On Levels in Arrangements of Curves, II: A Simple Inequality and Its Consequences

Timothy M. Chan*
School of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
tmchan@uwaterloo.ca

Abstract

We give a surprisingly short proof that in any planar arrangement of n curves where each pair intersects at most a fixed number (s) of times, the k-level has subquadratic ($O(n^{2-\frac{1}{2s}})$) complexity. This answers one of the main open problems from the author's previous paper (FOCS'00), which provided a weaker bound for a restricted class of curves (graphs of degree-s polynomials) only. When combined with existing tools (cutting curves, sampling, etc.), the new idea generates a slew of improved k-level results for most of the curve families studied earlier, including a near-$O(n^{3/2})$ bound for parabolas.

1. Introduction

The k-level of lines. We begin by re-examining an old result on a famous open problem in two-dimensional combinatorial geometry.

The problem, first investigated by Lovász [21] and Erdős *et al.* [16] in the early 1970s, is the following: for a set P of n points in the plane, how many different subsets of size k (called *k-sets*) can be formed by intersecting P with a halfplane, asymptotically in the worst case?

For $k = \lfloor n/2 \rfloor$, we are asking for the number of different ways to bisect a point set into two equal halves with a line. In the dual, the problem is equivalent to determining the worst-case complexity (number of vertices/edges) of the k-level in an arrangement of n lines, where the *k-level* is defined as the closure of all points that lie on exactly one line and strictly above exactly $k-1$ lines. Alternatively, as a "kinetic" problem in one dimension, we are seeking the maximum number of changes that can be undergone by the k-th smallest element, for n elements moving linearly on the real line. These different views of the problem help explain its central place in combinatorial and computational geometry. In particular, the problem is related to the analysis of geometric algorithms for ham-sandwich cuts [20], range searching [11], geometric optimization with violations [10], and kinetic/parametric minimum spanning trees [3]. See the many books [14, 22, 23, 27] and surveys [5, 7, 18] for more background information.

In the discussion below, we focus on the most important case, when $k = \Theta(n)$, because as Agarwal *et al.* [1] showed, an $O(f(n))$ upper bound for this case automatically implies a "k-sensitive" $O((n/k)f(k))$ bound, for any well-behaved function f.

The initial papers by Lovász [21] and Erdős *et al.* [16] established an $O(n^{3/2})$ upper bound and an $\Omega(n \log n)$ lower bound. The conjecture by Erdős *et al.* [16] was that the number is $o(n^{1+\varepsilon})$ for any fixed $\varepsilon > 0$. No progress on the upper bound was made for almost two decades until Pach, Steiger, and Szemerédi's $O(n^{3/2}/\log^* n)$ result from their (remarkably involved) FOCS'89 paper [24]. No significant progress was made for yet another decade until Dey's $O(n^{4/3})$ result from his (remarkably short) FOCS'97 paper [13]. The lower bound meanwhile has been increased to $n2^{\Omega(\sqrt{\log n})}$ by Toth in 2000 [32]; this lower bound was known some time ago by Klawe, Paterson, and Pippenger [14] for the weaker case of pseudo-lines (see below).

Before describing our contribution, we first comment on the earlier upper-bound approaches. A paper by Agarwal *et al.* [1] (proceedings version) already contained a collection of simple proofs of the $O(n^{3/2})$ bound. Agarwal *et al.* noticed that *all* known proofs can be carried over to establish the more general result that n *nonoverlapping concave chains* in an arrangement of n lines have $O(n^{3/2})$ vertices. Dey's breakthrough [13] was inspired by this observation: his $O(n^{4/3})$ proof was designed for this more general result. Many natural problems in combinatorial plane geometry have complexity $\Theta(n^{4/3})$, and indeed Dey's upper

*This work was supported in part by an NSERC Research Grant.

bound is tight for this multiple concave chain problem [15]. It would thus appear that the limit has been reached, unless somehow a substantially different approach for the k-set/k-level problem is discovered.

To renew hope for Erdős *et al.*'s original conjecture, we suggest a different proof of the old $O(n^{3/2})$ upper bound, unrelated to concave chains. It is based on the following naive intuition: if the k-level were huge, then nearby levels would be huge as well, but the total size of the arrangement is bounded (by $O(n^2)$). Technically, the proof amounts to a simple relationship between different levels in the arrangement; the bound follows by solving a difference equation. In contrast, previous proofs focused on a single k-level, and relationships between different levels were explored "after the fact" (for example, as in [1, 8, 19, 33]).

