ON DARBOUX PROPERTY OF FUZZY MULTIMEASURES

NIKOS E. MASTORAKIS
Military Institutes of University Education (ASEI), Hellenic Naval Academy
Terma Hatzikyriakou, 18593, GREECE
wseas_team@wseas.org
ALINA GAVRILUŢ, ANCA CROITORU, GABRIELA APREUTESEI
"Al.I. Cuza" University, Faculty of Mathematics
Bd. Carol I, No. 11, Iaşi, 700506, ROMANIA
gavrilut@uaic.ro, croitoru@uaic.ro, gabriela@uaic.ro

Abstract: We present some properties regarding Darboux property, non-atomicity, regular fuzziness of multimeasures taking values in the family of all closed nonvoid subsets of a real normed space.

Key–words: Darboux property, fuzzy multimeasure, atom, non-atomic, regular, diffused.

1 Introduction

Since 1965, when Zadeh [27] introduced fuzzy sets, as an extension of classical sets, the theory of fuzziness has proved its utility in a broad spectrum of areas, such as statistics, probability (e.g. Dempster [6], Shafer [24]), economy, theory of games (e.g. Aumann and Shapley [1]), computer and systems sciences, artificial intelligence (e.g. Mastorakis [19]), physics, biology, medicine (e.g. Pham, Brandl, Nguyen N.D. and Nguyen T.V. [22] in prediction of osteoporotic fractures), human decision making (e.g. Liginlal and Ow [18]). In the last decades, non-additive measure theory has been studied by many authors (e.g. Chiţescu [3, 4], Choquet [5], Denneberg [7], Dobrakov [8], Olejček [20], Pap [21], Precupanu [23], Sugeno [25], Suzuki [26]). Notions and theorems of non-additive measure theory (such as: continuity, regularity, extension, decomposition, measure, integral) have been studied in the set-valued case. Purely atomic measures where studied in literature in different forms. For instance, Chiţescu [3, 4] established interesting connections with different classic problems concerning L^p spaces. In Gavriluţ [11], Gavriluţ and Croitoru [12,13,14], we extended the concepts of atom, pseudo-atom and Darboux property for set multifunctions.

In this paper, we study the case of multimeasures, presenting some of their properties regarding Darboux property, non-atomicity, regular fuzziness.

2 Preliminaries

T will be an abstract nonvoid set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{C}$ a ring of subsets of T.

By $i = \overline{1,n}$ we mean $i \in \{1, 2, \dots, n\}$, for $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

Let (X, d) be a metric space. $\mathcal{P}_0(X)$ is the family of all non-empty subsets of X, $\mathcal{P}_f(X)$ the family of non-empty closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of non-empty closed bounded subsets of X and $\mathcal{P}_{bfc}(X)$ the family of all nonvoid closed bounded convex subsets of X.

For arbitrary sets $E, F \in \mathcal{P}_0(X)$, we denote $h(E,F) = \max\{e(E,F), e(F,E)\}$, where $e(E,F) = \sup_{x\in E} d(x,F)$ is the excess of E over F and $d(x,F)$ is the distance from x to F. It is known that h becomes an extended metric on $\mathcal{P}_f(X)$ (i.e. it is a metric which can also take the value $+\infty$) and h becomes a metric (called Hausdorff) on $\mathcal{P}_{bf}(X)$ (Hu and Papageorgiou [16]).

In the sequel, $(X, \|\cdot\|)$ will be a real normed space, with the distance d induced by its norm. We denote $|E| = h(E, \{0\})$, for every $E \in \mathcal{P}_0(X)$, where 0 is the origin of X. We have $|E| = \sup_{x\in E} \|x\|$, for every $E \in \mathcal{P}_0(X)$. If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a set multifunction, then by $|\mu| : \mathcal{C} \to \overline{\mathbb{R}}_+$ we mean the set function defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$. On $\mathcal{P}_0(X)$ we consider the Minkowski addition "$\dot{+}$", defined by: $E \dot{+} F = \overline{E+F}$, for every $E, F \in \mathcal{P}_0(X)$, where $E + F = \{x + y | x \in E, y \in F\}$ and $\overline{E+F}$ is the closure of $E + F$ with respect to the topology induced by the norm of X.

