Ramification of surfaces: sufficient jet order for wild jumps

Igor Zhukov

February 8, 2008

There exist different approaches to ramification theory of n-dimensional schemes with $n \geq 2$. One of these approaches is based on the following idea: to reduce the situation of an n-dimensional scheme $\mathcal{X}$ to a number of 1-dimensional settings, just by restricting to curves C properly crossing the ramification subscheme $R \subset \mathcal{X}$. This way seems to be very natural; however, it has not got much attention, except for [D] and [Br]. With the present paper we intend to initiate the systematic development of this idea.

Let us be more precise. Let L/K be a finite Galois extension of the function field of a connected normal n-dimensional scheme $\mathcal{X}$, and let $f : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ in L. Denote by $R \subset \mathcal{X}$ the reduced branch locus of this morphism. Let C be a curve, i. e., a closed integral 1-dimensional subscheme of $\mathcal{X}$. Since f is finite, there are several curves on $\mathcal{Y}$ lying over C; let D be any of them. Then the natural morphism $D \to C$ has familiar ramification invariants that can be non-vanishing only at the points where C meets R. The principle is to collect these invariants for all regular curves on C that are distinct from the components of R and to consider the data obtained this way as a system of invariants of f.

One of the distant goals is to develop a sufficiently rich system of ramification invariants that enables one to express numerous known ramification data in terms of this system. On the other hand, we hope to obtain a system of invariants with nice functorial properties. For the latter reason, we do not require C to be transversal to R, and this is a new instant in this paper. It seems that the consideration of curves tangent to the branch divisor gives deep information; this can be observed from consideration of Artin-Schreier extensions and their composites. (Note that for 1-dimensional schemes, Swan characters at ramified points form a system of invariants that enjoy both properties: it is sufficiently rich and functorial.)

In the present paper we restrict the setting as follows:

- $n = 2$;
- $\mathcal{X}$ is equicharacteristic, i. e., it can be equipped with a morphism to $\operatorname{Spec} k$, where k is an algebraically closed field of prime characteristic p;
- If P is any closed point of $\mathcal{X}$, the local ring $\mathcal{O}_{\mathcal{X},P}$ is a 2-dimensional excellent local ring with the residue field k;
- $\mathcal{X}$ is regular;
- L/K is solvable.

In section 2 we state a series of questions related to the behavior of ramification jumps as one varies the curve C. Some of these questions are new and some generalize the questions considered in the above-mentioned papers.

In the following sections we answer the first of these questions affirmatively; this is the main result of the paper. Namely, we prove that the wild ramification jumps of the extension of the function field of C determined by L/K at a point $P \in C \cap R$ depend only on the jet of C at P of certain order, and we can bound this order in a certain uniform way (see Corollary 4.1.1). Here $C \subset \mathcal{X}$ is a curve which is regular at P. Roughly speaking, if two curves C and C' have sufficiently high order of tangence at P, then the wild jumps for C and C' are the same. By the wild jumps we mean the usual ramification jumps (in lower numbering) divided by the index of tame ramification.

The proof of the main result is based on the work with arcs (parameterized algebroid curves) on $\mathcal{X}$, including the singular ones. One of the central ingredients is a comparison of invariants of singularity of an arc on $\mathcal{X}$ and those of an arc on $\mathcal{Y}$ above it in the case of cyclic L/K of prime degree; see Propositions 5.5 and 6.3.

It would be important to prove a stronger fact: not only the jumps but the whole ramification filtration is the same for C and C'. This statement (for curves transversal to R) is a step in Deligne's program [D] describing how to compute Euler-Poincaré characteristics of constructible étale sheaves on surfaces.

At present, we are able to prove the above fact for some class of Galois groups that includes, in particular, all abelian p-groups, see Theorem 9.1.

As for the other questions from section 2, in the case of Artin-Schreier extensions we can answer most of them affirmatively. This is the subject of another paper [Z].

I would like to thank P. Deligne, L. Illusie, G. Laumon, A. N. Parshin for the discussions that gave me inspiration for this work and for the whole program. I am sincerely grateful to I. B. Fesenko, B. Köck, T. Saito for numerous discussions and also for the fact that they found errors in earlier versions of the paper and gave me many suggestions on the improvement of exposition. I would like to thank M. V. Bondarko, N. L. Gordeev and I. A. Panin who answered my questions in commutative algebra and algebraic geometry that arose in the course of this work. I am deeply thankful to V. P. Snaith for the invitation to Southampton, where a big part of the work was done, as well as for showing me a lot of beautiful mathematics and for his all-embracing help during my visits to Southampton.

I acknowledge the hospitality of Southampton University and of Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and the financial support from Royal Society and from RFBR (projects 00-01-00140 and 01-01-00997.)

1 Terminology, notation, preliminary facts

General notation

k is always an algebraically closed field of characteristic $p > 0$.

For any commutative ring A denote

$$\operatorname{Spec}_1 A = \{\mathfrak{p} \in \operatorname{Spec} A | \operatorname{height} \mathfrak{p} = 1\}.$$

We denote by v_X the valuation in the discrete valuation ring $k[[X]]$.

For a prime ideal $\mathfrak{p} \in \operatorname{Spec}_1 A$, denote by $F_\mathfrak{p}$ the prime divisor $\operatorname{Spec}(A/\mathfrak{p})$. If $\mathfrak{p}$ is a principal ideal (t), we write F_t instead of $F_{(t)}$.

Let A be an equal characteristic regular 2-dimensional local ring with the maximal ideal $\mathfrak{m} = (T, U)$ and the residue field k, N a positive integer. We have a canonical isomorphism between the completion of A and $k[[T, U]]$. A map $\lambda : k = A/\mathfrak{m} \to A$ is said to be a section of level N if the diagram

$$\begin{array}{ccc} k & \xrightarrow{\lambda} & A \\ \downarrow & & \downarrow \\ k[[T,U]]/(T,U)^N & = & A/\mathfrak{m}^N \end{array}$$

commutes.

For any local ring A, we denote the completion of A by $\widehat{A}$.

The field of functions on an integral scheme S is denoted by $k(S)$.

Invariants of wild ramification

Let K be a complete discrete valuation field with perfect residue field of characteristic p, and let L/K be a finite Galois extension. Let v be the valuation on L and $\mathcal{O}_L = \{a \in L | v_L(a) \geq 0\}$.

Recall that for any integral $i \geq -1$ the ith ramification subgroup in the group $\operatorname{Gal}(L/K)$ is defined as

$$G_i = G_i(L/K) = \{g \in \operatorname{Gal}(L/K) | \forall a \in \mathcal{O}_L : v(g(a) - a) \geq i + 1\}.$$

We have $|G_0| = e_{L/K}$, the ramification index, whereas $|G_1|$ is the wild ramification index, i. e., the maximal power of p dividing $e_{L/K}$. One can also define the tame ramification index

$$e^t(L/K) = (G_0 : G_1).$$

Now we introduce the ramification jumps, numbered from the bigger to the smaller ones. Namely, for any $i \geq 1$, denote

$$h^{(i)} = h^{(i)}(L/K) = \min\{j \geq 1 | p^i \text{ does not divide } |G_j|\} - 1.$$

This definition takes into account the multiplicities of jumps. For example, if $G_a = G_1$ is of order p^2 for some $a \geq 1$ and G_j is trivial for any $j > a$, then $h^{(1)} = h^{(2)} = a$, and $h^{(i)} = 0$ for $i \geq 3$. We define also modified ("wild") ramification jumps $w^{(i)}(L/K) = h^{(i)}(L/K)/e^t(L/K)$, $i = 1, 2, \ldots$. Note that $w^{(i)}(L/K)$ are non-negative and integral in view of [S, Ch. IV, Cor. 1 to Prop. 9].

Next, we define the modified Hasse-Herbrand function $W_{L/K} : [1, \infty) \to [1, \infty)$ as

$$W_{L/K}(u) = \int_1^u \frac{dt}{(G_1 : G_t)} = \frac{\varphi_{L/K}(u)}{e^t(L/K)},$$

where $G_u = G_i$, i is the minimal integer such that $i \geq u$. Here $\varphi_{L/K}$ is the usual Hasse-Herbrand function as defined in [S, Ch. IV, §3].

[S, Ch. IV, Prop. 15] implies that for a Galois subextension M/K in L/K we have $W_{L/K} = W_{M/K} \circ W_{L/M}$.

The following obvious property justifies the introduction of modified jumps and modified Hasse-Herbrand function.

1.1 Lemma. *Let K'/K be a tamely ramified extension. Then $w^{(i)}(K'L/K') = w^{(i)}(L/K)$ for any i, and $W_{K'L/K'} = W_{L/K}$.*

Ramification of surfaces

A *surface* over k is a connected normal 2-dimensional scheme $\mathcal{X}$ with a morphism $\mathcal{X} \to \operatorname{Spec} k$ which induces an isomorphism of residue fields at all points of codimension 2 in $\mathcal{X}$.

Let C, C' be distinct prime divisors on a surface $\mathcal{X}$, and let P be a closed point of $\mathcal{X}$. We define the intersection multiplicity of C and C' at P as

$$(C.C')_P = \begin{cases} \dim_k \mathcal{O}_{\mathcal{X},P}/(\mathfrak{p}+\mathfrak{p}'), & P \in C \cap C', \\ 0, & P \notin C \cap C', \end{cases}$$

where $\mathfrak{p}$ and $\mathfrak{p}'$ are the prime ideals of $\mathcal{O}_{\mathcal{X},P}$ corresponding to C and C'. By linearity this definition can be extended to any two divisors C, C' with no common components.

Let $\mathcal{X}$ be a surface with the function field K. Let L/K be a finite separable extension, $\mathcal{Y}$ the normalization of $\mathcal{X}$ in L. We denote by $R_{L/K,\mathcal{X}} \subset \mathcal{X}$ the reduced branch locus of the corresponding finite morphism $\varphi : \mathcal{Y} \to \mathcal{X}$, i. e., $R_{L/K,\mathcal{X}} = \varphi(\operatorname{Supp} \Omega^1_{\mathcal{Y}/\mathcal{X}})$. If $\mathcal{X}$ is regular, then R is of pure dimension 1. This follows from the theorem of the purity of the branch locus in [N, §41].

The extension L/K is said to be *tame* (with respect to $\mathcal{X}$) if L/K is tamely ramified with respect to any extension of any discrete valuation of K associated with a prime divisor of $\mathcal{X}$. Similarly one defines *unramified* extensions.

A monoidal transformation means blowing up of a closed point. The following fact is not difficult to prove. (Note that $R_{L/K,\mathcal{X}_n}$ is a part of the total transform of $R_{L/K,\mathcal{X}}$.) This proposition is believed to be standard, though we have not found a suitable reference.

1.2 Proposition. *Let $\mathcal{X}$ be a regular surface over k, L/K a finite extension of its fraction field. Then there exists a sequence of monoidal transformations $\mathcal{X}_n \to \mathcal{X}$ such that $R_{L/K,\mathcal{X}_n}$ is a simply normal crossing divisor.*

From now on, let L/K be also normal.

Let C be a prime divisor on $\mathcal{X}$ such that $C \not\subset R_{L/K,\mathcal{X}}$. We can introduce D_{C_1}, the decomposition subgroup in $\operatorname{Gal}(L/K)$ at C_1, where C_1 is a component of $\varphi^{-1}(C)$. Since L/K was not ramified at the generic point of C, the group D_{C_1} can be identified with $\operatorname{Gal}(k(C_1)/k(C))$.

Next, let P be a regular point on C. Let v be the corresponding valuation on $k(C)$. Fix any extension v_1 of v onto the field $k(C_1)$. Let $K_{C,v}$ be the completion of $k(C)$ with respect to v, and let L_{C_1,v_1} be the completion of $k(C_1)$ with respect to v_1. Then the decomposition group D_{C_1,v_1} of v_1 can be identified with $\operatorname{Gal}(L_{C_1,v_1}/K_{C,v})$.

Now we introduce

$$w^{(i)}_{C,P} = w^{(i)}_{C,P}(L/K) = w^{(i)}(L_{C_1,v_1}/K_{C,v})$$

and

$$W_{C,P,L/K} = W_{L_{C_1,v_1}/K_{C,v}}.$$

These objects do not depend on the choice of C_1 over C and of v_1 since $\mathrm{Gal}(L/K)$ acts transitively on the set of such C_1, and $\mathrm{Gal}(k(C_1)/k(C))$ acts transitively on the set of possible v_1.

Let C and C' be prime divisors on a surface $\mathcal{X}$ and let P be a closed point in $C \cap C'$ such that both C and C' are regular at P. We say that L/K has *equal wild jumps with respect to C and C' at P*, if $w^{(i)}_{C,P}(L/K) = w^{(i)}_{C',P}(L/K)$ for any i. We say that L/K is *equally ramified with respect to C and C' at P in the strong sense* if for any C_1 and v_1 one can choose C'_1 and v'_1 such that $G_j(L_{C_1,v_1}/K_{C,v}) = G_j(L_{C'_1,v'_1}/K_{C',v'})$ for any $j \ge -1$.

Let M be a finite-dimensional representation of $\mathrm{Gal}(L/K)$ (over any field F, $\mathrm{char}\, F \ne p$). Denote by $\mathrm{Sw}_{C,P}(M)$ the Swan conductor of the restriction of this representation on the subgroup D_{C_1,v_1}. In other words,

$$\mathrm{Sw}_{C,P}(M) = \sum_{i=1}^{\infty} \frac{\dim_F (M/M^{G_i(L_{C_1,v_1}/K_{C,v})})}{(G_0(L_{C_1,v_1}/K_{C,v}) : G_i(L_{C_1,v_1}/K_{C,v}))}. \qquad (1)$$

It is easy to see from this formula that $\mathrm{Sw}_{C,P}(M)$ does not depend on the choice of C_1 over C and of v_1.

2 Questions

Let $\mathcal{X}$ and L/K be as in the introduction. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in L. Denote $R = R_{L/K,\mathcal{X}}$.

For a closed point P of $\mathcal{X}$ denote by U_P the set of all prime divisors C of $\mathcal{X}$ such that C is regular at P and $C \not\subset R$.

2.1 Question. (existence of a uniform sufficient jet order) *Does there exists an effective divisor R_0 supported at R such that for any $P \in R$ and any $C, C' \in U_P$ the condition $(C.C')_P \ge (C.R_0)_P$ implies that L/K has equal wild jumps with respect to C and C' at P?*

If the answer is positive, we have $R_0 = \sum_{i=1}^{s} m_i C_i$, where $C_1 \ldots, C_s$ are all prime components of R. Then we say that m_i is a uniform sufficient jet order at C_i, $i = 1, \ldots, s$.