The k-level of curves. Unfortunately, we see no obvious ways to obtain an $o(n^{3/2})$ bound with the new idea, let alone an improvement to Dey's result. However, as it turns out, the idea adapts beautifully to a generalization of the problem—the k-level in an arrangement of curves. (In the one-dimensional kinetic setting, this corresponds to tracking changes to the k-th smallest element, where each element moves according to a nonlinear function in time.) This generalized problem is equally natural and fundamental, and has been studied intensively by several authors over the years [4, 9, 26, 31].

For example, our $O(n^{3/2})$ proof works immediately for *pseudo-lines* (x-monotone curves going from $x = -\infty$ to $x = \infty$, where each pair intersects at most once) and *pseudo-segments* (x-monotone curve segments where each pair intersects at most once). Previous proofs also generalize to pseudo-lines, but their generalizations to arbitrary pseudo-segments require additional tools (to make pseudo-segments *extendible* [9]). As another feature of our proof, we can derive an $O(n^{3/2} + A)$ bound for the k-level in an arrangement of n x-monotone curve segments with A "bad" pairs, where every good pair intersects at most once but every bad pair may intersect $O(1)$ times. To our knowledge, a similar bound for these "almost" pseudo-segment arrangements cannot be obtained easily (if at all) through previous approaches.

More importantly, our proof gives the first subquadratic $O(n^{2-\frac{1}{2s}})$ bound for general x-monotone curve segments (called *s-intersecting* curve segments) where each pair intersects at most s times for a fixed constant s. Previously, a subquadratic bound was obtained in the predecessor of this paper [9] only when the curve segments are graphs of fixed-degree polynomials (or, slightly more generally, when repeated differentiation leads to a 1- or 2-intersecting family). For degree-s polynomials, the previous bound was much worse—$O(n^{2-\frac{1}{9\cdot 2^{s-3}}}\log^{2/3} n)$.

As Agarwal *et al.* [1] showed also for curves and curve segments, an $O(f(n))$ upper bound implies a k-sensitive $O((n/k)f(k)\beta(n/k))$ bound, for any well-behaved function f and some slow-growing function β.

The only major tool available for curves was pioneered by Tamaki and Tokuyama [31] and later improved and extended by Agarwal *et al.* [4] and in our previous paper [9]: the idea is to *cut* the curves into pseudo-segments, so that k-level results on these pseudo-segments can be applied. Unfortunately, for odd s, it is generally not possible to cut s-intersecting curves into subquadratically many $(s-1)$-intersecting segments, as a simple example illustrates. Our approach manages to obtain nontrivial k-level bounds directly, without cutting the curves.

This paper does not replace its predecessor [9], as bounds on the number of cuts turn out to have further applications, for example, in the combinatorial analysis of pseudo-concave chains [9], incidences [2, 4], and many faces [2, 4] in arrangements of curves, as well as parametric minimum spanning trees [9]. Our approach has no ramifications on these problems. This defect can be viewed as an advantage, if one believes that the k-level problem has lower complexity than these other problems.

While our approach does not require cutting, it can be combined with cutting to improve practically all existing k-level upper bounds for arrangements of curves other than pseudo-lines/segments. The previous [4, 9] and new results are summarized in Table 1; the notation $\widetilde{O}$ here hides polylogarithmic factors, and in some cases, factors of the form $(\log n)^{O(\alpha(n)^2)}$, where $\alpha(n)$ is the (slow-growing) inverse Ackermann function. The most notable of these results is perhaps in the case of axis-aligned parabolas (which arises in the tracking of the k-th smallest distance for a set of linearly moving points). The history of the parabolic case illustrates the evolution of techniques well: Tamaki and Tokuyama's proof [31], which introduced the cutting approach, yielded an $O(n^{23/12}) = O(n^{1.917})$ bound; Dey's result [13] implied a reduction to $O(n^{17/9}) = O(n^{1.889})$; in the previous paper [9], we utilized the cutting number more effectively and obtained an $\widetilde{O}(n^{16/9}) = O(n^{1.778})$ bound; Agarwal *et al.* [4] improved the cutting number itself and obtained an $\widetilde{O}(n^{5/3}) = O(n^{1.667})$ bound; the present paper finds the best way to use the cutting number and finally brings the upper bound to $\widetilde{O}(n^{3/2})$ (which, as the reader may recall, was the initial upper bound for the line case).