Definition 2.1. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is said to be:

(i) *continuous from below* if the equality $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$ holds for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ (that is, $A_n \subseteq A_{n+1}$, for every $n \in \mathbb{N}^*$), with $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{C}$ (denote this by $A_n \nearrow A$);

(ii) *continuous from above* if the equality $\lim_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$ holds for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ (that is, $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}^*$), with $\bigcap_{n=1}^{\infty} A_n = A \in \mathcal{C}$ (denote this by $A_n \searrow A$);

(iii) *continuous* if μ is continuous from below and from above;

(iv) *monotone* if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$;

(v) *a multimeasure* if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$;

(vi) *a multisubmeasure* if μ is monotone, $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$);

(vii) *fuzzy* if $\mu(\emptyset) = \{0\}$, μ is monotone and continuous;

(viii) *countably additive* if it satisfies $\mu(\emptyset) = \{0\}$ and the equality $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ (i.e. $\lim_{n\to\infty} h(\mu(\bigcup_{n=1}^{\infty} A_n), \sum_{k=1}^{n} \mu(A_k)) = 0$) holds for every sequence of pairwise disjoint sets $(A_n) \subset \mathcal{C}$, such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$).

Remark 2.2. If μ is $\mathcal{P}_f(X)$-valued, then in Definition 2.1-(v), (vi) and (viii) it usually appears "$\dot{+}$" instead of "$+$", because the sum of two closed sets is not always closed.

3 Darboux property

In this section, several results concerning Darboux property are established for multimeasures.

Definition 3.1. Let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function. One says that:

(i) m has *the Darboux property* if for every $A \in \mathcal{C}$, with $m(A) > 0$ and every $p \in (0, 1)$, there exists $B \in \mathcal{C}$ such that $B \subseteq A$ and $m(B) = p\, m(A)$.

(ii) m is *semi-convex* if for every $A \in \mathcal{C}$, with $m(A) > 0$, there is $B \in \mathcal{C}$ such that $B \subseteq A$ and $m(B) = \frac{1}{2}\, m(A)$.

(iii) A set $A \in \mathcal{C}$ is an *atom* of m if $m(A) > 0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $m(B) = 0$ or $m(A \backslash B) = 0$.

(iv) m is *non-atomic* if it has no atoms.

Definition 3.2. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction. We say that:

(i) μ has *the Darboux property* if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$ and every $p \in (0, 1)$, there exists $B \in \mathcal{C}$ such that $B \subseteq A$ and $\mu(B) = p\, \mu(A)$.

(ii) μ is *semi-convex* if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there is $B \in \mathcal{C}$ such that $B \subseteq A$ and $\mu(B) = \frac{1}{2}\, \mu(A)$.

(iii) μ is *diffused* if $\mu(\{t\}) = \{0\}$, for any $t \in T$.

(iv) A set $A \in \mathcal{C}$ is an *atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

(v) μ is *non-atomic* if it has no atoms.

Remark 3.3.

I. If μ is monotone, then μ is non-atomic if and only if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(A \backslash B) \supsetneq \{0\}$).

II. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be an additive set function and μ be the multimeasure induced by ν (that is $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$, defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$). Then μ has the Darboux property if and only if ν has the Darboux property.

III. Evidently, if μ has the Darboux property, then it is semi-convex.

Theorem 3.4. (Gavriluţ and Croitoru [14]) *If a multimeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ has the Darboux property, then the following properties hold:*

(i) for every $Z_1, Z_2 \in \mathcal{R}(\mu) = \{\mu(A) | A \in \mathcal{C}\}$ and every $\lambda \in (0, 1)$, we have $\lambda Z_1 + (1 - \lambda) Z_2 \in \mathcal{R}(\mu)$;

(ii) $R(\mu) = \bigcup_{A\in\mathcal{C}} \mu(A)$ is convex.

Theorem 3.5. *Let $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ be a multimeasure, with the Darboux property. Then μ is non-atomic.*

Proof. Suppose that, on the contrary, there exists an atom $A \in \mathcal{C}$. Since $\mu(A) \supsetneq \{0\}$ and μ has the Darboux property, there are $p \in (0, 1)$ and $B \in \mathcal{C}$ such that $B \subseteq A$ and $\mu(B) = p\, \mu(A)$. Because A is an atom of μ, then $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

I) If $\mu(B) = \{0\}$, then $p\, \mu(A) = \{0\}$, a contradiction, because $p \in (0, 1)$ and $\mu(A) \supsetneq \{0\}$.