To simplify the statements of the other questions, we make two technical assumptions.

(i) $\mathcal{X}$ is local, i. e., $\mathcal{X} = \mathrm{Spec}\, A$, where A is a 2-dimensional local ring. This enables us to write $(C.C')$ instead of $(C.C')_{(0)}$. We shall also abbreviate $w^{(i)}_{F_\mathfrak{p}}$ as $w^{(i)}_\mathfrak{p}$.

(ii) R is a simply normal crossing divisor in $\mathcal{X}$. (In particular, the components of R are regular.)

Since R is a simply normal crossing divisor, it consists of at most 2 irreducible components. We shall not consider the case $R = \emptyset$. If there is 1 component (resp. 2 components), we denote it by C_1 (resp. by C_1 and C_2); in the former case choose any regular C_2 with $(C_1.C_2) = 1$.

In both cases $(C_1.C_2) = 1$; $C_1 = F_{\mathfrak{p}_1}$, $C_2 = F_{\mathfrak{p}_2}$; we may introduce regular local parameters T and U in A such that $\mathfrak{p}_1 = (T)$, $\mathfrak{p}_2 = (U)$.

Denote

$$U_A = \{\mathfrak{p} \in \mathrm{Spec}_1\, A | A/\mathfrak{p} \text{ is regular, } F_\mathfrak{p} \not\subset R\}.$$

Let $\mathfrak{p} \in U_A$. The set

$$J_m(\mathfrak{p}) = \{\mathfrak{p}' \in U_A | (F_{\mathfrak{p}}.F_{\mathfrak{p}'}) \geq m\}$$

is said to be the jet of $\mathfrak{p}$ of order m. Also we introduce

$$T_r = \{\mathfrak{p} \in U_A | (F_{\mathfrak{p}}.C_1) = r, (F_{\mathfrak{p}}.R) \leq r+1\}$$

and

$$T_{r,m} = \{J_m(\mathfrak{p}) | \mathfrak{p} \in T_r\}.$$

2.1.1 Remark. We mentioned R in the definition of T_r in order to exclude the curves tangent to C_2 from T_1 in the case of two-component branch divisor.

For the following question we have to endow the sets $T_{r,m}$ with the structure of affine varieties.

Fix a positive integer n and a section $\lambda : A/\mathfrak{m} \to A$ of level n.

Let $\mathfrak{p} \in T_1$ (i. e., this is the ideal of the germ of a curve transversal to all components of R). Then $\mathfrak{p} = (f)$, where $f \equiv -U \mod (T, U^2)$. Without loss of generality, we may assume

$$f \equiv -U + \lambda(\alpha_1)T + \cdots + \lambda(\alpha_n)T^n \mod \deg n+1,$$

where $\alpha_1, \ldots, \alpha_n \in k$ are determined uniquely by $\mathfrak{p}$. (This follows from Weierstraß preparation theorem.) Thus, if $R = C_1$, we can identify $T_{1,n}$ with the set of closed points of $\mathbb{A}_k^n$ via:

$$(\alpha_1, \ldots, \alpha_n) \mapsto J_n((-U + \lambda(\alpha_1)T + \cdots + \lambda(\alpha_n)T^n)).$$

In the case $R = C_1 + C_2$, the same map identifies $T_{1,n}$ with the set of the closed points of $(\mathbb{A}_k^n)_{x_1 \neq 0}$ Observe that $\alpha_1, \alpha_2, \ldots$ are in fact the coefficients in the expansion

$$u = \alpha_1 t + \alpha_2 t^2 + \ldots,$$

where t and u are the images of T and U in the completion $\widehat{(A/\mathfrak{p})} \simeq k[[t]]$. Therefore, $\alpha_1, \alpha_2, \ldots$ are independent of n.

Similarly, if $\mathfrak{p} \in T_r$, $r \geq 2$, we have $\mathfrak{p} = (f)$,

$$f \equiv -T + \lambda(\beta_r)U^r + \cdots + \lambda(\beta_n)U^n \mod \deg n+1.$$

We have a bijection

$$\begin{aligned} (\mathbb{A}_k^{n+1-r})_{x_1 \neq 0} &\to T_{r,n}, \\ (\beta_r, \ldots, \beta_n) &\mapsto J_n((-T + \lambda(\beta_r)U^r + \cdots + \lambda(\beta_n)U^n)). \end{aligned}$$

As in the previous case, $\beta_1, \beta_2, \ldots$ are independent of n.

2.2 Question. (semi-continuity of a jump) *Let m_1, m_2 be uniform sufficient jet orders at C_1, C_2, fix any positive integers r, i, and $m \geq rm_1 + m_2$. Does the set*

$$\{J_m(\mathfrak{p}) | \mathfrak{p} \in T_r;\ w_{\mathfrak{p}}^{(i)}(L/K) \leq s\}$$

form a closed subset in $T_{r,m}$ for any $s \geq 0$?

2.3 Question. (generic value of a jump) *Fix any r and i. Is*

$$w_r^{(i)}(L/K) = \sup\{w_{\mathfrak{p}}^{(i)}(L/K) | \mathfrak{p} \in T_r\}$$

finite?

2.4 Question. (asymptotic of jumps) *Is the sequence $(w_r^{(i)}(L/K)/r)_r$ (for any fixed i) convergent and bounded by its limit?*

2.4.1 Remark. Consideration of Artin-Schreier extensions suggests affirmative answers to all these questions, see [Z].

Representation version

One can state similar questions where $w_{C,P}^{(i)}(L/K)$ is replaced with $\mathrm{Sw}_{C,P}(M)$, starting with

2.5 Question. (existence of a uniform sufficient jet order for Swan conductor) *Let M be a finite-dimensional representation of* $\mathrm{Gal}(L/K)$ *over any field F,* $\mathrm{char}\, F \neq p$. *Does there exists an effective divisor R_0 supported at R such that for any $P \in R$ and any $C, C' \in U_P$ the condition $(C.C')_P \geq (C.R_0)_P$ implies* $\mathrm{Sw}_{C,P}(M) = \mathrm{Sw}_{C',P}(M)$*?*

2.5.1 Remark. To answer this question affirmatively, it is sufficient, in view of (1), to prove the existence of uniform sufficient jet order for equal ramification in the strong sense.

2.5.2 Remark. Versions of some of the questions, in terms of Swan (or Artin) conductors, were considered in [D] and [Br]. Brylinski gives in [Br] the affirmative answer to Question 2.5 for cyclic p-extensions under the following assumptions:

- $\mathcal{X}$ is a smooth algebraic surface;
- The branch divisor is a regular curve D;
- If a character of an extension corresponds to the Witt vector $(x_0, \ldots, x_{n-1})$, then all x_i have neither poles nor zeroes other than D in the neighborhood of P (it follows that P is a non-exceptional point on the branch divisor in Deligne's terminology);
- $(C.D)_P = 1$ (i. e., for curves transversal to the branch divisor; Brylinski states that this condition is not necessary).

In the case $(C.D)_P = 1$ Brylinski computes the sufficient jet order and the generic value of Artin conductor in terms of Kato-Swan conductor.

2.5.3 Remark. A representation version of Question 2.4, after extension to non-local $\mathcal{X}$, includes the existence of

$$\mathrm{Sw}_\infty(M; D) = \lim_{r\to\infty} \frac{\mathrm{Sw}_r(M; D)}{r},$$

where D is any prime component of R, and $\mathrm{Sw}_r(M; D)$ is the generic value of $\mathrm{Sw}_{C,P}(M)$ over the $P \in D$ and $C \in U_P$ with $(C.D)_P = r$.

We expect that $\mathrm{Sw}_\infty(M; D)$ is an important invariant of ramification of M at the place D, generalizing Kato-Swan conductor, in a sense. However, at the moment we cannot suggest a precise statement.

3 Arcs

Here an arc is a parameterized algebroid curve drawn on a scheme. Consideration of arcs is the main tool to answer Question 2.1. We use [C] as a convenient reference source for basic facts about algebroid curves.

The terminology and notation are partially standard (see [C]) and partially customized for our needs.

Algebroid curves

An (irreducible) *algebroid curve* over k is a complete Noetherian 1-dimensional local domain $\mathcal{R}$ with a coefficient subfield k. By abuse of language, $\operatorname{Spec}\mathcal{R}$ is also said to be an algebroid curve.

Let $\mathcal{R}$ be an algebroid curve; the integral closure of $\mathcal{R}$ is a complete discrete valuation ring. Let v be the corresponding valuation.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{R}$. An element $x \in \mathfrak{m}$ is said to be a *transversal parameter* of $\mathcal{R}$, if $x + \mathfrak{m}^2$ is not nilpotent in $\operatorname{gr}_{\mathfrak{m}} \mathcal{R}$. (According to [C], this is equivalent to the following characterization property: $v(x)$ is minimal in $\mathfrak{m}$.)

The *multiplicity* of $\mathcal{R}$ is the only $e = e(\mathcal{R})$ such that $\dim_k(\mathcal{R}/\mathfrak{m}^n) = en+q$ for n large enough. It is known [C, Th. 1.4.7] that $e(\mathcal{R}) = v(u)$ for any transversal parameter u.

Obviously, $e(\mathcal{R}) = 1$ iff $\mathcal{R}$ is a regular ring.

We recall the definition of (strict) quadratic transform of an algebroid curve from [C].

Let $\mathcal{R}$ be an irreducible algebroid curve over k, $\mathfrak{m}$ the maximal ideal of $\mathcal{R}$. For $x \in \mathfrak{m}$, $x \neq 0$, denote

$$\mathcal{R}_x = \mathcal{R}[x^{-1}z | z \in \mathfrak{m}].$$

Now, the *quadratic transform* of $\mathcal{R}$ is defined as $\mathcal{R}_1 = \mathcal{R}_x$ for any transversal parameter x of $\mathcal{R}$. (It is known that $\mathcal{R}_1$ is well defined.)

Obviously, $\mathcal{R}_1 \neq \mathcal{R}$ unless $\mathfrak{m}$ is a principal ideal, i. e., $\mathcal{R}$ is regular.

Let $\overline{\mathcal{R}}$ be the integral closure of $\mathcal{R}$. Then $\mathcal{R} \subset \mathcal{R}_1 \subset \overline{\mathcal{R}}$. In particular,

$$e(\mathcal{R}_1) \leq e(\mathcal{R}). \tag{2}$$

Since $\overline{\mathcal{R}}$ is a Noetherian $\mathcal{R}$-module, the chain of quadratic transforms

$$\mathcal{R} \subset \mathcal{R}_1 \subset \mathcal{R}_2 = (\mathcal{R}_1)_1 \subset \dots$$

stabilizes, i. e., $\mathcal{R}_i = \overline{\mathcal{R}}$ for some i.

The minimal such i is denoted by $M(\mathcal{R})$; we put $M(\mathcal{R}) = 0$ for a regular R. Thus, if $M(\mathcal{R}) \geq 1$, we have $M(\mathcal{R}_1) = M(\mathcal{R}) - 1$.

3.1 Lemma. *If $\mathcal{R} \subset \mathcal{R}' \subset \overline{\mathcal{R}}$, then $M(\mathcal{R}') \leq M(\mathcal{R})$.*

This is obvious.

Arcs

Let $\mathcal{X}$ be any scheme over a field k. By an arc on $\mathcal{X}$ we mean a non-constant k-morphism $\mathcal{C} : \operatorname{Spec} k[[X]] \to \mathcal{X}$ which maps the closed point to a closed point.

If $\mathcal{X} = \operatorname{Spec} A$, where A is a local ring, an arc on $\mathcal{X}$ can be identified with a local homomorphism of k-algebras $f^{\mathcal{C}} : A \to k[[X]]$ such that $\operatorname{Ker} f^{\mathcal{C}}$ is not the maximal ideal of A.

The *center* of $\mathcal{C}$ is defined as $\mathcal{C}((X))$ and will be denoted by $P_{\mathcal{C}}$. For a fixed closed point $P \in \mathcal{X}$, the set of all arcs such that $P_{\mathcal{C}} = P$ can be identified with the set of arcs on $\operatorname{Spec} \mathcal{O}_{\mathcal{X},P}$. This enables us to concentrate on the case of local schemes.

Let $P = P_{\mathcal{C}}$. Then $f^{\mathcal{C}}$ determines a k-algebra homomomorphism $\widehat{f^{\mathcal{C}}} : \widehat{\mathcal{O}_{\mathcal{X},P}} \to k[[X]]$ which corresponds to an arc $\widehat{\mathcal{C}}$ on $\operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P}}$. It is easy to see that if B is a k-subalgebra (with 1) of $k[[X]]$ and B is complete with respect to induced topology, then either $B = k$, or B contains $k[[t]]$ for some $t \in Xk[[X]]$, $t \neq 0$. Since $\mathcal{C}$ is non-constant, $\mathcal{R}^{\mathcal{C}} = \operatorname{Im} \widehat{f^{\mathcal{C}}} \neq k$. Therefore, $\mathcal{R}^{\mathcal{C}}$ contains $k[[t]]$ with t as above. It is easy to see that $\mathcal{R}^{\mathcal{C}}$ is integral over $k[[t]]$, whence $\dim \mathcal{R}^{\mathcal{C}} = 1$. Thus, $\mathcal{R}^{\mathcal{C}}$ is an algebroid curve over k.

The *support* of $\mathcal{C}$ is defined as the image of $\widehat{\mathcal{C}}$ and will be denoted by $[\mathcal{C}]$. It is the closed subscheme of $\operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P}}$ defined by $\operatorname{Ker} \widehat{f^{\mathcal{C}}}$ and isomorphic to $\operatorname{Spec} \mathcal{R}^{\mathcal{C}}$.

Now let $\mathcal{X} = \operatorname{Spec} A$; A a two-dimensional Noetherian local ring admitting the coefficient field k; $\mathfrak{m}$ the maximal ideal of A.

Let $x_1, \dots, x_n$ be a system of generators of $\mathfrak{m}$. Then we can write $\widehat{A} = k[[X_1, \dots, X_n]]/I$ so that $x_i = X_i + I$, $i = 1, \dots, n$. To determine an arc on $\mathcal{X}$, it is necessary and sufficient to choose $f_i^{\mathcal{C}} = f^{\mathcal{C}}(x_i) \in Xk[[X]]$ such that $b(f_1^{\mathcal{C}}, \dots, f_n^{\mathcal{C}}) = 0$ for all $b \in I$ and $f_i^{\mathcal{C}} \neq 0$ for some i.