2. The Proof for Lines

In this section, we illustrate the main idea for the case of lines. We actually analyze the complexity of $O(i)$ consecutive levels and re-derive Welzl's old $O(n^{3/2}i^{1/2})$ result [33]. This bound has been improved by Dey [13] to $O(n^{4/3}i^{2/3})$. We remark that although Dey's proof is also simple, it relies on a "crossing lemma", whose shortest proof requires prob-

class of curves	best previous result	new result
lines/pseudo-lines/pseudo-segments	$\widetilde{O}(n^{4/3}) = O(n^{1.334})$	—
parabolas	$\widetilde{O}(n^{5/3}) = O(n^{1.667})$	$\widetilde{O}(n^{3/2}) = \widetilde{O}(n^{1.5})$
pseudo-parabolas	$\widetilde{O}(n^{26/15}) = O(n^{1.734})$	$O(n^{8/5}) = O(n^{1.6})$
pseudo-parabolic segments	$\widetilde{O}(n^{16/9}) = O(n^{1.778})$	$O(n^{5/3}) = O(n^{1.667})$
pairwise-intersecting pseudo-parabolas	$\widetilde{O}(n^{14/9}) = O(n^{1.556})$	$O(n^{13/9}) = O(n^{1.445})$
graphs of cubics	$\widetilde{O}(n^{17/9}) = O(n^{1.889})$	$O(n^{31/18}) = O(n^{1.723})$
graphs of quartics	$\widetilde{O}(n^{35/18}) = O(n^{1.945})$	$\widetilde{O}(n^{47/27}) = O(n^{1.741})$
graphs of quintics	$\widetilde{O}(n^{71/36}) = O(n^{1.973})$	$\widetilde{O}(n^{97/54}) = O(n^{1.797})$
s-intersecting curve segments (odd $s \geq 3$)	$O(n^2)$	$O(n^{2-\frac{1}{2s}})$
s-intersecting curve segments (even $s \geq 4$)	$O(n^2)$	$O(n^{2-\frac{1}{2(s-1)}})$

Table 1. Upper bounds on the complexity of the k-level.

abilistic arguments. Our proof is self-contained and very short.

For simplicity, we assume that the given arrangement is in general position.

Let L_k denote the k-level. (By default, set L_k to $y = -\infty$ if $k \leq 0$, or to $y = \infty$ if $k > n$.)

Let t_i be the number of vertices in the arrangement strictly between L_{k-i} and L_{k+i}.

Let $\Delta t_i = t_{i+1} - t_i$. (In other words, Δt_i is the total number of "concave" vertices of L_{k-i} and "convex" vertices of L_{k+i}.)

Lemma 2.1 *For an arrangement of lines,*

$$t_i \;\leq\; 2i\Delta t_i + O(i^2).$$

Proof: Take each vertex v strictly between L_{k-i} and L_{k+i}. Suppose v is defined by the lines ℓ_1 and ℓ_2. For each $j \in \{1, 2\}$, shoot a rightward ray from v along ℓ_j until the ray hits L_{k-i} or L_{k+i}. Let w_j be the point hit (a concave vertex of L_{k-i} or a convex vertex of L_{k+i}, or, if there is no hit, a point at $x = \infty$ on ℓ_j). If w_1 is to the left of w_2, charge v to the pair (w_1, ℓ_2); otherwise, charge v to the pair (w_2, ℓ_1).

Clearly, each pair is charged at most once. For a pair (w, ℓ) to receive a charge, ℓ must lie strictly between L_{k-i} and L_{k+i} at the x-coordinate of w. Thus, the number of charges is bounded by $(\Delta t_i + 2i) \cdot 2i$, and the inequality follows. □

Theorem 2.2 *For an arrangement of n lines, the k-level has $O(n^{3/2})$ vertices.*

Proof: The above inequality is equivalent to the recurrence

$$t_i \;\leq\; \frac{2i}{2i+1} t_{i+1} \;+\; O(i),$$

where $t_n = O(n^2)$. It is straightforward to verify that $t_i = O(n^{3/2} i^{1/2})$. □

Remark: Like previous proofs, a modification can directly show that the first k levels have $O(nk)$ vertices [6, 17] and the k-level has $O(n\sqrt{k})$ vertices. For line segments, we can obtain another, albeit suboptimal, proof that the 1-level (the *lower envelope* [27]) has $O(n \log n)$ vertices; contrast this with Tagansky's probabilistic proof [29].

3. The Generalization to Curves

The proof extends easily to pseudo-lines and pseudo-segments. In this section, we consider how the bound is affected for general curve segments. The generalization of Lemma 2.1 is straightforward. We state the inequality in a form that is sensitive to the number of "bad" pairs; this form will be beneficial in later proofs.