II) If $\mu(A\backslash B) = \{0\}$, then, since μ is a multimeasure, we have: $\mu(A) = \mu(A\backslash B) \overset{\bullet}{+} \mu(B) = \mu(B)$. So $\mu(A) = \mu(B) = p\,\mu(A)$. Since μ is $\mathcal{P}_{bf}(X)$-valued, it results $|\mu|$ is $\mathbb{R}_+$-valued, hence, in our above equality, $p = 0$ or $\mu(A) = \{0\}$, a contradiction. Therefore, μ is non-atomic. □

Theorem 3.6. *Suppose $\mathcal{C}$ is the Borel σ-algebra of $T = [0, +\infty)$ and $m : \mathcal{C} \to \mathbb{R}_+$ is a finitely additive set function satisfying the conditions:*

(i) m is continuous from below;

(ii) $m([x, y]) \leq y - x$, for any $x, y \in T$, such that $x \leq y$.

Let $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ be the set multifunction defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$. Then μ has the Darboux property.

Proof. Let $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$ and let $p \in (0, 1)$. Let $f : T \to \mathbb{R}$ be the real function defined by $f(t) = m(A \cap [0, t])$. Since m is finitely additive and according to (ii), we have:

$$|f(t) - f(t')| \leq |t - t'|, \;\; \forall t, t' \in T,$$

which implies f is continuous on T. We have $f(0) = 0$ and, since (i), it follows $\lim\limits_{t\to\infty} f(t) = m(A)$. Then there is $t_0 \in T$ such that $f(t_0) = pm(A)$. If we denote $B = A \cap [0, t_0]$, then $B \in \mathcal{C}$, $B \subseteq A$ and $m(B) = pm(A)$. This implies $\mu(B) = p\mu(A)$, so μ has the Darboux property. □

Remark 3.7. The Lebesgue measure on $[0, +\infty)$ satisfies the conditions (i) and (ii) of Theorem 3.6.

Definition 3.8. (Chiţescu [3]) A set function $m : \mathcal{C} \to \mathbb{R}_+$ is said to be *finitely purely atomic* if there is a finite, disjoint family $(A_i)_{i=\overline{1,n}}$ of atoms of m so that $T = \bigcup\limits_{i=1}^{n} A_i$.

Theorem 3.9. *Let $m : \mathcal{C} \to \mathbb{R}_+$ be a finitely additive measure and $\mu : \mathcal{C} \to \mathcal{P}_0(L^\infty(m))$ defined by $\mu(A) = [0, \chi_A]$, for every $A \in \mathcal{C}$, where χ_A is the characteristic function of A. Then μ is countably additive if and only if m is finitely purely atomic.*

Proof. If we denote $\nu(A) = \chi_A$, for every $A \in \mathcal{C}$, then $\nu : \mathcal{C} \to L^\infty(m)$ is a finitely additive measure. According to Chiţescu [3], ν is countably additive if and only if m is finitely purely atomic. Now, we remark that μ is countably additive if and only if ν is countably additive and the theorem is proved. □

In what follows, let T be a Hausdorff locally compact space, $\mathcal{B}_0$ the Baire δ-ring generated by the G_δ-compact subsets of T (that is, compact sets which are countable intersections of open sets) and $\mathcal{B}$ the Borel δ-ring generated by the compact subsets of T.

Theorem 3.10.

I. If $\mu : \mathcal{B}_0 \to \mathcal{P}_{bfc}(X)$ is a semi-convex fuzzy multimeasure, then $\forall\, t \in T, \exists\, A_t \in \mathcal{B}_0$, such that $t \in A_t$ and $\mu(A_t) = \{0\}$.

II. If $\mu : \mathcal{B} \to \mathcal{P}_{bfc}(X)$ is a semi-convex fuzzy multimeasure, then μ is diffused.

Proof. I. Consider an arbitrary $t \in T$. We prove there exists $A_t \in \mathcal{B}_0$ so that $t \in A_t$.

Since T is a Hausdorff locally compact space, then for t there exists a compact neighgbourhood V_t, for which there is a relatively compact, open set D_t so that $V_t \subseteq D_t$.

By Gavriluţ [11], there exists $A_t \in \mathcal{B}_0$ so that $V_t \subseteq A_t \subseteq D_t$. Evidently, $t \in A_t$.

Let now $A = A_t \in \mathcal{B}_0$ be so that $t \in A$.

If $\mu(A) = \{0\}$, the proof is finished.

