle \( V(M) \) is characterised by the horizontal part \( \nabla^H \) and the vertical part \( \nabla^V \) [6].

A Finsler connection \( \nabla \) on \( VM \) is said to be quarter symmetric if its torsion tensor \( T \) satisfies:

\[
\begin{align*}
[T(X^H, Y^H)]^H & = p^H(Y^H) tX^H - p^H(X^H) tY^H \\
[T(X^V, Y^V)]^V & = p^V(Y^V) tX^V - p^V(X^V) tY^V
\end{align*}
\]  

\( \forall X, Y \in T_u(VM) \) and \( p \in T^*_u(VM) \).

A Finsler connection \( \nabla \) on \( VM \) is said to be metric if and only if

\[
\nabla^H G^H = 0, \quad \nabla^V G^V = 0
\]

If \( D \) is the metrical Finsler connection on \( VM \), then any quarter symmetric metric Finsler connection \( \nabla \) on \( VM \) is given by:

\[
\nabla_X Y = D_X Y + p(Y) tX - t(X, Y) p
\]

\( \forall X, Y \in T_u(VM) \), where \( p \) is a 1-form and \( t \) is a Finsler tensor field of type \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).

Here,

\[
't(X, Y) = G(t(X), Y) = G(X, t(Y)) \text{ and } p(x) = G(P, X).
\]

A quarter symmetric Finsler connection \( \nabla \) on \( VM \) satisfying

\[
(\nabla_X p)(Y) = (\nabla_Y p)(X) \text{ i.e.,}
(\nabla^H_X p)(Y^H) = (\nabla^H_Y p)(X^H)
(\nabla^V_X p)(Y^V) = (\nabla^V_Y p)(X^V)
\]

is called a special quarter symmetric Finsler connection on \( VM \).

The Vector bundle \( VM \) with Riemannian Finsler metric \( G \) admitting a Finsler tensor field \( \phi \) of type \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying:

\[
\begin{align*}
(i) & \quad (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0. \\
(ii) & \quad (\nabla_X \nabla_Y \eta)(Z) = [-G(X, Z) + \eta(Z)] \eta(Y) \\
& \quad + [-G(X, Y) + \eta(X) \eta(Y)] \eta(Z) \\
(iii) & \quad \eta(X) = G(X, \xi) \\
(iv) & \quad \nabla_X \xi = X - \eta(X) \xi; \quad \forall X, Y \in T_u(VM).
\end{align*}
\]

is said to possess the P-Kenmotsu Finsler structure[8]. The torsion free Finsler connection \( \nabla \) satisfying (1.5) is called P-Kenmotsu Finsler connection on \( VM \).

The Vector bundle \( VM \) with Riemannian Finsler metric \( G \) admitting a Finsler tensor field \( \phi \) of type \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), a vector field \( \xi \) and a 1-form \( \eta \) satisfying:

\[
\begin{align*}
(i) & \quad (\nabla_X \eta)(Y) = G(X, Y) - \eta(X) \eta(Y)
\end{align*}
\]
(ii) \( \eta(X) = G(X, \xi) \) and

(iii) \(( \nabla_X \eta)(Y) = \Omega(X, Y) ; \forall X, Y \in T_u(VM) \)

(1.6)
is said to possess the SP-Kenmotsu Finsler structure[8]. The torsion free Finsler connection \( \nabla \) satisfying (1.6) is called SP-Kenmotsu Finsler connection on VM.

2. Quarter symmetric Finsler connections on Kenmotsu and para Kenmotsu vector bundles

**Definition(2.1):** On the total space VM of a Kenmotsu vector bundle[8] we introduce a quarter symmetric Finsler connection \( \nabla \) by identifying p and t of (1.1) respectively with \( \eta \) and \( \phi \), where \( \eta \) is a 1 – form and \( \phi \) is an almost contact Finsler structure on VM[11] given by the Finsler tensor field of type \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), that is, by setting

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \eta(Y) \phi X - \eta(X) \phi Y,
\]

which give

\[
[T(X^H,Y^H)]^H = \eta^H(Y^H) \phi X^H - \eta^H(X^H) \phi Y^H
\]

\[
[T(X^V,Y^V)]^V = \eta^V(Y^V) \phi X^V - \eta^V(X^V) \phi Y^V
\]

(2.1)

\( \forall X, Y \in T_u(VM) \) and \( \eta \in T_u^*(VM) \).