Throughout the paper, we assume that "curve segments" are x-monotone and each pair intersects at most $O(1)$ times.

Given two curve segments γ_1 and γ_2 that intersect at least $s+1$ times, an *s-lens* refers to the portion of $\gamma_1 \cup \gamma_2$ between some $s+1$ consecutive intersection points of $\gamma_1 \cap \gamma_2$.

Let $\Lambda_i^{(s)}$ be the collection of all s-lenses lying strictly between L_{k-i} and L_{k+i}.

Lemma 3.1 (Main Inequality)

$$t_i \;\leq\; 2si\Delta t_i \;+\; O(ni + |\Lambda_i^{(s)}|).$$

Proof: Take each vertex v strictly between L_{k-i} and L_{k+i}. Suppose v is defined by the curve segments γ_1 and γ_2. If γ_1 and γ_2 form an s-lens in $\Lambda_i^{(s)}$, charge v to the s-lens. Otherwise, for each $j \in \{1, 2\}$, walk rightward from v along γ_j until a point of L_{k-i} or L_{k+i}, or an endpoint, is hit. Let w_j be the point hit (a vertex in $L_{k-i} \setminus L_{k-i-1}$ or $L_{k+i} \setminus L_{k+i+1}$, or an endpoint, or a point of discontinuity in L_{k-i} or L_{k+i}). If w_1 is to the left of w_2, charge v to the pair (w_1, γ_2); otherwise, charge v to the pair (w_2, γ_1).

Each pair is charged at most s times, because if this were not the case, a vertex charged to the pair would be in an s-lens strictly between L_{k-i} and L_{k+i}. For a pair (w, γ) to receive a charge, γ must lie strictly between L_{k-i} and L_{k+i} at the x-coordinate of w. Thus, excluding charges to s-lenses, the number of charges is bounded by $(\Delta t_i + O(n)) \cdot 2i \cdot s$. □

Theorem 3.2 *For an arrangement of n curve segments with A s-lenses, the k-level has $O(n^{2-\frac{1}{2s}} + A)$ vertices.*

Proof: The recurrence

$$t_i \;\le\; \frac{2si}{2si+1} t_{i+1} \;+\; O\left(n + \frac{A}{i}\right),$$

with $t_n = O(n^2)$, solves to $t_i = O(n^{2-1/2s} i^{1/2s} + A)$. □

4. Improvements by (Not) Cutting

For *pseudo-parabolas* (2-intersecting curves going from $x = -\infty$ to $x = \infty$) or *pseudo-parabolic segments* (2-intersecting curve segments), Theorem 3.2 implies an $O(n^{7/4})$ bound. This bound can be improved further, by applying known results to *cut* (i.e., subdivide by insertions of new endpoints) the given curve segments into a pseudo-segment family. Observe that the k-level is unchanged after the cut. A natural first idea is to make the cuts and then apply the main inequality (with a larger n but a smaller s). We discover, though, that it is possible—and, in fact, more effective—to avoid the actual cuts but instead estimate the number of "bad" lenses $|\Lambda_i^{(s)}|$ by the known techniques.

Given a class of curve segments, let $\nu^{(s)}(n)$ be the maximum size of a collection of nonoverlapping s-lenses in an arrangement of n segments in the class. Here, a collection is *nonoverlapping* if each pair intersects only at a discrete set of points. The number of cuts required to turn n curve segments into an s-intersecting family is clearly $\Omega(\nu^{(s)}(n))$, but is also $O(\nu^{(s)}(n))$ by the technique of Tamaki and Tokuyama [31], assuming that $\nu^{(s)}(n)/n^{1+\varepsilon}$ is monotone increasing.

Lemma 4.1 $|\Lambda_i^{(s)}| = O(i^2 \nu^{(s)}(n/i))$.

Proof: Given a collection of s-lenses, its *depth* refers to the maximum number of s-lenses that a point can lie on. The key observation is that $\Lambda_i^{(s)}$ has depth at most $4i$: if a point w lies on an s-lens $\lambda \in \Lambda_i^{(s)}$, then either curve segment defining λ must lie strictly between L_{k-i} and L_{k+i} at the x-coordinate of w.

As one step of their technique, Tamaki and Tokuyama [31] have basically shown (at least for $s = 1$) that the maximum size of a depth-$O(i)$ collection is $O(i^2 \nu^{(s)}(n/i))$. For the sake of completeness, we quickly provide the proof, which employs random sampling in the style of Clarkson and Shor [12].