If $\mu(A) \supsetneq \{0\}$, since μ is semi-convex, there exists $A_1 \in \mathcal{B}_0$ so that $A_1 \subseteq A$ and $\mu(A_1) = \frac{1}{2}\mu(A)$. Then

$$\mu(A) = \frac{1}{2}\mu(A) \overset{\bullet}{+} \mu(A\backslash A_1) = \frac{1}{2}\mu(A) \overset{\bullet}{+} \frac{1}{2}\mu(A).$$

Using the cancellation law in $\mathcal{P}_{bfc}(X)$ (Godet-Thobie [15]), we get that $\mu(A\backslash A_1) = \frac{1}{2}\mu(A)$. Let $B_1 = A_1$ or $B_1 = A\backslash A_1$ be so that $t \in B_1$. Evidently, $\mu(B_1) = \frac{1}{2}\mu(A)$. We construct by induction a decreasing sequence of sets $(B_n)_n \subset \mathcal{B}_0$ such that $\mu(B_n) = \frac{1}{2^n}\mu(A), B_n \subseteq A$ and $t \in B_n$, for every $n \in \mathbb{N}^*$. We suppose that we have already obtained $B_1, B_2, ...B_n$. We construct B_{n+1} :

Because μ is semi-convex, there exists $A_{n+1} \in \mathcal{B}_0$ so that $A_{n+1} \subseteq B_n$ and

$$\mu(A_{n+1}) = \frac{1}{2}\mu(B_n) = ... = \frac{1}{2^{n+1}}\mu(A).$$

Then

$$\mu(B_n\backslash A_{n+1}) = \frac{1}{2}\mu(B_n) = \frac{1}{2^{n+1}}\mu(A).$$

Let $B_{n+1} = A_{n+1}$ or $B_{n+1} = B_n\backslash A_{n+1}$ be so that $t \in B_{n+1}$. Then $\mu(B_{n+1}) = \frac{1}{2^{n+1}}\mu(A)$. We consider $B = \bigcap\limits_{n=1}^{\infty} B_n$. Obviously, $t \in B, B \in \mathcal{B}_0$ and, for every $n \in \mathbb{N}^*$, we have

$$|\mu(B)| \leq h(\mu(B_n), \mu(B)) + |\mu(B_n)| =$$

$$= h(\mu(B_n), \mu(B)) + \frac{1}{2^n}|\mu(A)|.$$

Since μ is $\mathcal{P}_{bfc}(X)$-valued, it results $|\mu(A)| \in \mathbb{R}$. Also, because μ is continuous from above, we have

$\lim_{n\to\infty} h(\mu(B_n), \mu(B)) = 0$. Therefore, $|\mu(B)| = 0$, hence $\mu(B) = \{0\}$. So, for every $t \in T$, there exists $B \in \mathcal{B}_0$ so that $t \in B$ and $\mu(B) = \{0\}$, as claimed.

II. If μ is defined on $\mathcal{B}$, then $A_t = \{t\}$ and the conclusion follows. □

Corollary 3.11. *Let $\mathcal{C} = \mathcal{B}_0$ (or $\mathcal{B}$). If $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ is a fuzzy multimeasure having the Darboux property, then $\forall t \in T, \exists A_t \in \mathcal{C}$, such that $t \in A_t$ and $\mu(A_t) = \{0\}$.*

Since every monotone multimeasure is a multisubmeasure, by Gavriluţ [11] we get the following:

Theorem 3.12. (Gavriluţ [11]) *Consider $A \in \mathcal{B}$ with $\mu(A) \supsetneq \{0\}$ and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ a fuzzy multimeasure.*

I. If A is an atom of μ, then there is a compact set $K_0 \in \mathcal{B}$ so that $K_0 \subseteq A$ and $\mu(A \backslash K_0) = \{0\}$.

II. A is an atom of μ if and only if $\exists! a \in A$ such that $\mu(A \backslash \{a\}) = \{0\}$.

III. μ is non-atomic if and only if μ is diffused.

4 Regular fuzzy multimeasures

In this section, we study some properties of regular fuzzy multimeasures. In the sequel, T will be a Hausdorff locally compact space.

Definition 4.1. A set function $m : \mathcal{C} \to \mathbb{R}_+$ is said to be *regular* if for every $\varepsilon > 0$, there exist a compact set $K \subseteq A$, $K \in \mathcal{C}$ and an open set $D \supseteq A$, $D \in \mathcal{C}$ such that $m(D \backslash K) < \varepsilon$.

Definition 4.2. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be *regular* if for every $\varepsilon > 0$, there exist a compact set $K \subseteq A$, $K \in \mathcal{C}$ and an open set $D \supseteq A$, $D \in \mathcal{C}$ such that $|\mu(D \backslash K)| < \varepsilon$.