Then, for a quarter symmetric Finsler connection \( \nabla \) on VM of Kenmotsu vector bundle, we have

(a) \([T(\phi X^H, \phi Y^H)]^H = 0 ; [T(\phi X^V, \phi Y^V)]^V = 0\)

(b) \([T(\phi^2 X^H, Y^H)]^H + [T(X^H, Y^H)]^H = -[T(X^H, Y^H)]^H
\]

\([T(\phi^2 X^V, Y^V)]^V + [T(X^V, Y^V)]^V = -[T(X^V, Y^V)]^V\)

(c) \(\phi^2 [T(X^H, Y^H)]^H + [T(X^H, Y^H)]^H = 0
\]

\(\phi^2 [T(X^V, Y^V)]^V + [T(X^V, Y^V)]^V = 0\)

(d) \(\eta^H [T(X^H, Y^H)]^H = 0 ; \eta^V [T(X^V, Y^V)]^V = 0\)

(2.2)

\( \forall X, Y \in T_u(VM) \) and \( \eta \in T_u^*(VM) \).

**Definition (2.2):** On the total space VM of a Para Kenmotsu (or) SP-Kenmotsu vector bundle[8] we introduce a quarter symmetric Finsler connection \( \nabla \) by identifying p and t of (1.1) respectively with \( \eta \) and \( \phi \), where \( \eta \) is a 1 – form and \( \phi \) is an almost para contact Finsler structure on VM[9] given by the Finsler tensor field of type \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), that is, by setting

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \eta(Y) \phi X - \eta(X) \phi Y,
\]

which give

\[
[T(X^H,Y^H)]^H = \eta^H(Y^H) \phi X^H - \eta^H(X^H) \phi Y^H
\]

\[
[T(X^V,Y^V)]^V = \eta^V(Y^V) \phi X^V - \eta^V(X^V) \phi Y^V
\]

(2.3)

\( \forall X, Y \in T_u(VM) \) and \( \eta \in T_u^*(VM) \).
It may be noted that to specify the clear difference between the symbols $\phi$ and $\eta$ of (2.1) and (2.3), in this paper we replace $\phi$, $\eta$, $\xi$, $G$ of (2.3), concerning almost para contact Finsler structures, respectively by $\tilde{\phi}$, $\tilde{\eta}$, $\tilde{\xi}$, $\tilde{G}$.

Hence (2.3) can be written as

$$[T(X^H,Y^H)]^H = \tilde{\eta}(Y^H) \phi X^H - \tilde{\eta}(X^H) \phi Y^H$$
$$[T(X^V,Y^V)]^V = \tilde{\eta}(Y^V) \phi X^V - \tilde{\eta}(X^V) \phi Y^V$$

(2.4)

$\forall X, Y \in T_u(VM)$ and $\tilde{\eta} \in T_{u^*}(VM)$.

Then for a quarter symmetric Finsler connection $\nabla$ on VM of P-Kenmotsu (or) SP-Kenmotsu vector bundle, we get

(a) $$\left[ T\left( \tilde{\phi} X^H, \tilde{\phi} Y^H \right) \right]^H = 0; \left[ T\left( \tilde{\phi} X^V, \tilde{\phi} Y^V \right) \right]^V = 0$$

(b) $$\left[ T\left( \phi X^H, Y^H \right) \right]^H + \left[ T\left( X^H, \phi Y^H \right) \right]^H = [T(X^H, Y^H)]^H$$
$$\left[ T\left( \tilde{\phi} X^V, Y^V \right) \right]^V + \left[ T\left( X^V, \phi Y^V \right) \right]^V = [T(X^V, Y^V)]^V$$

(c) $$\tilde{\phi}[T(X^H, Y^H)]^H - [T(X^H, Y^H)]^H = 0$$
$$\tilde{\phi}[T(X^V, Y^V)]^V - [T(X^V, Y^V)]^V = 0$$

(d) $$\tilde{\eta}[T(X^H, Y^H)]^H = 0; \tilde{\eta}[T(X^V, Y^V)]^V = 0$$

(2.5)

$\forall X, Y \in T_u(VM)$ and $\tilde{\eta} \in T_{u^*}(VM)$.