Pick a random subset R of the given curve segments with $|R| = n/i$. For each lens $\lambda \in \Lambda_i^{(s)}$ defined by curve segments γ_1 and γ_2, put λ in a subcollection $\Lambda(R)$ iff γ_1 and γ_2 are in R but all other curve segments forming lenses in $\Lambda_i^{(s)}$ that contain an *endpoint* (the leftmost or rightmost point) of λ are not in R.

Then $\Lambda(R)$ is nonoverlapping, because if two lenses overlap, one lens must contain an endpoint of the other. Thus, $E|\Lambda(R)| = O(\nu^{(s)}(n/i))$. On the other hand, because $\Lambda_i^{(s)}$ has depth $O(i)$, for a fixed lens $\lambda \in \Lambda_i^{(s)}$, $\Pr\{\lambda \in \Lambda(R)\} \approx (1/i)^2(1-1/i)^{O(i)} = \Omega(1/i^2)$. Thus, $E|\Lambda(R)| = \Omega(|\Lambda_i^{(s)}|/i^2)$. The lemma follows. □

Theorem 4.2

(a) *For an arrangement of n pseudo-parabolic segments, the k-level has $O(n^{5/3})$ vertices.*

(b) *For an arrangement of n pseudo-parabolas, the k-level has $O(n^{8/5})$ vertices.*

(c) *For an arrangement of n pseudo-parabolas that posess a "3-parameter algebraic representation"* [4] *(e.g., axis-aligned parabolas), the k-level has $O(n^{3/2}(\log n)^{O(\alpha(n)^2)})$ vertices.*

(d) *For an arrangement of n s-intersecting curve segments for an even $s \ge 4$, the k-level has $O(n^{2-\frac{1}{2(s-1)}})$ vertices.*

Proof:

(a) Tamaki and Tokuyama [31] originally obtained $\nu^{(1)}(n) = O(n^{5/3})$ for pseudo-parabolas. In the previous paper [9], we showed that the bound holds for pseudo-parabolic segments as well. Combined with Lemma 4.1, the main inequality becomes

$$t_i \;\le\; 2i\Delta t_i \;+\; O(n^{5/3} i^{1/3}),$$

which solves to $t_i = O(n^{5/3} i^{1/3})$.

(b) Agarwal *et al.* [4] observed that a recent result by Pinchasi and Radoičić [25] implies the improved bound $\nu^{(1)}(n) = O(n^{8/5})$ for pseudo-parabolas. The main inequality now becomes

$$t_i \;\le\; 2i\Delta t_i \;+\; O(n^{8/5} i^{2/5}),$$

which solves to $t_i = O(n^{8/5} i^{2/5})$.

(c) Agarwal *et al.* [4] also established the bound $\nu^{(1)}(n) = O(n^{3/2}(\log n)^{O(\alpha(n)^2)})$ for pseudo-parabolas admitting a 3-parameter algebraic representation. For these curves,

$$t_i \le 2i\Delta t_i + O(n^{3/2}i^{1/2}(\log n)^{O(\alpha(n)^2)}),$$

which solves to $t_i = O(n^{3/2}i^{1/2}(\log n)^{O(\alpha(n)^2)})$.

(d) In the previous paper [9], we observed that a modification of Tamaki and Tokuyama's technique [31] implies $\nu^{(s-1)}(n) = O(n^{2-\frac{1}{s+1}})$ for s-intersecting curve segments when s is even. Here,

$$t_i \le 2(s-1)i\Delta t_i + O(n^{2-\frac{1}{s+1}} i^{\frac{1}{s+1}}),$$

which solves to $t_i = O(n^{2-\frac{1}{2(s-1)}} i^{\frac{1}{2(s-1)}})$, provided that $2(s-1) > s+1$. □

5. Improvements by Bootstrapping

For certain curve families, further improvements can be obtained by switching to the following inequality, which uses level bounds on smaller arrangements to obtain better level bounds overall. The proof of this inequality is quite similar to a "k-sensitizing" trick by Agarwal *et al.* [1] but is described here in a self-contained manner.

Given a class of curve segments, let $\tau_i(n)$ be the maximum number of vertices strictly between the $(k-i)$- and $(k+i)$-level in an arrangement of n segments in the class.

Lemma 5.1 *For an arrangement of n curve segments and $j \ge 2i$,*

(a) $t_i = O((n + \Delta t_j)\tau_i(O(j))/j)$.

(b) $t_i = O((n + t_j/j)\tau_i(O(j))/j)$.