We have $\mathcal{R}^{\mathcal{C}} = k[[f_1^{\mathcal{C}}, \dots, f_n^{\mathcal{C}}]]$. We can decompose $f^{\mathcal{C}}$ as

$$A \xrightarrow{\alpha_A} k[[X_1, \dots, X_n]]/I \xrightarrow{\beta_{\mathcal{C}}} \mathcal{R}^{\mathcal{C}} \xrightarrow{\gamma_{\mathcal{C}}} k[[t_{\mathcal{C}}]] \xrightarrow{\delta_{\mathcal{C}}} k[[X]],$$

where α_A is the completion homomorphism, and $\delta_{\mathcal{C}} \circ \gamma_{\mathcal{C}} \circ \beta_{\mathcal{C}} = \widehat{f^{\mathcal{C}}}$. Further, $\beta_{\mathcal{C}}$ is surjective, $\gamma_{\mathcal{C}}$ embeds $\mathcal{R}^{\mathcal{C}}$ into its integral closure in the field of fractions, $\delta_{\mathcal{C}}$ is injective and makes $k[[X]]$ into a finite $k[[t_{\mathcal{C}}]]$-algebra.

The *degree* of $\mathcal{C}$ is defined as $d_{\mathcal{C}} = [k((X)) : k((t_{\mathcal{C}}))] = v_X(t_{\mathcal{C}})$. A *primitive* arc is an arc of degree 1. The *multiplicity* of $\mathcal{C}$ is defined as

$$E_{\mathcal{C}} = e(\mathcal{R}^{\mathcal{C}}) \cdot d_{\mathcal{C}}.$$

Finally, we put $M_{\mathcal{C}} = M(\mathcal{R}^{\mathcal{C}})$. An arc is said to be *regular* if $E_{\mathcal{C}} = 1$. Obviously, $\mathcal{C}$ is regular iff it is primitive and $M_{\mathcal{C}} = 0$.

Two arcs $\mathcal{C}, \mathcal{C}'$ on a k-scheme $\mathcal{X}$ are said to be *weakly equivalent*, if $P_{\mathcal{C}} = P_{\mathcal{C}'}$, and $[\mathcal{C}] = [\mathcal{C}']$. Informally, weakly equivalent arcs are merely different parameterizations of the same algebroid curve on $\mathcal{X}$. To make this formal, we define a *parameterization* of an embedded algebroid curve $\operatorname{Spec} \mathcal{R} \to \mathcal{X}$ as a composition $\operatorname{Spec} k[[X]] \xrightarrow{\operatorname{Spec} \alpha} \operatorname{Spec} \mathcal{R} \to \mathcal{X}$, where α is any injective local homomorphism of k-algebras $\mathcal{R} \to k[[X]]$. A parameterization is said to be primitive if it is a primitive arc.

Next, $\mathcal{C}$ and $\mathcal{C}'$ are said to be *equivalent*, if $f^{\mathcal{C}'} = \lambda \circ f^{\mathcal{C}}$, where λ is a k-algebra automorphism of $k[[X]]$. Obviously, any arc is weakly equivalent to a primitive one which is defined uniquely up to equivalence.

Now we define the intersection multiplicity of two arcs $\mathcal{C}$ and $\mathcal{D}$ on any surface $\mathcal{X}$ such that $P_{\mathcal{C}}$ is a regular point on $\mathcal{X}$. This intersection multiplicity is either a non-negative integer or ∞.

First, we put $(\mathcal{C}.\mathcal{D}) = 0$ if $P_\mathcal{C} \neq P_\mathcal{D}$. In the remaining case we replace $\mathcal{X}$ with $\operatorname{Spec} \mathcal{O}_{\mathcal{X},P_\mathcal{C}}$ and assume that $\mathcal{X} = \operatorname{Spec} A$, where A is a two-dimensional regular local domain.

Assume that $\mathcal{D}$ is primitive. Then we define

$$(\mathcal{C}.\mathcal{D}) = v_X(\widehat{f^\mathcal{C}}(g_\mathcal{D})),$$

where $g_\mathcal{D}$ is a generator of $\operatorname{Ker} \beta_\mathcal{D}$. The definition in [C, 2.3.1] is just a particular case of this one when $\mathcal{C}$ is also primitive.

It is easy to see that $(\mathcal{C}.\mathcal{D})$ is unchanged if $\mathcal{C}$ or $\mathcal{D}$ is replaced with an equivalent arc.

If $\mathcal{D}$ is not necessarily primitive, put

$$(\mathcal{C}.\mathcal{D}) = d_\mathcal{D} \cdot (\mathcal{C}.\widetilde{\mathcal{D}}),$$

where $\widetilde{\mathcal{D}}$ is defined by $f^{\widetilde{\mathcal{D}}} = \delta_{\widetilde{\mathcal{D}}} \circ \gamma_\mathcal{D} \circ \beta_\mathcal{D} \circ \alpha_A$, and $\delta_{\widetilde{\mathcal{D}}}$ maps $t_\mathcal{D}$ to X. In view of the remark in the previous paragraph, $(\mathcal{C}.\mathcal{D})$ is independent of the choice of $t_\mathcal{D}$. The relation to the intersection multiplicities of "embedded plane curves" $(\mathcal{R}^\mathcal{C}, \mathcal{R}^\mathcal{D})$ from [C, 2.3] is as follows:

$$(\mathcal{C}.\mathcal{D}) = d_\mathcal{C} d_\mathcal{D} (\mathcal{R}^\mathcal{C}, \mathcal{R}^\mathcal{D}).$$

Next, we define $(\mathcal{C}.D)$ where $\mathcal{C}$ is an arc on $\mathcal{X}$ as above, and D is a Weil divisor on $\mathcal{X}$. Assume first that $\mathcal{X} = \operatorname{Spec} A$, where A is a *complete* local ring. We require linearity on D and assume, therefore, that D is a prime divisor: $D = \operatorname{Spec} A/\mathfrak{p}$. Note that $A/\mathfrak{p}$ is an algebroid curve. Then by definition $(\mathcal{C}.D) = (\mathcal{C}.\mathcal{D})$, where $\mathcal{D}$ is any primitive parameterization of D. (If $(\mathcal{C}.\mathcal{D}) = \infty$, $(\mathcal{C}.D)$ is assumed undefined.)

For general $\mathcal{X}$, let $\mathcal{X}' = \operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P_\mathcal{C}}}$, and let $f : \mathcal{X}' \to \mathcal{X}$ be the natural morphism. Then by definition $(\mathcal{C}.D) = (\mathcal{C}_1.f^*D)$, where $\mathcal{C}_1$ is the unique arc on $\mathcal{X}'$ such that $f \circ \mathcal{C}_1 = \mathcal{C}$.

3.2 Proposition. *Let P be a regular point of codimension 2 on a surface $\mathcal{X}$. Let C and C' be any distinct irreducible curves on $\mathcal{X}$ such that P is a regular point on both C and C', and let D be a divisor on $\mathcal{X}$ such that C is not a component of D. Let g_C and $g_{C'}$ be local equations of C and C' at P. Denote by $\mathcal{C}$ and $\mathcal{C}'$ any primitive parameterizations of $\widehat{\mathcal{O}_{\mathcal{X},P}}/(g_C)$ and $\widehat{\mathcal{O}_{\mathcal{X},P}}/(g_{C'})$ respectively. Then $(C.C')_P = (\mathcal{C}.\mathcal{C}')$, and $(C.D)_P = (\mathcal{C}.D)$.*

Proof. None of $(C.C')_P$, $(\mathcal{C}.\mathcal{C}')$, $(C.D)_P$, $(\mathcal{C}.D)$ changes, if one replaces $\mathcal{X}$ with $\operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P}}$, replacing C, C', and D with their pullbacks. Therefore, we may assume without loss of generality that $\mathcal{X} = \operatorname{Spec} A$, where A is a complete 2-dimensional local ring with a coefficient field k. We may also assume that D is a prime divisor, and the second equality to prove is reduced to the first one. Since C is regular, and $\mathcal{C}$ is primitive, $f^\mathcal{C}$ is surjective. Therefore,

$$\begin{aligned}(\mathcal{C}.\mathcal{C}') &= v_X(f^\mathcal{C}(g_{C'})) \\ &= \dim_k k[[X]]/(f^\mathcal{C}(g_{C'})) \\ &= \dim_k A/(\operatorname{Ker} f^\mathcal{C} + (g_{C'})) \\ &= \dim_k A/(\operatorname{Ker} f^\mathcal{C} + \operatorname{Ker} f^{\mathcal{C}'}) \\ &= \dim_k A/(g_C, g_{C'}) \\ &= (C.C')_P. \quad \square\end{aligned}$$

3.3 Lemma. *Let $\mathcal{C}$ be an arc on* $\operatorname{Spec} A$*, where A is a two-dimensional regular local ring; let T, U be local parameters of A. Then*

1. $E_{\mathcal{C}} = \min((\mathcal{C}.F_T), (\mathcal{C}.F_U))$.

2. There exist $\alpha, \beta \in k$ such that $(\mathcal{C}.F_{\alpha T + \beta_U}) > E_{\mathcal{C}}$; $[\alpha : \beta]$ is a uniquely defined point on $\mathbb{P}^1_k$.

Proof. 1. Let $t = f^{\mathcal{C}}(T)$, $u = f^{\mathcal{C}}(U)$. We have $(\mathcal{C}.F_T) = v_X(t)$, and $(\mathcal{C}.F_U) = v_X(u)$. It remains to note that either t or u is a transversal parameter of $\mathcal{R}^{\mathcal{C}}$, and $v_X = d_{\mathcal{C}} v$, where v is the valuation associated with the integral closure of $\mathcal{R}^{\mathcal{C}}$.

2. We have $v_X(\alpha t + \beta u) > \min(v_X(t), v_X(u))$, where $[\alpha : \beta]$ is a uniquely defined point on $\mathbb{P}^1_k$. $\square$

Monoidal transformations

Consider an arc $\mathcal{C} : \operatorname{Spec} k[[X]] \to \mathcal{X}$ on a surface $\mathcal{X}$. Assume that $O = P_{\mathcal{C}}$ is a regular point. Let $\mathcal{X}_1$ be the blowing up of $\mathcal{X}$ at the point O. By the second part of Lemma 3.3, $\mathcal{C}$ determines a unique point O_1 in the exceptional divisor, and we denote $A = \mathcal{O}_{\mathcal{X},O}$, $A_1 = \mathcal{O}_{\mathcal{X}_1,O_1}$. We can write down a commutative diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{\alpha_A} & k[[X_1, X_2]] & \xrightarrow{\beta_{\mathcal{C}}} & \mathcal{R}^{\mathcal{C}} & \xrightarrow{\gamma_{\mathcal{C}}} & k[[t_{\mathcal{C}}]] & \xrightarrow{\delta_{\mathcal{C}}} & k[[X]] \\
\downarrow & & \downarrow & & \downarrow & & \| & & \| \\
A_1 & \xrightarrow{\alpha_{A_1}} & k[[X_1', X_2']]/I_1 & \xrightarrow{\beta_{\mathcal{C}_1}} & (\mathcal{R}^{\mathcal{C}})_1 & \xrightarrow{\gamma_{\mathcal{C}_1}} & k[[t_{\mathcal{C}}]] & \xrightarrow{\delta_{\mathcal{C}_1}} & k[[X]],
\end{array}$$

where $(\mathcal{R}^{\mathcal{C}})_1$ is the quadratic transform of $\mathcal{R}^{\mathcal{C}}$, $\beta_{\mathcal{C}_1}$ comes from the embedding of strict quadratic transform of an embedded algebroid curve β_C into the formal quadratic transformation of $\operatorname{Spec} k[[X_1, X_2]]$ in the direction defined by O_1, see the discussion in [C, pp. 35–38]; $\gamma_{\mathcal{C}_1}$ is induced by $\gamma_{\mathcal{C}}$; $\delta_{\mathcal{C}_1} = \delta_{\mathcal{C}}$. The composition $\delta_{\mathcal{C}_1} \circ \gamma_{\mathcal{C}_1} \circ \beta_{\mathcal{C}_1} \circ \alpha_{A_1}$ determines an arc $\mathcal{C}_1$ on $\operatorname{Spec} A_1$ and on $\mathcal{X}_1$. This arc is said to be the *strict transform* of $\mathcal{C}$. It follows from (2) that $E_{\mathcal{C}_1} \leq E_{\mathcal{C}}$.

If $P_{\mathcal{C}} = P_{\mathcal{D}}$, then it is immediate from [C, 2.3.2] that

$$(\mathcal{C}.\mathcal{D}) = (\mathcal{D}.\mathcal{C}) = E_{\mathcal{C}} \cdot E_{\mathcal{D}} + (\mathcal{C}_1.\mathcal{D}_1), \tag{3}$$

where $\mathcal{C}_1$ and $\mathcal{D}_1$ are the strict transforms of $\mathcal{C}$ and $\mathcal{D}$ after blowing up $\mathcal{X}$.

The point $O_1 \in \mathcal{X}_1$ is said to be *the first infinitely near point* for $\mathcal{C}$. The $(i+1)$*th infinitely near point* for $\mathcal{C}$ is defined as the ith infinitely near point for $\mathcal{C}_1$.

The lemma below follows easily from the definitions.

3.4 Lemma. *Let $\mathcal{C}$ be an arc such that $P_{\mathcal{C}}$ is a regular point. Then $E_{\mathcal{C}} = (\mathcal{C}_1.E)$, where $\mathcal{C}_1$ is the strict transform of $\mathcal{C}$, and E is the exceptional divisor.*

Hamburger-Noether expansion

Let $\mathcal{C}$ be an arc on $\operatorname{Spec} A$, where A is a two-dimensional regular local ring; let T, U be local parameters of A. Let $t = f^{\mathcal{C}}(T)$ and $u = f^{\mathcal{C}}(U)$. Assume

$v_X(t) \geq v_X(u)$. Then one can write, in a unique way,

$$
\begin{aligned}
t = a_{01}u + & a_{02}u^2 + \cdots + a_{0h_0}u^{h_0} + u^{h_0}z_1, \\
u = \quad & a_{12}z_1^2 + \cdots + a_{1h_1}z_1^{h_1} + z_1^{h_1}z_2, \\
& \dots \\
z_{r-2} = \quad & a_{r-1,2}z_{r-1}^2 + \cdots + a_{r-1,h_{r-1}}z_{r-1}^{h_{r-1}} + z_{r-1}^{h_{r-1}}z_r, \\
z_{r-1} = \quad & a_{r2}z_r^2 + a_{r3}z_r^3 + \dots,
\end{aligned}
\tag{4}
$$

where $r \geq 0$, $h_0, \dots, h_{r-1}$ are positive integers, $a_{ij} \in k$, $z_i \in k[[X]]$, $E_{\mathcal{C}} = v(u) > v(z_1) > \cdots > v(z_r) = d_{\mathcal{C}}$, and $a_{rj} \neq 0$ for some j if $r > 0$. This is the Hamburger-Noether expansion of $\mathcal{C}$ in parameters T, U.