Remark 4.3.

I. If $m : \mathcal{C} \to \mathbb{R}_+$ is regular, then the following set multifunctions $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ are also regular:

$\mu(A) = [0, m(A)]$ and $\mu(A) = \{m(A)\}$, for every $A \in \mathcal{C}$.

II. (Gavriluţ [11]) Let $\mathcal{C} = \mathcal{B}_0$ (or $\mathcal{B}$) and $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(\mathbb{R})$ be the monotone multimeasure generated by a regular σ-finite measure $m : \mathcal{C} \to \mathbb{R}_+$ (i.e. $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$). The following statements are equivalent:

(i) μ has the Darboux property;

(ii) μ is non-atomic;

(iii) $\forall t \in T, \exists A_t \in \mathcal{C}$ such that $t \in A_t$ and $\mu(A_t) = \{0\}$.

Since every Baire measure $m : \mathcal{B}_0 \to \mathbb{R}_+$ is regular (Gavriluţ [11]), Remark 4.3-II is still valid if $\mu : \mathcal{B}_0 \to \mathcal{P}_{bfc}(\mathbb{R})$ is the monotone multimeasure generated by a σ-finite measure $m : \mathcal{B}_0 \to \mathbb{R}_+$.

Remark 4.4. (Gavriluţ [10]) If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a regular monotone multimeasure, then it is continuous from above.

Theorem 4.5. *If $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a diffused regular monotone multimeasure, then $\mu_{/\mathcal{B}_0}$ satisfies the condition: $\forall t \in T, \exists A_t \in \mathcal{B}_0$ such that $t \in A_t$ and $\mu(A_t) = \{0\}$.*

Proof. Because for every $t \in T$, $\{t\}$ is compact, by the regularity of μ, for every $n \in \mathbb{N}^*$, there is an open set $\widetilde{D}_n \supseteq \{t\}$, $\widetilde{D}_n \in \mathcal{B}$ such that $|\mu(\widetilde{D}_n \backslash \{t\})| < \frac{1}{n}$. Since for every $n \in \mathbb{N}^*$, $\{t\} \subseteq \widetilde{D}_n$, according to Gavriluţ [11], there exists an open set $D'_n \in \mathcal{B}_0$ so that $\{t\} \subseteq D'_n \subseteq \widetilde{D}_n$. Because $|\mu(\widetilde{D}_n \backslash \{t\})| < \frac{1}{n}$, we get $|\mu(D'_n \backslash \{t\})| < \frac{1}{n}$, for every $n \in \mathbb{N}^*$. So, since μ is diffused, we have:

$$\begin{aligned} |\mu(D'_n)| &= h(\mu(D'_n), \{0\}) = h(\mu(D'_n), \mu(\{t\})) = \\ &= h(\mu(D'_n \backslash \{t\}) \overset{\bullet}{+} \mu(\{t\}), \mu(\{t\})) \leq \\ &\leq |\mu(D'_n \backslash \{t\})| < \frac{1}{n}, \end{aligned}$$

for every $n \in \mathbb{N}^*$.

Denote $D_n = \bigcap_{i=1}^{n} D'_i$, for every $n \in \mathbb{N}^*$ and let $D = \bigcap_{n=1}^{\infty} D'_n$. Then $D_n \in \mathcal{B}_0$, for every $n \in \mathbb{N}^*$, $D \in \mathcal{B}_0$, $D_n \searrow D$ and $|\mu(D_n)| \leq |\mu(D'_n)| < \frac{1}{n}$, for every $n \in \mathbb{N}^*$. Also, $t \in D$. It only remains to prove that $\mu(D) = \{0\}$. Indeed, since μ is continuous from above, $\mu(D) = \lim_{n\to\infty} \mu(D_n)$ with respect to h.

On the other hand, for every $n \in \mathbb{N}^*$, we have:

$$\begin{aligned} |\mu(D)| &\leq h(\mu(D), \mu(D_n)) + |\mu(D_n)| < \\ &< h(\mu(D), \mu(D_n)) + \frac{1}{n}. \end{aligned}$$

Consequently, $|\mu(D)| = 0$, so $\mu(D) = \{0\}$, as claimed. □

Corollary 4.6. *Let $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ be a regular monotone multimeasure. Then μ is diffused on $\mathcal{B}$ if and only if for every $t \in T$, there is $A_t \in \mathcal{B}_0$, such that $t \in A_t$ and $\mu(A_t) = \{0\}$.*

Concluding remarks. In this paper, we present some results regarding the Darboux property for multimeasures. We give some relationships among the Darboux property, non-atomicity and continuity of set multifunctions. Also, we study some properties of regular fuzzy multimeasures.