Proof: Divide the plane into $O((n+\Delta t_j)/j)$ vertical slabs such that for each slab σ, the set S_σ of all segments that define vertices of $L_{k-j} \setminus L_{k-j-1}$ and $L_{k+j} \setminus L_{k+j+1}$ and endpoints within σ has size at most j. Now, the set T_σ of all segments that lie strictly between L_{k-j} and L_{k+j} at the left wall of σ has size at most $2j$. Within the slab σ, each level in the original arrangement that is strictly between L_{k-j} and L_{k+j} is a level of $S_\sigma \cup T_\sigma$. Thus, t_i is bounded by $\tau_i(O(j))$ times the number of slabs. This proves (a).

(b) follows by choosing an index $j' \in \{i, \ldots, j-1\}$ with $\Delta t_{j'} \le t_j/(j-i)$ (which exists because $\sum_{j'=i}^{j-1} \Delta t_{j'} \le t_j$). □

This inequality is best applied by cutting the given segments first (with a larger n but a smaller τ_i function), as in the following example:

Theorem 5.2 *For an arrangement of n pseudo-parabolas where each pair intersects exactly twice, the k-level has $O(n^{13/9}\log^{1/3} n)$ vertices.*

Proof: Agarwal *et al.* [4] established the bound $\nu^{(1)}(n) = O(n^{4/3})$ in this particular case. Combined with Lemma 4.1, the main inequality becomes

$$t_i \le 2i\Delta t_i + O(n^{4/3}i^{2/3}),$$

which yields $t_i = O(n^{3/2}i^{1/2})$ only.

To obtain a better bound for small i, cut the arrangement into $O(n^{4/3}\log n)$ extendible pseudo-segments using Agarwal *et al.*'s result in conjunction with the extendibility result from our previous paper [9]. Next, apply Lemma 5.1 to these segments, with Dey's level bound $\tau_i(j) = O(j^{4/3}i^{2/3})$ [13], which holds for extendible pseudo-segments [30]:

$$t_i = O\left(\left(n^{4/3}\log n + \frac{n^{3/2}}{j^{1/2}}\right) j^{1/3} i^{2/3}\right).$$

Setting $j = \lfloor n^{1/3}/\log^2 n \rfloor$ yields $t_i = O(n^{13/9}i^{2/3}\log^{1/3} n)$ for $i = O(n^{1/3}/\log^2 n)$. □

Remark: Curiously, Theorem 5.2 is the only result in this paper that relies on Dey's k-level bound and the concept of extendible pseudo-segments.

We now apply this "bootstrapping" strategy to improve k-level bounds for graphs of low-degree polynomials. For cubics, Theorem 3.2 alone yields only an $O(n^{11/6})$ bound, while the best bound for cutting n cubics to pseudo-segments ($\nu^{(1)}(n)$) is also $O(n^{11/6})$. Yet, a successful combination can lead to the result below. Here, F' refers to the family of curve segments corresponding to the derivatives of the univariate functions whose graphs form the family F.

Theorem 5.3 *For an arrangement of n curve segments F such that F' is pseudo-parabolic (e.g., graphs of cubics), the k-level has $O(n^{31/18})$ vertices.*

Proof: In the previous paper [9], we showed that $\nu^{(2)}(n) = O(n^{3/2})$ for a family F where F' is pseudo-parabolic. Combined with Lemma 4.1, the main inequality becomes

$$t_i \le 4i\Delta t_i + O(n^{3/2}i^{1/2}),$$

which solves to $t_i = O(n^{7/4}i^{1/4})$.

To refine the bound for small i, cut the arrangement into $O(n^{3/2})$ pseudo-parabolic segments by a result from the previous paper, and apply Lemma 5.1 to these pseudo-parabolic segments, using the bound $\tau_i(j) = O(j^{5/3}i^{1/3})$

established for pseudo-parabolic segments in the proof of Theorem 4.2(a):

$$t_i = O\left(\left(n^{3/2} + \frac{n^{7/4}}{j^{3/4}}\right) j^{2/3} i^{1/3}\right).$$

Setting $j = \lfloor n^{1/3} \rfloor$ yields $t_i = O(n^{31/18} i^{1/3})$ for $i = O(n^{1/3})$. □

Further bootstrapping steps can in fact yield improved results for degree-s polynomials for any s. To illustrate the strategy, we only give bounds for the $s = 4$ and $s = 5$ cases.