Conversely, any system (4) is the Hamburger-Noether expansion of some arc $\mathcal{C}$ on A in parameters T, U. The data $(r; (h_i); (a_{ij}))$ correspond bijectively to the classes of weakly equivalent arcs on $\operatorname{Spec} A$.

We have $r = 0$ iff $e(\mathcal{R}^{\mathcal{C}}) = 1$. In this case the Hamburger-Noether expansion is just an *equation* of $[\mathcal{C}]$ in parameters T, U.

If $r > 0$, we have $h_0 + h_1 + \cdots + h_{r-1} = M_{\mathcal{C}}$.

Let $\mathcal{C}$, $\mathcal{D}$ be two arcs on $\operatorname{Spec} A$. Then the length of the "common part" of Hamburger-Noether expansions of $\mathcal{C}$ and $\mathcal{D}$ is equal to the number of common infinitely near points for $\mathcal{C}$ and $\mathcal{D}$.

A detailed discussion of these and related facts on Hamburger-Noether expansions is given in [C].

Ramification invariants associated with arcs

Let $\mathcal{X}$ be a surface over k; L/K a finite Galois extension of its function field; $\mathcal{Y}$ the normalization of $\mathcal{X}$ in L; $R = R_{L/K,\mathcal{X}}$.

Let $\mathbf{U}_{\mathcal{X}} = \mathbf{U}_{\mathcal{X},L/K}$ consist of all arcs $\mathcal{C}$ on $\mathcal{X}$ such that $(\mathcal{C}.R)$ is defined, i. e., $P_{\mathcal{C}}$ is a regular point, and $[\mathcal{C}]$ is not a component of the pullback of R to $\operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}}$. We shall define wild ramification jumps $w_{\mathcal{C}}^{(i)}(L/K)$ for any $\mathcal{C} \in \mathbf{U}_{\mathcal{X}}$, $i = 1, 2, \dots$.

Let $\mathcal{C} \in \mathbf{U}_{\mathcal{X}}$. Replacing $\mathcal{X}$ with $\widehat{\mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}}$, we may assume that $\mathcal{X} = \operatorname{Spec} A$, where A is a complete local ring. Then $[\mathcal{C}]$ is a prime divisor on $\operatorname{Spec} A$. The prime divisors D of $\mathcal{Y}$ lying over $[\mathcal{C}]$ are associated with all extensions to L of the valuation on K determined by $[\mathcal{C}]$. Therefore, $\operatorname{Gal}(L/K)$ acts transitively on the set of all such D.

Fix any D, and let $H \subset \operatorname{Gal}(L/K)$ be the stabilizer of D. Then $H = \operatorname{Gal}(k(D)/k([\mathcal{C}]))$. Let $\mathcal{O}$ and $\mathcal{O}_1$ be the valuation rings in the complete discrete valuation fields $k([\mathcal{C}])$ and $k(D)$ respectively. Consider a co-Carthesian square

$$
\begin{array}{ccc}
k([\mathcal{C}]) & \longrightarrow & k((X)) \\
\downarrow & & \downarrow \\
k(D) & \longrightarrow & k(D) \otimes_{k([\mathcal{C}])} k((X)),
\end{array}
$$

where the upper arrow is induced by $\delta_{\mathcal{C}}$. Then $k(D) \otimes_{k([\mathcal{C}])} k((X))$ is an H-Galois algebra over $k((X))$. It follows that $k(D) \otimes_{k([\mathcal{C}])} k((X)) \simeq \prod_{i=1}^d L_{\mathcal{C}}$ for some d and some Galois extension $L_{\mathcal{C}}/k((X))$ of degree $|H|/d$.

We introduce

$$w_{\mathcal{C}}^{(i)}(L/K) = w^{(i)}(L_{\mathcal{C}}/k((X))), \quad i = 1, 2, \ldots$$

and

$$W_{\mathcal{C},L/K} = W_{L_{\mathcal{C}}/k((X))}.$$

3.5 Lemma. *Let C be a prime divisor on $\mathcal{X}$, P a regular closed point on C. Let $g \in \mathcal{O}_{\mathcal{X},P}$ be a local equation of C at P. Denote by $\mathcal{C}$ any primitive parameterization of $\widehat{\mathcal{O}_{\mathcal{X},P}}/(g)$. Then $w_{C,P}^{(i)}(L/K) = w_{\mathcal{C}}^{(i)}(L/K)$.*

Proof. The extension $L_{C_1,v_1}/K_{C,v}$ from the definition of $w_{C,P}^{(i)}(L/K)$ coincides with $k(D)/k([\mathcal{C}])$ from the definition of $w_{\mathcal{C}}^{(i)}(L/K)$. The latter extension is isomorphic to $L_{\mathcal{C}}/k((X))$ since $\mathcal{C}$ is a primitive arc. □

4 Main theorem

Let $\mathcal{X}$ be a surface over k; L/K a finite solvable Galois extension of its function field; $\mathcal{Y}$ the normalization of $\mathcal{X}$ in L; $R = R_{L/K,\mathcal{X}}$.

We say that L/K *has equal wild jumps at* $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X},L/K}$, if $w_{\mathcal{C}}^{(i)}(L/K) = w_{\mathcal{D}}^{(i)}(L/K)$ for all $i = 1, 2, \ldots$. This is equivalent to $W_{\mathcal{C},L/K} = W_{\mathcal{D},L/K}$.

4.1 Theorem. *In the above setting, there exists a non-decreasing sequence of positive integers $\Delta_{\mathcal{X}}(L/K, i)$, $i = 1, 2, \ldots$, such that $\Delta_{\mathcal{X}}(L/K, i) \geq i^2$, and if $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$, $P_{\mathcal{C}} = P_{\mathcal{D}}$ is a regular point on $\mathcal{X}$, $\mathcal{C}, \mathcal{D}$ are primitive, and*

$$(\mathcal{C}.\mathcal{D}) \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})),$$

then L/K has equal wild jumps at $\mathcal{C}$ and $\mathcal{D}$.

Theorem 4.1 will be proved in the following 4 sections.

4.1.1 Corollary. *The answer to Question 2.1 is affirmative.*

Proof. It is sufficient to put

$$R_0 = \Delta_{\mathcal{X}}(L/K, 1)R$$

and to apply Prop. 3.2 and Lemma 3.5. □

4.1.2 Remark. If C is a regular curve on a surface $\mathcal{X}$, and $\mathcal{Y}$ is the normalization of $\mathcal{X}$ in a finite extension of its function field, then the irreducible curves on $\mathcal{Y}$ over C ("liftings" of C) are, in general, not regular. This is why we have to introduce $\Delta_{\mathcal{X}}(L/K, i)$ with $i > 1$ and to prove Theorem 4.1 which is more general than Corollary 4.1.1.

5 Lifting of arcs

Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite surjective morphism of k-schemes, $\mathcal{C}$ an arc on $\mathcal{X}$. An arc $\mathcal{C}'$ on $\mathcal{Y}$ is said to be a *lifting* of $\mathcal{C}$ onto $\mathcal{Y}$ if $\mathcal{C}' : \operatorname{Spec} k[[X]] \to \mathcal{Y}$ can be factored as $\mathcal{C}' = \mathcal{C}_f \circ g$, where $\mathcal{C}_f$ is determined by a Carthesian square

$$\begin{array}{ccc} \operatorname{Spec} k[[X]] \times_{\mathcal{X}} \mathcal{Y} & \xrightarrow{\mathcal{C}_f} & \mathcal{Y} \\ \big\downarrow & & f \big\downarrow \\ \operatorname{Spec} k[[X]] & \xrightarrow{\mathcal{C}} & \mathcal{X} \end{array}$$

and g is the normalization of an irreducible component of $\operatorname{Spec} k[[X]] \times_{\mathcal{X}} \mathcal{Y}$ with reduced scheme structure.

5.0.1 Remark. Let $\mathcal{C}$ be a primitive arc. Analyzing a more detailed diagram

$$\begin{array}{ccccccc} \operatorname{Spec} k[[X]] \times_{\mathcal{X}} \mathcal{Y} & \longrightarrow & [\mathcal{C}] \times_{\mathcal{X}} \mathcal{Y} & \longrightarrow & \operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}} \times_{\mathcal{X}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & f \big\downarrow \\ \operatorname{Spec} k[[X]] & \xrightarrow{\operatorname{Spec} \gamma_{\mathcal{C}}} & [\mathcal{C}] & \xrightarrow{\operatorname{Spec} \beta_{\mathcal{C}}} & \operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}} & \longrightarrow & \mathcal{X} \end{array}$$

we see that the following numbers coincide:

1) the number of non-equivalent liftings of $\mathcal{C}$ onto $\mathcal{Y}$;

2) the number of non-equivalent liftings of $\widehat{\mathcal{C}} : \operatorname{Spec} k[[X]] \to \operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}}$ onto $\operatorname{Spec} \widehat{\mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}} \times_{\mathcal{X}} \mathcal{Y}$;

3) the number of irreducible components in $[\mathcal{C}] \times_{\mathcal{X}} \mathcal{Y}$ (note that $[\mathcal{C}] \times_{\mathcal{X}} \mathcal{Y}$ is the spectrum of a finite $\mathcal{R}^{\mathcal{C}}$-algebra);

4) the number of irreducible components in $\operatorname{Spec} k[[X]] \times_{\mathcal{X}} \mathcal{Y}$.

(See also proof of Prop. 5.1 below.)

5.0.2 Remark. It is also clear from the above diagram that any arc $\mathcal{D}$ on $\mathcal{Y}$ is a lifting of the arc $f \circ \mathcal{D}$ on $\mathcal{X}$. Moreover, if $\mathcal{D}$ is primitive, and $\mathcal{C}$ is a primitive and arc on $\mathcal{X}$ weakly equivalent to $f \circ \mathcal{D}$, then $\mathcal{D}$ is also a lifting of $\mathcal{C}$. Thus, any primitive arc on $\mathcal{Y}$ is a lifting of a primitive arc on $\mathcal{X}$.

We examine liftings in the case of some special finite extensions of two-dimensional local rings.

5.1 Proposition. *Let $A = k[[T,U]]$, $A_1 = k[[T_1,U]]$, and A is embedded into A_1 by mapping T to $\xi(T_1,U) \equiv T_1^l \mod (T_1^{l+1}, T_1 U)$, where l is a prime number. Assume also that there exists an A-automorphism σ of A_1 such that $\sigma^l = 1$, and $A_1^{\langle\sigma\rangle} = A$.*

1. Let $\mathcal{C}$ be a primitive arc on $\operatorname{Spec} A$ with $[\mathcal{C}] \neq F_T$. Then the number $n_{\mathcal{C}}$ of non-equivalent liftings of $\mathcal{C}$ onto $\operatorname{Spec} A_1$ is either 1 or l, and these liftings are primitive arcs.

2. If $\mathcal{C}'$ is a lifting of a primitive arc $\mathcal{C}$ as above, we have

$$\begin{aligned} (\mathcal{C}'.F_{T_1}) &= (\mathcal{C}.F_T)/n_{\mathcal{C}}, \\ (\mathcal{C}'.F_U) &= l(\mathcal{C}.F_U)/n_{\mathcal{C}}, \\ E_{\mathcal{C}'} &= \min((\mathcal{C}.F_T), l(\mathcal{C}.F_U))/n_{\mathcal{C}}. \end{aligned}$$

If $n_{\mathcal{C}} = l$, we have $M_{\mathcal{C}'} \leq M_{\mathcal{C}}$.

3. Let $\mathcal{C}$ and $\mathcal{D}$ be primitive arcs on $\operatorname{Spec} A$ with $[\mathcal{C}], [\mathcal{D}] \neq F_T$. Let $\mathcal{C}'$ and $\mathcal{D}'$ be their liftings. If $n_{\mathcal{C}} = 1$ or $n_{\mathcal{D}} = 1$, then $(\mathcal{C}'.\mathcal{D}') = l(\mathcal{C}.\mathcal{D})/(n_{\mathcal{C}} n_{\mathcal{D}})$. If $n_{\mathcal{C}} = n_{\mathcal{D}} = l$, then $\sum_{i=1}^{l} (\mathcal{C}_i.\mathcal{D}') = (\mathcal{C}.\mathcal{D})$, where $\mathcal{C}_1, \ldots, \mathcal{C}_l$ are all pairwise non-equivalent liftings of $\mathcal{C}$.

Proof. Let $\mathcal{C}$ be an arc on $\operatorname{Spec} A$ with $[\mathcal{C}] \neq F_T$. It is clear from Remark 5.0.1 that $n_{\mathcal{C}}$ coincides with the number of primes in $\operatorname{Spec} A_1$ over the generic point of $[\mathcal{C}]$. By [B, Ch. V, §2, Th. 2], $\langle\sigma\rangle$ acts transitively on the set of such primes, whence $n_{\mathcal{C}}$ is either 1 or l. Denote by g any generator of $\operatorname{Ker} f^{\mathcal{C}}$.

Let first $n_{\mathcal{C}} = 1$. Since $[\mathcal{C}] \neq F_T$, there are no nilpotents in $\mathcal{R}^{\mathcal{C}} \otimes_A A_1$. The unique lifting $\mathcal{C}'$ of $\mathcal{C}$ corresponds to the bottom row of the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\beta_{\mathcal{C}}} & A/(g) = \mathcal{R}^{\mathcal{C}} & \xrightarrow{\gamma_{\mathcal{C}}} & k[[t_{\mathcal{C}}]] & = & k[[X]] \\ \downarrow & & \downarrow & & \downarrow & & \\ A_1 & \longrightarrow & A_1/(g) = \mathcal{R}^{\mathcal{C}} \otimes_A A_1 & \xrightarrow{\gamma_{\mathcal{C}} \otimes_A A_1} & B & \longrightarrow & B/\operatorname{Nil} B \end{array} \tag{5}$$

extended to the right with the embedding of $B/\operatorname{Nil} B$ into its integral closure $B' \simeq k[[X]]$.