References:

[1] Aumann, R.J., Shapley, L.S. – *Values of Non-atomic Games*, Princeton University Press, Princeton, New Jersey, 1974.

[2] Bandyopadhyay, U. - *On vector measures with the Darboux property*, Quart. J. Oxford Math. (1974), 57-61.

[3] Chiţescu, I. - *Finitely purely atomic measures and Lp-spaces*, An. Univ. Bucureşti Şti. Natur. 24 (1975), 23-29.

[4] Chiţescu, I. - *Finitely purely atomic measures: coincidence and rigidity properties*, Rend. Circ. Mat. Palermo (2) 50 (2001), no. 3, 455-476.

[5] Choquet, G. – *Theory of capacities*, Ann. Inst. Fourier (Grenoble), 5 (1953-1954), 131-292.

[6] Dempster, A.P. – *Upper and lower probabilties induced by a multivalued mapping*, Ann. Math. Statist. 38(1967), 325-339.

[7] Denneberg, D. – *Non-additive Measure and Integral*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1994.

[8] Dobrakov, I. - *On submeasures, I,* Dissertationes Math. 112 (1974), 5-35.

[9] Gavriluţ, A. - *Properties of regularity for multisubmeasures*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Tomul L, f.2 (2004), 373-392.

[10] Gavriluţ, A. – *Regularity and o-continuity for multisubmeasures*, An. Ştiinţ. Univ. "Al.I. Cuza" Iaşi, Tomul L, f.2 (2004), 393-406.

[11] Gavriluţ, A. - *Non-atomicity and the Darboux property for fuzzy and non-fuzzy Borel/Baire multivalued set functions*, Fuzzy Sets and Systems (2008), doi: 10.1016/j.fss.2008.06.009.

[12] Gavriluţ, A., Croitoru, A. - *Fuzzy multisubmeasures and applications*, the 9th International Conference on Fuzzy Systems (FS'08), WSEAS, Sofia, Bulgaria, 2008.

[13] Gavriluţ, A., Croitoru, A. - *Non-atomicity for fuzzy and non-fuzzy multivalued set functions*, Fuzzy Sets and Systems (2008), doi: 10.1016/j.fss.2008.11.023.

[14] Gavriluţ, A., Croitoru, A. – *Pseudo-atoms and Darboux property for set multifunctions*, submitted for publication.

[15] Godet-Thobie, C. – *Multimesures et multimesures de transition*, Thèse de Doctorat, Univ. des Sci. et Techn. du Languedoc, Montpellier, 1975.

[16] Hu, S., Papageorgiou, N. S. – *Handbook of Multivalued Analysis*, vol. I, Kluwer Acad. Publ., Dordrecht, 1997.

[17] Klimkin, V.M., Svistula, M.G. - *Darboux property of a non-additive set function*, Sb. Math. 192 (2001), 969-978.

[18] Liginlal, D., Ow, T.T. – *Modelling attitude to risk in human decision process: An application of fuzzy measures*, Fuzzy Sets and Systems 157(2006), 3040-3054.

[19] Mastorakis, N.E. – *Fuzziness in Multidimensional Systems*, Computational Intelligence and Applications, WSEAS-Press, 1999, 3-11.

[20] Olejček, V. - *Darboux property of regular measures*, Mat. Cas. 24 (1974), no 3, 283-288.

[21] Pap, E. - *Null-additive Set Functions*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[22] Pham, T.D., Brandl, M., Nguyen, N.D., Nguyen, T.V. – *Fuzzy measure of multiple risk factors in the prediction of osteoporotic fractures*, Proceedings of the 9th WSEAS International Conference on Fuzzy Systems [FS'08], Sofia, Bulgaria, May 2-4, 2008, 171-177.

[23] Precupanu, A.M. - *On the set valued additive and subadditive set functions*, An. Ştiinţ. Univ. "Al.I. Cuza" Iaşi, 29 (1984), 41-48.

[24] Shafer, G. – *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, N.J., 1976.

[25] Sugeno, M. - *Theory of fuzzy integrals and its applications*, Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[26] Suzuki, H. – *Atoms of fuzzy measures and fuzzy integrals*, Fuzzy Sets and Systems 41 (1991), 329-342.

[27] Zadeh, L.A. – *Fuzzy sets*, Information and Control 8 (1965), 338-353.