Theorem 5.4 *For an arrangement of n curve segments F such that F'' is pseudo-parabolic (e.g., graphs of quartics), the k-level has $O(n^{47/27} \log^{26/27} n)$ vertices.*

Proof: The repeated differentiation technique from our previous paper [9] gives $\nu^{(3)}(n) = O(n^{3/2})$ and $\nu^{(2)}(n) = O(n^{7/4})$. Combined with Lemma 4.1, the main inequality for $s = 2$ becomes

$$t_i \le 4i\Delta t_i + O(n^{7/4} i^{1/4}).$$

which solves to $t_i = O(n^{7/4} i^{1/4} \log(n/i))$.

To refine the bound, cut the arrangement into $O(n^{3/2})$ curve segments so that F' becomes pseudo-parabolic, by the result from the previous paper [9]. Next, apply Lemma 5.1 to these segments, using the bound $\tau_i(j) = O(j^{31/18} i^{1/3})$ established for this type of segments in the proof of Theorem 5.3:

$$t_i = O\left(\left(n^{3/2} + \frac{n^{7/4}\log n}{j^{3/4}}\right) j^{13/18} i^{1/3}\right).$$

Setting $j = \lfloor n^{1/3} \log^{4/3} n \rfloor$ yields $t_i = O(n^{47/27} i^{1/3} \log^{26/27} n)$ for $i = O(n^{1/3})$. □

Theorem 5.5 *For an arrangement of the graphs of n curve segments F such that F''' is 2-intersecting (e.g., graphs of quintics), the k-level has $O(n^{97/54} \log^{26/27} n)$ vertices.*

Proof: The repeated differentiation technique [9] gives $\nu^{(4)}(n) = O(n^{3/2})$, $\nu^{(3)}(n) = O(n^{7/4})$, and $\nu^{(2)}(n) = O(n^{15/8})$. Combined with Lemma 4.1, the main inequality for $s = 3$ becomes

$$t_i \le 6i\Delta t_i + O(n^{7/4} i^{1/4}).$$

which solves to $t_i = O(n^{11/6} i^{1/6})$.

To refine the bound, cut the arrangement into $O(n^{3/2})$ curve segments so that F'' becomes pseudo-parabolic, by the result from the previous paper [9]. Next, apply Lemma 5.1 to these segments, using the bound $\tau_i(j) = O(j^{47/27} i^{1/3} \log^{26/27} n)$ established for this type of segments in the proof of Theorem 5.4:

$$t_i = O\left(\left(n^{3/2} + \frac{n^{11/6}}{j^{5/6}}\right) j^{20/27} i^{1/3} \log^{26/27} j\right).$$

Setting $j = \lfloor n^{2/5} \rfloor$ yields $t_i = O(n^{97/54} i^{1/3} \log^{26/27} n)$ for $i = O(n^{2/5})$. □

6. Additional Remarks

Toth's $n2^{\Omega(\sqrt{\log n})}$ lower bound [32] remains the current record, even for s-intersecting curves. At this point, it is conceivable that Erdős *et al.*'s $o(n^{1+\varepsilon})$ conjecture [16] might hold for curves.

We do not know whether the main idea here, of exploiting relationship between nearby levels, could help at all to resolve the original k-level problem for lines in the plane. (For example, is it possible to reduce the factor 2 in Lemma 2.1 by increasing the overhead term, perhaps by combining with Dey's technique?) It is also unclear if the idea could be useful in higher dimensions [28].

References

[1] P. K. Agarwal, B. Aronov, T. M. Chan, and M. Sharir. On levels in arrangements of lines, segments, planes, and triangles. *Discrete Comput. Geom.*, 19:315–331, 1998. (Preliminary version in *Proc. 13th ACM Sympos. Comput. Geom.*, pages 30–38, 1997.)

[2] P. K. Agarwal, B. Aronov, and M. Sharir. On the complexity of many faces in arrangements of circles. To appear in *Discrete and Computational Geometry: The Goodman-Pollack Festschrift* (B. Aronov, S. Basu, J. Pach, M. Sharir, eds.), Springer-Verlag, Berlin. (Also in *Proc. 42nd IEEE Sympos. Found. Comput. Sci.*, pages 74–83, 2001.)

[3] P. K. Agarwal, D. Eppstein, L. J. Guibas, and M. R. Henzinger. Parametric and kinetic minimum spanning trees. In *Proc. 39th IEEE Sympos. Found. Comput. Sci.*, pages 596–605, 1998.

[4] P. K. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir, and S. Smorodinsky. Lenses in arrangements of pseudo-circles and their applications. *J. ACM*, to appear. (Preliminary version in *Proc. 18th ACM Sympos. Comput. Geom.*, pages 123–132, 2002.)

[5] P. K. Agarwal and M. Sharir. Arrangements and their applications. In *Handbook of Computational Geometry* (J. Urrutia and J. Sack, eds.), North-Holland, pages 49–119, 2000.