Since A' is a free A-module of rank l, it is clear that $\mathcal{R}^{\mathcal{C}} \otimes_A A_1$ is of finite k-codimension in B and in B', whence $\mathcal{C}'$ is a primitive arc. Next, it is obvious that $g_1 = g(\xi(T_1, U), U) \in k[[T_1, U]]$ is an equation of $\mathcal{C}'$. By the definition and Lemma 3.3 we have

$$\begin{aligned} (\mathcal{C}'.F_{T_1}) &= v_X(g_1(0, X)) = v_X(g(0, X)) = (\mathcal{C}.F_T)/n_{\mathcal{C}}, \\ (\mathcal{C}'.F_U) &= v_X(g_1(X, 0)) = l v_X(g(X, 0)) = l(\mathcal{C}.F_U)/n_{\mathcal{C}}, \\ E_{\mathcal{C}'} &= \min((\mathcal{C}.F_{T_1}), (\mathcal{C}.F_U)) = \min((\mathcal{C}.F_T), l(\mathcal{C}.F_U))/n_{\mathcal{C}}. \end{aligned}$$

Next, let $n_{\mathcal{C}} = l$. Then $g(\xi(T_1, U), U) = g_1 \dots g_l$ in A_1, and $(g_1), \dots, (g_l)$ are distinct prime ideals. Consider $(A_1/(g_1 \dots g_l)) \otimes_{A/(g)} k((t_{\mathcal{C}}))$. This is a $k((t_{\mathcal{C}}))$-algebra of dimension l with l minimal prime ideals; therefore, it is isomorphic to the direct product of l copies of $k((t_{\mathcal{C}}))$. Applying $\otimes_{A/(g)} k((t_{\mathcal{C}}))$ to the right square of the diagram (5), we see that non-equivalent liftings of $\mathcal{C}$ are, up to equivalence, the primitive arcs $\mathcal{C}_1, \dots, \mathcal{C}_l$ with $\mathcal{R}^{\mathcal{C}_i} = A_1/(g_i)$.

The group $\langle\sigma\rangle$ acts transitively on $\{(g_1), \dots, (g_l)\}$, therefore, on $\{\mathcal{C}_1, \dots, \mathcal{C}_l\}$. It follows that $(\mathcal{C}_i.F_{T_1})$ and $(\mathcal{C}_i.F_U)$ are independent of i. On the other hand,

$$\begin{aligned} \sum_{i=1}^{l} (\mathcal{C}_i.F_{T_1}) &= \sum_{i=1}^{l} v(g_i(0, X)) = v(g(0, X)) = (\mathcal{C}.F_T), \\ \sum_{i=1}^{l} (\mathcal{C}_i.F_U) &= \sum_{i=1}^{l} v(g_i(X, 0)) = l v(g(X, 0)) = l(\mathcal{C}.F_U), \end{aligned}$$

and the desired formulae for $(\mathcal{C}'.F_{T_1})$, $(\mathcal{C}'.F_U)$, $E_{\mathcal{C}'}$ follow.

Since $A_1/(g_i) \simeq \mathcal{R}^{\mathcal{C}_i}$ is integral over $A/(g) \simeq \mathcal{R}^{\mathcal{C}}$ with the same field of fractions, it follows from Lemma 3.1 that $M_{\mathcal{C}_i} \leq M_{\mathcal{C}}$.

To prove the remaining assertion, we have to relate $f^{\mathcal{D}}$ and $f^{\mathcal{D}'}$. Let ψ be the map that makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{f^{\mathcal{D}}} & k[[X]] \\ \downarrow & & \downarrow \psi \\ A_1 & \xrightarrow{f^{\mathcal{D}'}} & k[[X]] \end{array}$$

commutative. If $n_{\mathcal{D}} = 1$, ψ is an embedding of $k[[X]]$ into a totally ramified extension of degree l. In particular,

$$\psi(f^{\mathcal{D}}(T)) = \xi(f^{\mathcal{D}'}(T_1), f^{\mathcal{D}'}(U)),$$
$$\psi(f^{\mathcal{D}}(U)) = f^{\mathcal{D}'}(U).$$

In this case $(\mathcal{C}'.\mathcal{D}')$ is computed exactly in the same way as $(\mathcal{C}'.F_U)$.

If $n_{\mathcal{D}} = l$, $n_{\mathcal{C}} = 1$, we change the roles of $\mathcal{C}$ and $\mathcal{D}$.

Finally, if $n_{\mathcal{C}} = n_{\mathcal{D}} = l$, then ψ is the identity map, and

$$f^{\mathcal{D}}(T) = \xi(f^{\mathcal{D}'}(T_1), f^{\mathcal{D}'}(U)),$$
$$f^{\mathcal{D}}(U) = f^{\mathcal{D}'}(U).$$

We obtain

$$\begin{aligned}\sum_{i=1}^{l}(\mathcal{C}_i.\mathcal{D}') &= \sum_{i=1}^{l} v_X(g_i(f^{\mathcal{D}'}(T_1), f^{\mathcal{D}'}(U))) \\ &= v_X(g(\xi(f^{\mathcal{D}'}(T_1), f^{\mathcal{D}'}(U)), f^{\mathcal{D}'}(U))) \\ &= v_X(g(f^{\mathcal{D}}(T), f^{\mathcal{D}}(U))) = (\mathcal{C}.\mathcal{D}). \quad \square\end{aligned}$$

5.1.1 Corollary. *In the setting of Proposition we have $E_{\mathcal{C}'} \leq lE_{\mathcal{C}}$.*

Proof. Use Lemma 3.3. □

Growth of $M_{\mathcal{C}}$ in cyclic extensions of prime degree

In this subsection $\mathcal{X}$ is a regular k-surface with the function field K; L/K is a cyclic extension of prime degree l; $\mathcal{Y} \to \mathcal{X}$ is the normalization of $\mathcal{X}$ in L. For any arc $\mathcal{C}$ on $\mathcal{X}$ we denote by $n_{\mathcal{C}}$ the number of non-equivalent liftings of $\mathcal{C}$ onto $\mathcal{Y}$.

Up to the end of the section we assume that $l \neq p$. Similar statements with $l = p$ are deferred until the analysis of Artin-Schreier coverings in the section 6.

5.2 Lemma. *Let $\mathcal{C} \in \mathbf{U}_{\mathcal{X}}$ be a primitive arc. Assume that $L = K(x)$, $x^l = a$, where $a \in \mathcal{O}_{\mathcal{X},P_{\mathcal{C}}}$. Then $n_{\mathcal{C}} = 1$ iff $l \nmid v_X(f^{\mathcal{C}}(a))$.*

Proof. In view of Remark 5.0.1, we may assume $\mathcal{X} = \operatorname{Spec} A$, where A is a local ring. We have $\mathcal{Y} = \operatorname{Spec} B$, where B is the integral closure of $A[x]$. Let U be the complement of $R_{L/K,\mathcal{X}}$ in $\mathcal{X}$. The unramified morphism $\mathcal{Y} \times_{\mathcal{X}} U \to U$ factors through $\operatorname{Spec} A[x] \times_{\mathcal{X}} U$, whence $\operatorname{Spec} A[x] \times_{\mathcal{X}} U$ is regular, and $\mathcal{Y} \times_{\mathcal{X}} U = \operatorname{Spec} A[x] \times_{\mathcal{X}} U$.

By Remark 5.0.1, $n_{\mathcal{C}}$ coincides with the number of irreducible components in $[\mathcal{C}] \times_{\mathcal{X}} \mathcal{Y}$, i. e., the number of primes in $B \otimes_A \hat{A}$ over the generic point $\mathfrak{p}$ of $[\mathcal{C}]$. Since $\mathcal{C} \in \mathbf{U}_{\mathcal{X}}$, we obtain that $\mathfrak{p}$ is in $\operatorname{Spec} \hat{A} \times_{\mathcal{X}} U$, and $n_{\mathcal{C}}$ is the number of primes in $A[x] \otimes_A \hat{A}$ over $\mathfrak{p}$, i. e., the number of minimal primes in

$$(\hat{A}/\mathfrak{p})[Y]/(Y^l - a) \simeq k[[X]][Y](Y^l - f^{\mathcal{C}}(a)) \simeq k[[X]][Y](Y^l - X^m),$$

where $m = f^{\mathcal{C}}(a)$. If $l|m$, then $Y^l - X^m$ is a product of l irreducible factors and $n_{\mathcal{C}} = l$. If $l \nmid m$, it is easy to check that $Y^l - X^m$ is irreducible, and $n_{\mathcal{C}} = 1$. (Indeed, apply induction on $\max(m, l)$, and substitute either XZ for Y or YZ for X.) □

5.3 Lemma. *Assume that $R_{L/K,\mathcal{X}}$ is a simply normal crossing divisor. Let $\mathcal{C} \in \mathbf{U}_{\mathcal{X}}$ be a primitive arc with $n_{\mathcal{C}} = 1$, not regular. Let $\mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ be a primitive arc such that $E_{\mathcal{D}} < lE_{\mathcal{C}}$, and $(\mathcal{C}.\mathcal{D}) > (M_{\mathcal{C}}+1)E_{\mathcal{C}}E_{\mathcal{D}}$. Then*

1. *$n_{\mathcal{D}} = 1$;*
2. *$E_{\mathcal{D}} \geq E_{\mathcal{C}}$.*

Proof. Let $n = M_{\mathcal{C}}$. Then $\mathcal{C}$ and $\mathcal{D}$ have at least n common infinitely near points $O_1, \dots, O_n$. Let $\mathcal{C}_n$ and $\mathcal{D}_n$ be the nth strict transforms of $\mathcal{C}$ and $\mathcal{D}$ respectively. Then $\mathcal{C}_n$ is regular and the only line of the Hamburger-Noether expansion of $\mathcal{C}_n$ (in suitable local parameters t and u at O_n) coincides with the last line of the Hamburger-Noether expansion of $\mathcal{C}$ (in some local parameters t_0, u_0):

$$z_{r-1} = a_{r2}z_r^2 + a_{r3}z_r^3 + \dots,$$

where $z_{r-1} = f^{\mathcal{C}_n}(t)$, $z_r = f^{\mathcal{C}_n}(u)$. Let $j = \min\{j | a_{rj} \neq 0\}$. Then $j = v_X(z_{r-1}) = E_{\mathcal{C}_{n-1}} \leq E_{\mathcal{C}}$, and

$$(\mathcal{C}_n.\mathcal{D}_n) \geq (\mathcal{C}.\mathcal{D}) - M_{\mathcal{C}}E_{\mathcal{C}}E_{\mathcal{D}} > E_{\mathcal{C}}E_{\mathcal{D}} \geq jE_{\mathcal{C}_n}E_{\mathcal{D}_n}.$$

Therefore, $\mathcal{C}_n$ and $\mathcal{D}_n$ have at least j common infinitely near points. It follows that the beginning of the first line in the Hamburger-Noether expansion of $\mathcal{D}_n$ with respect to t and u is

$$f^{\mathcal{D}_n}(t) = a_{rj}f^{\mathcal{D}_n}(u)^j + \dots$$

Let $\lambda = v_X(f^{\mathcal{D}_n}(u))$. Then $v_X(f^{\mathcal{D}_n}(t)) = \lambda j$, and, applying the equalities from Hamburger-Noether expansions of $\mathcal{C}$ and $\mathcal{D}$ from bottom to top, we deduce that $v_X(f^{\mathcal{D}}(t_0)) = \lambda v_X(f^{\mathcal{C}}(t_0))$ and $v_X(f^{\mathcal{D}}(u_0)) = \lambda v_X(f^{\mathcal{C}}(u_0))$. It follows that $E_{\mathcal{D}} = \lambda E_{\mathcal{C}}$, whence $\lambda < l$ and $E_{\mathcal{D}} \geq E_{\mathcal{C}}$.

We may choose t_0 and u_0 so that the local equations of the components of $R_{L/K,\mathcal{X}}$ passing through $P_{\mathcal{C}}$ are within $\{t_0, u_0\}$. Then the same is true for $R_{L/K,\mathcal{X}_n}$, O_n and $\{t, u\}$ respectively, where $\mathcal{X}_n$ is the nth monoidal transformation of $\mathcal{X}$. It follows $L = K(x)$, $x^l = t^q u^s \varepsilon$, where ε is invertible in a neighborhood of O_n. We obtain

$$\begin{aligned} n_{\mathcal{C}} = 1 &\implies n_{\mathcal{C}_n} = 1 \\ &\implies l \nmid qj + s = qv_X(f^{\mathcal{C}_n}(t)) + sv_X(f^{\mathcal{C}_n}(u)) \\ &\implies l \nmid \lambda(qj+s) = qv_X(f^{\mathcal{D}_n}(t)) + sv_X(f^{\mathcal{D}_n}(u)) \\ &\implies n_{\mathcal{D}_n} = 1 \\ &\implies n_{\mathcal{D}} = 1. \quad \square \end{aligned}$$

5.4 Lemma. *Let $\mathcal{C}, \mathcal{D}$ be primitive arcs on $\mathcal{X}$, at least one of them being not regular. Assume that $(\mathcal{C}.\mathcal{D}) > (M_{\mathcal{C}}+1)E_{\mathcal{C}}E_{\mathcal{D}}$. Then for any regular arc $\mathcal{F}$ on $\mathcal{X}$ we have*

$$(\mathcal{D}.\mathcal{F}) = (\mathcal{C}.\mathcal{F}) \cdot \frac{E_{\mathcal{D}}}{E_{\mathcal{C}}}.$$

Proof. Without loss of generality we may assume that $P_{\mathcal{C}} = P_{\mathcal{D}} = P_{\mathcal{F}} =: O$. Let t_0, u_0 be a system of local parameters in O such that t_0 is a local equation of $R^{\mathcal{F}}$. Then $(\mathcal{C}.\mathcal{F}) = v_X(f^{\mathcal{C}}(t_0))$, and $(\mathcal{D}.\mathcal{F}) = v_X(f^{\mathcal{D}}(t_0))$. As in the proof of Lemma 5.3, we obtain $v_X(f^{\mathcal{D}}(t_0)) = \lambda v_X(f^{\mathcal{C}}(t_0))$ and $v_X(f^{\mathcal{D}}(u_0)) = \lambda v_X(f^{\mathcal{C}}(u_0))$ for some λ. It follows $E_{\mathcal{D}} = \lambda E_{\mathcal{C}}$. $\square$