**Theorem (2.1):** Let $\nabla$ be a quarter symmetric Finsler connection on the total space VM of a Kenmotsu vector bundle. Then for every $X,Y \in T_u(VM)$ there exists a Finsler tensor field $A$ such that

$$\phi\left[ A(X^H, Y^H) \right]^H + \left[ A(X^H, Y^H) \right]^H = 0$$
$$\phi\left[ A(X^V, Y^V) \right]^V + \left[ A(X^V, Y^V) \right]^V = 0$$

**Proof:** If we put $[A(X^H, Y^H)]^H = \phi[T(Y^H, X^H)]^H$, then from (2.2) c we have

$$\phi\left[ A(X^H, Y^H) \right]^H = \phi^2[T(Y^H, X^H)]^H = -[T(Y^H, X^H)]^H$$

Therefore,

$$\phi^2[A(X^H, Y^H)]^H = -\phi[T(Y^H, X^H)]^H = -[A(X^H, Y^H)]^H$$

which implies

$$\phi^2[A(X^H, Y^H)]^H + [A(X^H, Y^H)]^H = 0$$

Similarly the other part holds.
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**Theorem (2.2):** Let $\nabla$ be a quarter symmetric Finsler connection on $\text{VM}$ of a P-Kenmotsu (or) SP-Kenmotsu vector bundle. Then for every $X, Y \in T_0(\text{VM})$, there exists a Finsler tensor field $A$ such that

$$
\tilde{\phi} \left[ A(X^H, Y^H) \right]^H = \left[ A(X^H, Y^H) \right]^H
$$

**Proof:** If we put $A(X^H, Y^H) = \phi \left[ T(Y^H, X^H) \right]^H$, then from (2.5)c, we get

$$
\tilde{\phi} \left[ A(X^H, Y^H) \right]^H = \left[ T(Y^H, X^H) \right]^H.
$$

Therefore,

$$
\tilde{\phi} \left[ A(X^H, Y^H) \right]^H = \phi \left[ T(Y^H, X^H) \right]^H = \left[ A(X^H, Y^H) \right]^H.
$$

Similarly the second part.

### 3. Quarter symmetric metric Finsler connection on Kenmotsu vector bundle

Let $D$ be a metrical Finsler connection and $\nabla$ be quarter symmetric Finsler connection on $\text{VM}$ of Kenmotsu vector bundle. Then

\[ \nabla_X Y = D_X Y + H(X, Y) \quad \forall \ X, Y \in T_0(\text{VM}) \quad (3.1) \]

where

\[ H(X, Y) = \frac{1}{2} \left[ T(X, Y) + P(X, Y) + P(Y, X) \right] \quad (3.2) \]

and

\[ G(P(X,Y), Z) \text{ def } G(T(Z, X), Y) \quad (3.3) \]

From (2.1) and (3.3) we get

\[ P(X, Y) = - \eta(X) \phi Y - \Omega(X, Y) \xi \quad (3.4) \]

where \( \eta(X) = G(X, \xi) \) and \( \Omega(X, Y) = G(\phi X, Y) = -G(X, \phi Y) \)

Now from (2.1), (3.2), (3.4) and (3.5) we get

\[ H(X, Y) = - \eta(X) \phi Y \quad (3.6) \]

Then from (3.1) \( \nabla \) is given as:

\[ \nabla_X Y = D_X Y - \eta(X) \phi Y \quad (3.7) \]
**Theorem (3.1):** The quarter symmetric Finsler connection $\nabla$ on the total space $VM$ of a Kenmotsu vector bundle, given by (3.7), is metric.

**Proof:** We have
$$\nabla_X (G(Y, Z)) = (\nabla_X G)(Y, Z) + G(\nabla_X Y, Z) + G(Y, \nabla_X Z)$$
Using (3.7) in the above equation, we get $(\nabla_X G)(Y, Z) = 0$, which shows that $\nabla$ is metric.

**Proposition (3.1):** The Finsler tensor field $H(X, Y)$ satisfies:

$$[H(\xi^H, Y^H)]^H = [T(\xi^H, Y^H)]^H$$
$$[H(\xi^V, Y^V)]^V = [T(\xi^V, Y^V)]^V$$

$$G[T(X^H, Y^H)^H, \xi^H] = 0 = G[T(X^V, Y^V)^V, \xi^V]$$

(3.8)

**Proof:** From equations (2.1), (3.5), and (3.6), we have (3.8).

**Theorem (3.2):** The quarter symmetric metric Finsler connection $\nabla$ on the total space $VM$ of Kenmotsu vector bundle with Kenmotsu Finsler connection $D$ satisfies

$$\nabla_X Y = D_X Y + H(X, Y); \quad \forall X, Y \in Tu(VM)$$
(4.1)

where
$$H(X, Y) = \frac{1}{2} \{T(X, Y) + P(X,Y) + P(Y, X)\}$$
(4.2)

and
$$G(T(Z,X), Y) = G(P(X,Y), Z)$$
(4.3)