[6] N. Alon and E. Győri. The number of small semispaces of a finite set of points in the plane. *J. Combin. Theory Ser. A*, 41:154–157, 1986.

[7] A. Andrzejak and E. Welzl. k-sets and j-facets: a tour of discrete geometry. Manuscript, 1997.

[8] T. M. Chan. Remarks on k-level algorithms in the plane. Manuscript, 1999.

[9] T. M. Chan. On levels in arrangements of curves. *Discrete Comput. Geom.*, 29:375–393, 2003. (Preliminary version in *Proc. 41st IEEE Sympos. Found. Comput. Sci.*, pages 219–227, 2000.)

[10] T. M. Chan. Low-dimensional linear programming with violations. In *Proc. 43rd IEEE Sympos. Found. Comput. Sci.*, pages 570–579, 2002.

[11] B. Chazelle and F. P. Preparata. Halfspace range search: an algorithmic application of k-sets. *Discrete Comput. Geom.*, 1:3–93, 1986.

[12] K. L. Clarkson and P. W. Shor. Applications of random sampling in computational geometry, II. *Discrete Comput. Geom.*, 4:387–421, 1989.

[13] T. K. Dey. Improved bounds on planar k-sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.

[14] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer-Verlag, Berlin, 1987.

[15] D. Eppstein. Geometric lower bounds for parametric matroid optimization. *Discrete Comput. Geom.*, 20:463–476, 1998.

[16] P. Erdős, L. Lovász, A. Simmons, and E. Straus. Dissection graphs of planar point sets. In *A Survey of Combinatorial Theory* (J. N. Srivastava, ed.), North-Holland, Amsterdam, Netherlands, pages 139–154, 1973.

[17] J. E. Goodman and R. Pollack. On the number of k-subsets of a set of n points in the plane. *J. Combin. Theory Ser. A*, 36:101–104, 1984.

[18] D. Halperin. Arrangements. In *Handbook of Discrete and Computational Geometry* (J. E. Goodman and J. O'Rourke, eds.), pages 389–412, CRC Press, 1997.

[19] N. Katoh, H. Tamaki, and T. Tokuyama. Parametric polymatroid optimization and its geometric applications. *Int. J. Comput. Geom. Appl.*, 12:429–443, 2002.

[20] C.-Y. Lo, J. Matoušek, and W. L. Steiger. Algorithms for ham-sandwich cuts. *Discrete Comput. Geom.*, 11:433–452, 1994.

[21] L. Lovász. On the number of halving lines. *Ann. Univ. Sci. Budapest, Eötvös, Sec. Math.*, 14:107–108, 1971.

[22] J. Matoušek. *Lectures on Discrete Geometry*. Springer-Verlag, 2002.

[23] J. Pach and P. K. Agarwal. *Combinatorial Geometry*. Wiley-Interscience, New York, 1995.

[24] J. Pach, W. Steiger, and E. Szemerédi. An upper bound on the number of planar k-sets. *Discrete Comput. Geom.*, 7:109–123, 1992.

[25] R. Pinchasi and R. Radoičić. Topological graphs with no self-intersecting cycle of length 4. In *Proc. 19th ACM Sympos. Comput. Geom.*, pages 98–103, 2003.

[26] M. Sharir. On k-sets in arrangements of curves and surfaces. *Discrete Comput. Geom.*, 6:593–613, 1991.

[27] M. Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications*. Cambridge University Press, New York, 1995.

[28] M. Sharir, S. Smorodinsky, and G. Tardos. An improved bound for k-sets in three dimensions. *Discrete Comput. Geom.*, 26:195–204, 2001.

[29] B. Tagansky. A new technique for analyzing substructures in arrangements of piecewise linear surfaces. *Discrete Comput. Geom.*, 16:455–479, 1996.

[30] H. Tamaki and T. Tokuyama. A characterization of planar graphs by pseudo-line arrangements. In *Proc. 8th Int. Sympos. Algorithms and Computation*, Lect. Notes in Comput. Sci., vol. 1350, Springer-Verlag, pages 123–132, 1997.

[31] H. Tamaki and T. Tokuyama. How to cut pseudoparabolas into segments. *Discrete Comput. Geom.*, 19:265–290, 1998.

[32] G. Toth. Point sets with many k-sets. *Discrete Comput. Geom.*, 26:187–194, 2001.

[33] E. Welzl. More on k-sets of finite sets in the plane. *Discrete Comput. Geom.*, 1:95–100, 1986.