5.5 Proposition. *Let $\mathcal{X} = \operatorname{Spec} A$, $\mathcal{Y} = \operatorname{Spec} A_1$, where A and A_1 are as in Prop. 5.1, $l \neq p$. Let $\mathcal{C}$ be a primitive arc on $\mathcal{X}$ with $[\mathcal{C}] \neq F_T$, and $\mathcal{C}'$ its lifting. Then*

$$M_{\mathcal{C}'} \leq \frac{l}{2}(M_{\mathcal{C}}+1)E_{\mathcal{C}}^2.$$

Proof. If $n_{\mathcal{C}} = l$, we have $M_{\mathcal{C}'} \leq M_{\mathcal{C}}$ by Lemma 3.1. We may, therefore, assume that $n_{\mathcal{C}} = 1$. Let t, u be local parameters of A_1 such that $v_X(f^{\mathcal{C}'}(t)) > v_X(f^{\mathcal{C}'}(u))$, namely, t, u are just T_1, U in a suitable order. Let

$$\begin{aligned}
f^{\mathcal{C}'}(t) =& a_{01}f^{\mathcal{C}'}(u) + a_{02}f^{\mathcal{C}'}(u)^2 + \cdots + a_{0h_0}f^{\mathcal{C}'}(u)^{h_0} + f^{\mathcal{C}'}(u)^{h_0}z_1,\\
f^{\mathcal{C}'}(u) =& \qquad a_{12}z_1^2 + \cdots + a_{1h_1}z_1^{h_1} + z_1^{h_1}z_2,\\
&\dots\\
z_{r-2} =& \qquad a_{r-1,2}z_{r-1}^2 + \cdots + a_{r-1,h_{r-1}}z_{r-1}^{h_{r-1}} + z_{r-1}^{h_{r-1}}z_r,\\
z_{r-1} =& \qquad a_{r2}z_r^2 + a_{r3}z_r^3 + \dots
\end{aligned}$$

be the Hamburger-Noether expansion of $\mathcal{C}'$ in the basis t, u. Define a new arc $\mathcal{D}'$ by means of equalities:

$$\begin{aligned}
f^{\mathcal{D}'}(t) =& a_{01}f^{\mathcal{D}'}(u) + a_{02}f^{\mathcal{D}'}(u)^2 + \cdots + a_{0h_0}f^{\mathcal{D}'}(u)^{h_0} + f^{\mathcal{D}'}(u)^{h_0}y_1,\\
f^{\mathcal{D}'}(u) =& \qquad a_{12}y_1^2 + \cdots + a_{1h_1}y_1^{h_1} + y_1^{h_1}y_2,\\
&\dots\\
y_{r-3} =& \qquad a_{r-2,2}y_{r-2}^2 + \cdots + a_{r-2,h_{r-1}}y_{r-2}^{h_{r-2}} + y_{r-2}^{h_{r-2}}y_{r-1},\\
y_{r-2} =& \qquad a_{r-1,2}y_{r-1}^2 + \cdots + a_{r-1,h_{r-1}}y_{r-1}^{h_{r-1}} + y_{r-1}^{h_{r-1}+1},\\
y_{r-1} =& X.
\end{aligned}$$

We see immediately that $\mathcal{C}'$ and $\mathcal{D}'$ have exactly $h_0 + \cdots + h_{r-1} = M_{\mathcal{C}'}$ common infinitely near points, whence $(\mathcal{C}'.\mathcal{D}') \geq 2M_{\mathcal{C}'}$. Next, we see that $v_X(y_{r-1}) < v_X(z_{r-1})$, $v_X(y_{r-1}) < v_X(z_{r-1})$, whence

$$\begin{aligned}
v_X(f^{\mathcal{D}'}(u)) < v_X(f^{\mathcal{C}'}(u)),\\
v_X(f^{\mathcal{D}'}(t)) < v_X(f^{\mathcal{C}'}(t)).
\end{aligned} \tag{6}$$

It follows $E_{\mathcal{D}'} < E_{\mathcal{C}'}$.

By Remark 5.0.2, $\mathcal{D}'$ is a lifting of a primitive arc $\mathcal{D}$ on $\mathcal{X}$. By Prop. 5.1 it follows from (6) that $E_{\mathcal{D}}/n_D < E_{\mathcal{C}}/n_{\mathcal{C}} = E_{\mathcal{C}}$.

Let first $n_{\mathcal{D}} = 1$. Then $E_{\mathcal{C}} > E_{\mathcal{D}}$, in particular, $\mathcal{C}$ is not regular. Then by Prop. 5.1 we have $(\mathcal{C}'.\mathcal{D}') = l(\mathcal{C}.\mathcal{D})$. Assume that Proposition does not hold for $\mathcal{C}$ and $\mathcal{C}'$, then

$$(\mathcal{C}.\mathcal{D}) > (M_{\mathcal{C}}+1)E_{\mathcal{C}}^2 > (M_{\mathcal{C}}+1)E_{\mathcal{C}}E_{\mathcal{D}}.$$

It follows from Lemma 5.3 that $E_{\mathcal{D}} \geq E_{\mathcal{C}}$, a contradiction.

Let finally $n_{\mathcal{D}} = l$. Then by Prop. 5.1 we have $(\mathcal{C}'.\mathcal{D}') = (\mathcal{C}.\mathcal{D})$. We have $E_{\mathcal{D}} < lE_{\mathcal{C}}$, and by Lemma 5.3 we obtain

$$(\mathcal{C}.\mathcal{D}) \leq (M_{\mathcal{C}}+1)E_{\mathcal{C}}E_{\mathcal{D}} < l(M_{\mathcal{C}}+1)E_{\mathcal{C}}^2,$$

and

$$M_{\mathcal{C}'} < \frac{l}{2}(M_{\mathcal{C}}+1)E_{\mathcal{C}}^2. \quad \square$$

6 Artin-Schreier extension

In this section we give an explicit shape of a uniform sufficient jet order in the case of a cyclic extension of degree p.

6.1 Proposition. *Let $\mathcal{X}$ be a regular k-surface with the function field K, $L = K(x)$, where $x^p - x = a \in K$. Assume that $R = R_{L/K,\mathcal{X}}$ is a simply normal crossing divisor. Let $F_1, \dots, F_s$ be the components of R. Put*

$$m_i = -v_i(a),$$

where v_i is the valuation on K associated with F_i, $i = 1, \dots, s$. Put

$$R_0 = R_0(\mathcal{X}, L/K) = (m_1 + 1)F_1 + \dots + (m_s + 1)F_s.$$

Let $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ be such that $P_{\mathcal{C}} = P_{\mathcal{D}}$, and

$$(\mathcal{C}.\mathcal{D}) \geq (\mathcal{C}.R_0)E_{\mathcal{D}} + E_{\mathcal{C}}E_{\mathcal{D}} \max(M_{\mathcal{C}}, M_{\mathcal{D}}).$$

Then L/K has equal wild jumps at $\mathcal{C}$ and $\mathcal{D}$.

Proof. Throughout the proof, we denote $O = P_{\mathcal{C}} = P_{\mathcal{D}}$.

Apply induction on $\max(M_{\mathcal{C}}, M_{\mathcal{D}})$. Let $M_{\mathcal{C}} = M_{\mathcal{D}} = 0$. If none of the components of R passes through O, we have $w_{\mathcal{C}}^{(i)} = 0 = w_{\mathcal{D}}^{(i)}$ for any i.

If 1 component of R passes through O, we may assume that this is F_1. Let $t = 0$ be a local equation of F_1 at O. Choose any $u \in \mathcal{O}_{\mathcal{X},O}$ such that t, u are local parameters at O.

If there are 2 components of R through O, we may assume that these are F_1 and F_2. Let $t = 0$, $u = 0$ be their local equations at O.

Since F_i are components of R, each of the valuations v_i must be totally or fiercely ramified in L/K; therefore, $m_i > 0$. We have $(\mathcal{C}.R_0) \geq m_1 + 1 \geq 2$, whence $N := (\mathcal{C}.\mathcal{D}) \geq 2$. Assume first that $\mathcal{C}$ is not tangent to the curve locally defined as $u = 0$. Applying Weierstraß preparation theorem, we can write $\mathcal{R}^{\mathcal{C}} = k[[t,u]]/(f)$, $\mathcal{R}^{\mathcal{D}} = k[[t,u]]/(g)$, where

$$f = -t + \beta_1 u + \beta_2 u^2 + \dots,$$
$$g = -t + \beta_1' u + \beta_2' u^2 + \dots,$$

with $\beta_i, \beta_i' \in k$, $i = 1, 2, \dots$. We have $\min\{i | \beta_i \neq \beta_i'\} = N$.

If F_2 passes through O, we have

$$a = t^{-m_1} u^{-m_2} \varepsilon(t, u),$$

in the completion of $\mathcal{O}_{\mathcal{X},O}$, where $\varepsilon \in k[[T,U]]$. Now, $w_{\mathcal{C}}^{(1)}$ and $w_{\mathcal{D}}^{(1)}$ are exactly the jump of the Artin-Schreier equation

$$x^p - x = (\beta_1 u + \beta_2 u^2 + \dots)^{-m_1} u^{-m_2} \varepsilon(\beta_1 u + \beta_2 u^2 + \dots, u), \qquad (7)$$

and that of

$$x^p - x = (\beta_1' u + \beta_2' u^2 + \dots)^{-m_1} u^{-m_2} \varepsilon(\beta_1' u + \beta_2' u^2 + \dots, u). \qquad (8)$$

Let $j = (\mathcal{C}.F_1)$. Then $j = \min\{i | \beta_i \neq 0\}$. We have

$$(\beta_1 u + \beta_2 u^2 + \dots)^{-m_1} u^{-m_2} \equiv$$
$$\equiv u^{-m_1 j - m_2} (\beta_j + \dots + \beta_{j+m_1 j + m_2 - 1} u^{m_1 j + m_2 - 1})^{-m_1} \mod k[[u]],$$

and we see that the RHS of (7) is congruent to that of (8) $\mod k[[u]]$, provided that $N \geq j + m_1 j + m_2$. This condition holds in view of $(\mathcal{C}.\mathcal{D}) \geq (\mathcal{C}.R_0) = (m_1+1)j + m_2 + 1$. It follows $w_{\mathcal{C}}^{(1)} = w_{\mathcal{D}}^{(1)}$.

If F_2 does not pass through O, the argument is the same; we have only to omit the factor u^{-m_2} in all the formulae.

Finally, let $\mathcal{C}$ be tangent to the curve $u = 0$. If $u = 0$ is the local equation of F_2, this case is reduced to the previous one by changing the roles of F_1 and F_2. If there is no component of R passing through O except for F_1, this case can be also reduced to the previous one by a substitution $u := u + t$.

Next, let $\max(M_{\mathcal{C}}, M_{\mathcal{D}}) > 0$. Let $\mathcal{X}' \to \mathcal{X}$ be the blowing up at O. Let $\mathcal{C}', \mathcal{D}'$ be the strict transforms of $\mathcal{C}$ and $\mathcal{D}$ respectively. Let F_i' be the strict transform of F_i, $i = 1, \dots, s$, and let E be the exceptional divisor.

Assume first that two components of R (say, F_1 and F_2) pass through O. The formula

$$(\mathcal{C}.\mathcal{D}) \geq ((m_1+1)(\mathcal{C}.F_1) + (m_2+1)(\mathcal{C}.F_2))E_{\mathcal{D}} + E_{\mathcal{C}} E_{\mathcal{D}} \max(M_{\mathcal{C}}, M_{\mathcal{D}})$$

implies

$$\begin{aligned}(\mathcal{C}'.\mathcal{D}') + E_{\mathcal{C}} E_{\mathcal{D}} \geq ((m_1+1)&((\mathcal{C}'.F_1') + E_{\mathcal{C}}) \\ &+ (m_2+1)((\mathcal{C}'.F_2') + E_{\mathcal{C}}))E_{\mathcal{D}} \\ &+ E_{\mathcal{C}} E_{\mathcal{D}} (\max(M_{\mathcal{C}'}, M_{\mathcal{D}'}) + 1).\end{aligned}$$

and

$$\begin{aligned}(\mathcal{C}'.\mathcal{D}') \geq ((m_1+1)&(\mathcal{C}'.F_1') + (m_2+1)(\mathcal{C}'.F_2') + (m_1+m_2+1)(\mathcal{C}'.E))E_{\mathcal{D}} \\ &+ E_{\mathcal{C}} E_{\mathcal{D}} (\max(M_{\mathcal{C}'}, M_{\mathcal{D}'}) + 1).\end{aligned}$$

Notice that $-v_E(a) \leq m_1 + m_2$, where v_E is the valuation associated with E. We obtain

$$(\mathcal{C}'.\mathcal{D}') \geq (\mathcal{C}'.R_0(\mathcal{X}', L/K))E_{\mathcal{D}'} + E_{\mathcal{C}'} E_{\mathcal{D}'} (\max(M_{\mathcal{C}'}, M_{\mathcal{D}'}) + 1).$$

Therefore, by the induction hypothesis, L/K has equal wild jumps at $\mathcal{C}'$ and $\mathcal{D}'$. Since $W_{\mathcal{C},L/K} = W_{\mathcal{C}',L/K}$, and $W_{\mathcal{D},L/K} = W_{\mathcal{D}',L/K}$, L/K has equal wild jumps at $\mathcal{C}$ and $\mathcal{D}$ as well.

In the case when only one component of R (say, F_1) passes through O the argument is similar. □

6.1.1 Corollary. *If L/K is a cyclic extension of degree p, Theorem 4.1 holds for $\mathcal{X}$ and L/K.*

Proof. Take

$$\Delta_{\mathcal{X}}(L/K, i) = i \max(m_1+1, \dots, m_s+1, i). \quad \square$$

A slight modification of the above argument enables us to prove the following analog of Lemma 5.3. The assumptions about $\mathcal{X}$ and L/K are as in Prop. 6.1; $n_{\mathcal{C}}$ denotes the number of non-equivalent liftings of $\mathcal{C}$.