Then from (2.4) and (4.3), we get

4. Quarter symmetric metric Finsler connection on P-Kenmotsu vector bundle

Let $D$ be a metrical Finsler connection and let $\nabla$ be quarter symmetric Finsler connection on $VM$ of P-Kenmotsu (or) SP–Kenmotsu vector bundle. Then

$$\nabla_X Y = D_X Y + H(X, Y); \quad \forall X, Y \in Tu(VM)$$
(4.1)

where
$$H(X, Y) = \frac{1}{2} \{T(X, Y) + P(X,Y) + P(Y, X)\}$$
(4.2)

and
$$G(T(Z,X), Y) = G(P(X,Y), Z)$$
(4.3)
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\[ P(X, Y) = \tilde{\eta}(X) \tilde{\phi} Y - \tilde{\Omega}(X, Y) \tilde{\xi}. \]  

(4.4)

where

\[ \tilde{\eta}(X) = \tilde{G}(X, \tilde{\xi}) \] and

\[ \tilde{\Omega}(X, Y) = \tilde{G}(\tilde{\phi} X, Y) = \tilde{G}(X, \tilde{\phi} Y). \]  

(4.5)

Now from equations (2.4), (4.2), (4.4) and (4.5) we get

\[ H(X, Y) = \tilde{\eta}(Y) \tilde{\phi} X = \tilde{\Omega}(X, Y) \tilde{\xi}. \]  

(4.6)

Then from (4.1) and (4.6), \( \nabla \) can be written as:

\[ \nabla_X Y = D_X Y + \tilde{\eta}(Y) \tilde{\phi} X = \tilde{\Omega}(X, Y) \tilde{\xi}. \]  

(4.7)

**Theorem (4.1):** The quarter symmetric Finsler connection on the total space VM of a P-Kenmotsu (or) SP-Kenmotsu vector bundle, given by (4.7), is metric.

**Proof:** We have

\[ \nabla_X (G(Y, Z)) = (\nabla_X G)(Y, Z) + G(\nabla_X Y, Z) + G(Y, \nabla_X Z) \]

Using (4.7) in the above equation, we get \( (\nabla_X G) = 0 \) which shows that \( \nabla \) is metric.

**Proposition(4.1):** The Finsler tensor field \( H(X, Y) \), given by (4.6), satisfies:

\[ [H(X^H, \tilde{\xi}^H)]^H = [T(X^H, \tilde{\xi}^H)]^H \]

\[ [H(X^V, \tilde{\xi}^V)]^V = [T(X^V, \tilde{\xi}^V)]^V. \]

\[ G([T(X^H, Y^H)]^H, \tilde{\xi}^H) = 0 = G([T(X^V, Y^V)]^V, \tilde{\xi}^V). \]  

(4.8)

**Proof:** From (2.4), (4.5) and (4.6) we have (4.8).

**Theorem (4.2):** On the total space VM of SP-Kenmotsu vector bundle with quarter symmetric metric Finsler connection \( \nabla \), we have

\[ (\nabla_X^{H} \tilde{\eta})(Y^H) - (\nabla_Y^{H} \tilde{\eta})(X^H) = 0 \]

\[ (\nabla_X^{V} \tilde{\eta})(Y^V) - (\nabla_Y^{V} \tilde{\eta})(X^V) = 0 \]  

(4.9)

\( \forall X, Y \in T_u(VM) \)

**Proof:** From (4.7), we have

\[ (\nabla_X \tilde{\eta})(Y) = (D_X \tilde{\eta})(Y) - \tilde{\Omega}(X, Y) \]

Hence for SP-Kenmotsu Finsler vector bundle, using (4.5), we get

\[ (\nabla_X \tilde{\eta})(Y) = \tilde{G}(\tilde{\phi} X, \tilde{\phi} Y) - \tilde{G}(\tilde{\phi} X, Y). \]

Similarly,

\[ (\nabla_Y \tilde{\eta})(X) = \tilde{G}(\tilde{\phi} Y, \tilde{\phi} X) - \tilde{G}(\tilde{\phi} Y, X). \]

On subtracting these two and using (4.5), we get (4.9).
Conclusion. It is found from the above result that $\nabla$ is a special quarter symmetric metric Finsler connection on SP-Kenmotsu vector bundle. Similarly, we can also prove that “the quarter symmetric metric Finsler connection $\nabla$ on the total space $VM$ of a P-Kenmotsu vector bundle is special”.

References


Received: October 12, 2007