6.2 Lemma. *Let $\mathcal{C}$ be a primitive arc on $\mathcal{X}$ with $n_{\mathcal{C}} = 1$. Let $\mathcal{D}$ be a primitive arc on $\mathcal{X}$ such that $E_{\mathcal{D}} < pE_{\mathcal{C}}$, and*

$$(\mathcal{C}.\mathcal{D}) > (\mathcal{C}.R_0(\mathcal{X}, L/K))E_{\mathcal{D}} + E_{\mathcal{C}}E_{\mathcal{D}}M_{\mathcal{C}},$$

where $R_0(\mathcal{X}, L/K)$ is as in Prop. 6.1. Then

1. *$n_{\mathcal{D}} = 1$;*
2. *$E_{\mathcal{D}} \geq E_{\mathcal{C}}$.*

Proof. Let $n = M_{\mathcal{C}}$. Then $\mathcal{C}$ and $\mathcal{D}$ have at least n common infinitely near points $O_1, \ldots, O_n$. Choose any basis of local parameters at $O_0 = P_{\mathcal{C}} = P_{\mathcal{D}}$ such that local equations of all components of $R_{L/K,\mathcal{X}}$ passing through O_0 are within this basis. This choice determines the choice of a distinguished basis of local parameters with the same property at each of the O_i. Let $\mathcal{X}_i$ be the blowing up of $\mathcal{X}_{i-1}$ at O_{i-1}, $\mathcal{X}_0 = \mathcal{X}$. Let r be the number of lines in the Hamburger-Noether expansion of $\mathcal{C}$. Considering Hamburger-Noether expansions in the chosen basis, we see from $(\mathcal{C}.\mathcal{D}) > E_{\mathcal{C}}E_{\mathcal{D}}M_{\mathcal{C}}$ that the first $r-1$ lines in the expansions of $\mathcal{C}$ and $\mathcal{D}$ are identical.

Let $\mathcal{C}_n$ and $\mathcal{D}_n$ be the arcs which are the nth strict transforms in $\mathcal{X}_n$ of $\mathcal{C}$ and $\mathcal{D}$ respectively. Since $n_{\mathcal{C}} = 1$, we see that $w^{(1)}_{\mathcal{C}_n} = w^{(1)}_{\mathcal{C}} > 0$. In particular, at least one component of $R_{L/K,\mathcal{X}_n}$ (say, F_1) passes through O_n. Let t, u be the distinguished basis at O_n. Then we may assume without loss of generality that t is the local equation of F_1, and u is the local equation of F_2, if there exists another component F_2 of $R_{L/K,\mathcal{X}_n}$ passing through O_n.

An argument similar to the step of induction in the previous proof shows that

$$(\mathcal{C}_n.\mathcal{D}_n) > (\mathcal{C}_n.R_0(\mathcal{X}_n, L/K))E_{\mathcal{D}}. \tag{9}$$

Consider the Hamburger-Noether expansion of the regular arc $\mathcal{C}_n$ in t, u:

$$f^{\mathcal{C}_n}(t) = \beta_1 f^{\mathcal{C}_n}(u) + \beta_2 f^{\mathcal{C}_n}(u)^2 + \ldots$$

Let

$$j = \min\{i | \beta_i \neq 0\} = (\mathcal{C}_n.F_1).$$

We see from (9) that $(\mathcal{C}_n.\mathcal{D}_n) > jE_{\mathcal{C}_n}E_{\mathcal{D}_n}$, whence $\mathcal{C}_n$ and $\mathcal{D}_n$ have at least j common infinitely near points. In particular, the first line of the Hamburger-Noether expansion of $\mathcal{D}_n$ in t, u begins as

$$f^{\mathcal{D}_n}(t) = \beta_j f^{\mathcal{D}_n}(u)^j + \ldots,$$

and

$$v_X(f^{\mathcal{D}_n}(t)) = jv_X(f^{\mathcal{D}_n}(u)).$$

Looking at the first $r-1$ lines of the Hamburger-Noether expansion of $\mathcal{C}$ and $\mathcal{D}$, we conclude that $E_{\mathcal{D}} = E_{\mathcal{C}}v_X(f^{\mathcal{D}_n}(u))$, whence $E_{\mathcal{D}} \geq E_{\mathcal{C}}$, and $v_X(f^{\mathcal{D}_n}(u)) < p$.

Note that $R^{\mathcal{C}_n} = k[[T, U]]/(-T + B_0(U))$, where

$$B_0(U) = \beta_j U^j + \beta_{j+1}U^{j+1} + \ldots$$

By the definition,

$$(\mathcal{C}_n.\mathcal{D}_n) = v_X(-f^{\mathcal{D}_n}(t) + B_0(f^{\mathcal{D}_n}(u))),$$

and we can write

$$f^{\mathcal{D}_n}(t) = B_0(f^{\mathcal{D}_n}(u)) + \delta,$$

where $v_X(\delta) > (\mathcal{C}_n.R_0(\mathcal{X}_n, L/K))E_{\mathcal{D}}$.

Assume that both F_1 and F_2 pass through O_n. (The case when only F_1 passes through O_n is similar.) Then L/K corresponds to an Artin-Schreier equation

$$x^p - x = t^{-m_1}u^{-m_2}\varepsilon(t,u),$$

in the completion of $\mathcal{O}_{\mathcal{X}_n,O_n}$, where $\varepsilon \in k[[T,U]]$. Then $w^{(1)}_{\mathcal{C}_n}$ and $w^{(1)}_{\mathcal{D}_n}$ are the ramification jumps of the equations

$$x^p - x = B_0^{-m_1}X^{-m_2}\varepsilon(B_0, X),$$

and

$$x^p - x = f^{\mathcal{D}_n}(t)^{-m_1} f^{\mathcal{D}_n}(u)^{-m_2}\varepsilon(f^{\mathcal{D}_n}(t), f^{\mathcal{D}_n}(u)),$$

respectively. Note that

$$\begin{aligned} f^{\mathcal{D}_n}(t)^{-m_1} f^{\mathcal{D}_n}(u)^{-m_2}\varepsilon(f^{\mathcal{D}_n}(t), f^{\mathcal{D}_n}(u)) &= \\ = (B_0(f^{\mathcal{D}_n}(u)) + \delta)^{-m_1} f^{\mathcal{D}_n}(u)^{-m_2}\varepsilon(B_0(f^{\mathcal{D}_n}(u)) + \delta, f^{\mathcal{D}_n}(u)) \\ \equiv B_0(f^{\mathcal{D}_n}(u))^{-m_1} f^{\mathcal{D}_n}(u)^{-m_2}\varepsilon(B_0(f^{\mathcal{D}_n}(u)), f^{\mathcal{D}_n}(u)) \mod k[[X]]. \end{aligned}$$

It follows that $w^{(1)}_{\mathcal{D}_n}$ is equal to the ramification jump of

$$x^p - x = \Lambda(f^{\mathcal{D}_n(u)}),$$

where $\Lambda = B_0^{-m_1}X^{-m_2}\varepsilon(B_0, X)$. Taking into account that $v_X(f^{\mathcal{D}_n(u)}) < p$, we conclude that $w^{(1)}_{\mathcal{D}_n}$ is exactly $v_X(f^{\mathcal{D}_n(u)})$ times the ramification jump of $x^p - x = \Lambda$, which is $w^{(1)}_{\mathcal{C}_n}$. In particular, $w^{(1)}_{\mathcal{D}_n} > 0$. □

Now we prove an analog of Prop. 5.5.

6.3 Proposition. *Let $\mathcal{X} = \operatorname{Spec} A$, $\mathcal{X}' = \operatorname{Spec} A_1$, where A and A_1 are as in Prop. 5.1, $l = p$. Let $\mathcal{C}$ be a primitive arc on $\mathcal{X}$ with $[\mathcal{C}] \neq F_T$, and $\mathcal{C}'$ its lifting. Then*

$$M_{\mathcal{C}'} < \frac{p}{2}(\mathcal{C}.R_0(\mathcal{X}, L/K))E_{\mathcal{C}} + \frac{p}{2}M_{\mathcal{C}}E_{\mathcal{C}}^2.$$

Proof. If $n_{\mathcal{C}} = p$, we have $M_{\mathcal{C}'} \leq M_{\mathcal{C}}$ by Prop. 5.1. We may, therefore, assume that $n_{\mathcal{C}} = 1$. Introduce $\mathcal{D}'$ and $\mathcal{D}$ exactly as in the proof of 5.5. Again, we have $E_{\mathcal{D}}/n_{\mathcal{D}} < E_{\mathcal{C}}$.

Let first $n_{\mathcal{D}} = 1$. Then $E_{\mathcal{C}} > E_{\mathcal{D}}$. By Lemma 6.2, we have

$$(\mathcal{C}.\mathcal{D}) \leq (\mathcal{C}.R_0(\mathcal{X}, L/K))E_{\mathcal{D}} + M_{\mathcal{C}}E_{\mathcal{C}}E_{\mathcal{D}}.$$

Next, by Prop. 5.1 we have $(\mathcal{C}'.\mathcal{D}') = p(\mathcal{C}.\mathcal{D})$, whence

$$M_{\mathcal{C}'} \leq \frac{1}{2}(\mathcal{C}'.\mathcal{D}') < \frac{p}{2}(\mathcal{C}.R_0(\mathcal{X}, L/K))E_{\mathcal{C}} + \frac{p}{2}M_{\mathcal{C}}E_{\mathcal{C}}^2. \tag{10}$$

Let finally $n_{\mathcal{D}} = p$. Then by Prop. 5.1 we have $(\mathcal{C}'.\mathcal{D}') = (\mathcal{C}.\mathcal{D})$. On the other hand, $E_{\mathcal{D}} < pE_{\mathcal{C}}$, and Lemma 6.2 implies

$$\begin{aligned}(\mathcal{C}.\mathcal{D}) &\leq (\mathcal{C}.R_0(\mathcal{X}, L/K))E_{\mathcal{D}} + M_{\mathcal{C}}E_{\mathcal{C}}E_{\mathcal{D}} \\ &< p(\mathcal{C}.R_0(\mathcal{X}, L/K))E_{\mathcal{C}} + pM_{\mathcal{C}}E_{\mathcal{C}}^2,\end{aligned}$$

and we have (10) again. □

7 Some reductions

In this section we show that Theorem 4.1 holds for $\mathcal{X}$ if it holds for some natural modifications of $\mathcal{X}$.

Throughout this section $\mathcal{X}$ is a surface over k, and L/K is a finite solvable Galois extension of the field of functions on $\mathcal{X}$.

Zariski covering

7.1 Proposition. *Let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$, $\mathcal{X}_2$ are open subschemes in $\mathcal{X}$. Then Theorem 4.1 holds for $\mathcal{X}$ and L/K if it holds for $\mathcal{X}_1$ and L/K as well as for $\mathcal{X}_2$ and L/K.*

7.1.1 Remark. If, say, $\mathcal{X}_1$ is not 2-dimensional, there are no arcs on it. We agree that Theorem 4.1 is satisfied for $\mathcal{X}_1$ in this case.

Proof. It is sufficient to put

$$\Delta_{\mathcal{X}}(L/K, i) = \max(\Delta_{\mathcal{X}_1}(L/K, i), \Delta_{\mathcal{X}_2}(L/K, i))$$

for any i. If, say, $\mathcal{X}_1$ is not 2-dimensional, we assume $\Delta_{\mathcal{X}_1}(L/K, i) = 0$. □

Monoidal transformation

Here we shall deduce the existence of a uniform sufficient jet order for a surface $\mathcal{X}$ from that for its monoidal transformation, $\mathcal{X}_1$.

7.2 Proposition. *Let $\mathcal{X}_1 \to \mathcal{X}$ be a monoidal transformation. Then Theorem 4.1 holds for $\mathcal{X}$ and L/K if it holds for $\mathcal{X}_1$ and L/K.*

Proof. Denote by $O \in \mathcal{X}$ the closed point being blown up and by E the exceptional curve. Let $F_1, \ldots, F_n$ be all irreducible components of $R_{L/K,\mathcal{X}}$. Then, obviously, the irreducible components of $R_{L/K,\mathcal{X}_1}$ are $F_1^1, \ldots, F_n^1$ and, possibly, E, where F_j^1 is the strict transform of F_j, $j = 1, \ldots, n$.

Assume that Theorem 4.1 holds for $\mathcal{X}_1$, and $\Delta_{\mathcal{X}_1}(L/K, -)$ is as in Theorem 4.1. Assume first that O lies on at least one of F_i, say, on F_1. Introduce

$$\Delta_{\mathcal{X}}(L/K, i) = 2\Delta_{\mathcal{X}_1}(L/K, i) + i^2.$$

Now let $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ be such that

$$(\mathcal{C}.\mathcal{D}) \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})),$$

and $P_{\mathcal{C}} = P_{\mathcal{D}} =: P$. Let $\mathcal{C}'$ and $\mathcal{D}'$ be the strict transforms of $\mathcal{C}$ and $\mathcal{D}$ respectively. If $P \neq O$, we have

$$\begin{aligned}(\mathcal{C}'.\mathcal{D}') &= (\mathcal{C}.\mathcal{D}) \\ &> ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\ &\geq ((\mathcal{C}.R_{L/K,\mathcal{X}_1}) + \max(M_{\mathcal{C}'}, M_{\mathcal{D}'}))\Delta_{\mathcal{X}_1}(L/K, \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})).\end{aligned}$$

Let $P = O$. Then it follows from (3) that

$$(\mathcal{C}'.\mathcal{D}') \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) - E_{\mathcal{C}}E_{\mathcal{D}}.$$

We have (e. g., also by (3)) that $(\mathcal{C}.F_1) \geq E_{\mathcal{C}}$, and $E_{\mathcal{C}} = (\mathcal{C}'.E)$ by Lemma 3.4. Next,

$$\begin{aligned}&(\mathcal{C}.R)\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\ &\qquad \geq 2(\mathcal{C}.R)\Delta_{\mathcal{X}_1}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) + E_{\mathcal{C}}E_{\mathcal{D}},\end{aligned}$$

whence

$$\begin{aligned}(\mathcal{C}'.\mathcal{D}') &> 2((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}_1}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\ &\geq ((\mathcal{C}.F_1) + \cdots + (\mathcal{C}.F_n) + E_{\mathcal{C}} + \max(M_{\mathcal{C}}, M_{\mathcal{D}})) \\ &\quad \times \Delta_{\mathcal{X}_1}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\ &\geq ((\mathcal{C}'.R_{L/K,\mathcal{X}_1}) + \max(M_{\mathcal{C}'}, M_{\mathcal{D}'})) \\ &\quad \times \Delta_{\mathcal{X}_1}(L/K, \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})).\end{aligned}$$

Since Theorem 4.1 holds for $\mathcal{X}_1$ and L/K, we conclude

$$w_{\mathcal{C}}^{(i)} = w_{\mathcal{C}'}^{(i)} = w_{\mathcal{D}'}^{(i)} = w_{\mathcal{D}}^{(i)}$$

for all $i = 1, 2, \ldots$

Consider the remaining case, when O lies on none of F_i. In this case introduce

$$\Delta_{\mathcal{X}}(L/K, i) = \Delta_{\mathcal{X}_1}(L/K, i).$$

If $P \neq O$, we proceed as in the former case. Finally, if $P = O$, both $\mathcal{C}$ and $\mathcal{D}$ do not meet the branch locus, whence $w_{\mathcal{C}}^{(i)} = 0 = w_{\mathcal{D}}^{(i)}$ for $i = 1, 2, \ldots$ □

Unramified extension

7.3 Proposition. *Let $\mathcal{X}$ be a surface over k, L/K a finite Galois extension of the fraction field of $\mathcal{X}$, K'/K an unramified extension, and $\mathcal{X}'$ the normalization of $\mathcal{X}$ in K'. Then Theorem 4.1 holds for $\mathcal{X}$ and L/K iff it holds for $\mathcal{X}'$ and $K'L/K'$ (with $\Delta_{\mathcal{X}}(L/K, -) = \Delta_{\mathcal{X}'}(K'L/K', -)$).*

Proof. Denote by $f : \mathcal{X}' \to \mathcal{X}$ the normalization morphism. Then $f^*R_{L/K,\mathcal{X}} = R_{K'L/K',\mathcal{X}'}$. Let $\mathcal{C}, \mathcal{D}$ be arcs on $\mathcal{X}'$ with $P_{\mathcal{C}} = P_{\mathcal{D}} =: P'$, and let $P = f(P')$ be a regular point. The ring $\mathcal{O}_{\mathcal{X}',P'}$ is unramified over $\mathcal{O}_{\mathcal{X},P}$, whence $\widehat{\mathcal{O}_{\mathcal{X}',P'}} = \widehat{\mathcal{O}_{\mathcal{X},P}}$, and

$$
\begin{aligned}
(\mathcal{C}.\mathcal{D}) &= (f \circ \mathcal{C}.f \circ \mathcal{D}),\\
E_{\mathcal{C}} &= E_{f\circ\mathcal{C}},\\
M_{\mathcal{C}} &= M_{f\circ\mathcal{C}},\\
w^{(i)}_{\mathcal{C}}(K'L/K') &= w^{(i)}_{f\circ\mathcal{C}}(L/K),\\
w^{(i)}_{\mathcal{D}}(K'L/K') &= w^{(i)}_{f\circ\mathcal{D}}(L/K).
\end{aligned}
$$

Note that for any component F of $R_{L/K,\mathcal{X}}$, f^*F is the sum of $[K' : K]$ distinct prime divisors with no common points. Then $(f \circ \mathcal{C}.R_{L/K,\mathcal{X}}) = (\mathcal{C}.R_{K'L/K',\mathcal{X}'})$ for any arc $\mathcal{C}$ on $\mathcal{X}'$.

Let Theorem 4.1 hold for $\mathcal{X}$ and L/K. Put

$$\Delta_{\mathcal{X}'}(K'L/K', i) = \Delta_{\mathcal{X}}(L/K, i),$$

and we see that Theorem 4.1 holds for $\mathcal{X}'$ and $K'L/K'$.

Conversely, let Theorem 4.1 be true for $\mathcal{X}'$ and L/K. Introduce

$$\Delta_{\mathcal{X}}(L/K, i) = \Delta_{\mathcal{X}'}(K'L/K', i).$$

Any arc $\mathcal{C}$ on $\mathcal{X}$ can be written as $\mathcal{C} = f \circ \mathcal{C}'$, where $\mathcal{C}'$ is an arc on $\mathcal{X}'$. We conclude that Theorem 4.1 holds for $\mathcal{X}$ and L/K. □

8 Proof of main theorem

8.1 Lemma. *Let $\mathcal{X}$ be a surface over k and L/K a finite tame extension of the fraction field of $\mathcal{X}$. Then Theorem 4.1 holds for $\mathcal{X}$ and L/K.*

Proof. In this case we have $w^{(i)}_{\mathcal{C}} = 0$ for any i and any suitable $\mathcal{C}$. □

A morphism of k-surfaces $\mathcal{X}_2 \to \mathcal{X}_1$ is said to be *tame* if it is dominant and proper and the corresponding extension of function fields is tame. (Such morphisms are in fact compositions of normalizations in tame extensions and birational morphisms.)

8.2 Lemma. *Let $\mathcal{X}_2 \to \mathcal{X}_1$ be a tame morphism, and let $L/k(\mathcal{X}_1)$ be a finite extension. Denote by $\mathcal{X}'_1$ (resp., $\mathcal{X}'_2$) the normalization of $\mathcal{X}_1$ (resp., $\mathcal{X}_2$) in L (resp., $Lk(\mathcal{X}_2)$). Then there exists a tame morphism $\mathcal{X}'_2 \to \mathcal{X}'_1$ such that the corresponding field extension is $Lk(\mathcal{X}_2)/L$.*

Proof. Since $\mathcal{X}'_1 \to \mathcal{X}_1$ is finite, any valuation associated with a prime divisor on $\mathcal{X}'_1$ s an extension of a valuation associated with a prime divisor on $\mathcal{X}_1$. Therefore, $Lk(\mathcal{X}_2)/L$ is tame with respect to $\mathcal{X}'_1$. It remains to note that the composition $\mathcal{X}'_2 \to \mathcal{X}_2 \to \mathcal{X}_1$ factors through $\mathcal{X}'_1$. □

8.3 Proposition. *Let $\mathcal{X}$ be a surface over k and L/K a finite solvable Galois extension of the fraction field of $\mathcal{X}$. Let $\mathcal{X}'$ be the normalization of $\mathcal{X}$ in a finite solvable Galois extension K' of K. Assume that for any tame morphism of k-surfaces $\mathcal{X}'' \to \mathcal{X}'$, Theorem 4.1 holds for $\mathcal{X}''$ and $k(\mathcal{X}'')L/k(\mathcal{X}'')$. Then Theorem 4.1 holds for $\mathcal{X}$ and L/K.*

Proof. The proof consists of 3 steps.

1 We apply induction on $[K' : K]$.

Proposition is trivial for $[K' : K] = 1$. Take $N > 1$, and assume that Proposition holds for all extensions of degree smaller than N.

Denote by K_1/K any cyclic subextension in K'/K of prime degree and by $\mathcal{X}_1$ the normalization of $\mathcal{X}$ in K_1. By the induction hypothesis, Theorem 4.1 holds for $\mathcal{X}_1$ and K_1L/K_1. In view of Lemma 8.2, the same is true if one replaces $\mathcal{X}_1$ with any $\mathcal{X}_1''$ such that there exists a tame morphism $\mathcal{X}_1'' \to \mathcal{X}_1$.

Thus, we have reduced Proposition to the following case: K'/K is a cyclic extension of prime degree l. We may also assume that $\mathcal{X}$ is regular just removing all singular points from it.

2 Assume first that $l \neq p$.

First of all, we may assume without loss of generality that

$$R_{L/K,\mathcal{X}} \cup R_{K'/K,\mathcal{X}} = R_{K'L/K,\mathcal{X}} =: R$$

is a simply normal crossing divisor. Indeed, let $\mathcal{X}_n$ be as in Proposition 1.2, and $\mathcal{X}_n'$ be the normalization of $\mathcal{X}_n$ in K'. Then the proper morphism $\mathcal{X}_n' \to \mathcal{X}$ factors through a proper birational morphism $\mathcal{X}_n' \to \mathcal{X}'$. If $\mathcal{X}'' \to \mathcal{X}_n'$ is a proper dominant morphism of k-surfaces such that $k(\mathcal{X}'') = K''$ is a finite tame extension of K' with respect to $\mathcal{X}_n'$, then $\mathcal{X}'' \to \mathcal{X}'$ is also a proper dominant morphism of k-surfaces, and K''/K' is obviously tame with respect to $\mathcal{X}'$. If Proposition holds for $\mathcal{X}_n$, it holds also for $\mathcal{X}$ by Proposition 7.2.

In view of 7.1, we may assume that all the components of $R_{K'L/K,\mathcal{X}}$ pass through some $P \in \mathcal{X}$, and $\mathcal{X} = \operatorname{Spec} A$ is a sufficiently small affine neighborhood of P. Let $t, u \in A$ be a system of local parameters at P which includes the local equations of all components of $R_{K'L/K,\mathcal{X}}$. Taking $\mathcal{X}$ small, we may also assume that t and u are prime elements in A, i. e., the Cartier divisor of t (resp., u) is a prime divisor F_t (resp., F_u).

If $R_{K'/K,\mathcal{X}}$ is empty, Proposition follows immediately from Prop. 7.3. Next, consider the case when $R_{K'/K,\mathcal{X}}$ consists of one component, and let $t \in A \subset \mathcal{O}_{\mathcal{X},P}$ be the local equation of this component. By Kummer theory, $K' = K(\sqrt[l]{a})$ for some $a \in K^*$. Let

$$a = \varepsilon t^{n_0} p_1^{n_1} \dots p_s^{n_s}$$

be the canonical factorization of a in $\mathcal{O}_{\mathcal{X},P}$, where ε is invertible. The extension K'/K is ramified at (t) and unramified at all (p_i), whence p divides (resp., does not divide) n_i for $i > 0$ (resp., $i = 0$). Choosing another generator of the subgroup $\langle a\rangle (K^*)^p/(K^*)^p$ and another generator of the prime ideal (t), we may assume that $a = t$.

Let $t_1^l = t$. It is easy to see that all the singular points of $\operatorname{Spec} A[t_1]$ lie over singular points of the curve $\operatorname{Spec} A/(t)$. Note that this curve coincides locally with $R_{K'/K,\mathcal{X}}$ and is therefore regular at P. If we replace $\mathcal{X}$ with a smaller affine neighborhood of P, we may assume that $\operatorname{Spec} A[t_1]$ is regular, whence $\mathcal{X}' = \operatorname{Spec} A[t_1]$ is the normalization of $\mathcal{X}$ in K'. We may also assume that t_1 is a prime element of $A[t_1]$, i. e., the Cartier divisor of t_1 is a prime divisor F_{t_1}.

Denote by $f : \mathcal{X}' \to \mathcal{X}$ the normalization morphism. Then

$$f^* R \supset R_{K'L/K',\mathcal{X}'}.$$

Let Q be a point on R, and let $Q' \in \mathcal{X}'$ be a point above Q. One of the components of R passing through Q has t as a local equation. If there is another component of R through Q, let $u_Q = u$ be its local equation (we have $Q = P$ in this case); otherwise choose any $u_Q \in A$ such that t, u_Q are local parameters at Q. It follows that t_1, u_Q are local parameters at Q'. Since f^*R is locally the zero locus of either t_1 or $t_1 u_Q$, we conclude that $R_{K'L/L,\mathcal{X}'}$ is a simply normal crossing divisor.

Since Theorem 4.1 holds for $\mathcal{X}'$ and $K'L/K'$, there exist some integers $\Delta_{\mathcal{X}'}(K'L/K', i)$, $i = 1, 2, \ldots$, such that if $\mathcal{C}', \mathcal{D}' \in \mathbf{U}_{\mathcal{X}'}$ are primitive arcs with $P_{\mathcal{C}'} = P_{\mathcal{D}'}$ and

$$\begin{aligned}(\mathcal{C}'.\mathcal{D}') \geq &((\mathcal{C}'.F_{t_1}) + (\mathcal{C}'.F_u) + \max(M_{\mathcal{C}'}, M_{\mathcal{D}'})) \\ &\times \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}}, E_{\mathcal{D}})),\end{aligned} \tag{11}$$

then $w^{(i)}_{\mathcal{C}'}(K'L/K') = w^{(i)}_{\mathcal{D}'}(K'L/K')$ for any i. (If, say, $R_{K'L/K',\mathcal{X}'}$ is just F_{t_1}, we omit the term $(\mathcal{C}'.F_u)$.)

Now introduce

$$\Delta_{\mathcal{X}}(L/K, i) = l^2 i^2 \Delta_{\mathcal{X}'}(K'L/K', li)$$

for all positive integers i.

Let $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ be primitive arcs such that

$$(\mathcal{C}.\mathcal{D}) \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})).$$

Let $Q = P_{\mathcal{C}} = P_{\mathcal{D}}$, $Q' \in \mathcal{X}'$ be such that $Q = f(Q')$. Let $Q \in F_t$. (We omit the trivial case $Q \notin F_t$.) Localization and completion at Q and Q' respectively make f into the morphism $\operatorname{Spec} k[[T_1, U]] \to \operatorname{Spec} k[[T, U]]$ from Prop. 5.1. It follows from Prop. 5.1 that there exist primitive arcs $\mathcal{C}', \mathcal{D}'$ which are liftings of $\mathcal{C}, \mathcal{D}$ respectively, and $P_{\mathcal{C}'} = P_{\mathcal{D}'} = Q'$. Fix $\mathcal{D}'$ and choose such lifting $\mathcal{C}'$ of $\mathcal{C}$ that $(\mathcal{C}'.\mathcal{D}')$ is maximal. We have $(\mathcal{C}'.F_u) = l(\mathcal{C}.F_u)$. (This is Prop. 5.1 if $Q = P$, and $0 = 0$ if $Q \neq P$.)

Applying further Prop. 5.1, we obtain

$$\begin{aligned} n_{\mathcal{C}} n_{\mathcal{D}}(\mathcal{C}'.\mathcal{D}') &\geq l(\mathcal{C}.\mathcal{D}) \\ &\geq l((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\ &\geq (n_{\mathcal{C}}(\mathcal{C}'.R(K'L/K', \mathcal{X}')) + l\max(M_{\mathcal{C}}, M_{\mathcal{D}}))l^2 \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \\ &\quad \times \Delta_{\mathcal{X}'}(K'L/K', l\max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\ &\geq (n_{\mathcal{C}}(\mathcal{C}'.R(K'L/K', \mathcal{X}')) + l\max(M_{\mathcal{C}}, M_{\mathcal{D}}))l^2 \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \\ &\quad \times \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})),\end{aligned}$$

whence

$$\begin{aligned} (\mathcal{C}'.\mathcal{D}') &\geq ((\mathcal{C}'.R(K'L/K', \mathcal{X}')) + \max(M_{\mathcal{C}}, M_{\mathcal{D}})) \\ &\qquad \times l\max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})) \\ &\geq (l(\mathcal{C}'.R(K'L/K', \mathcal{X}')) + l\max(M_{\mathcal{C}}, M_{\mathcal{D}})\max(E_{\mathcal{C}}, E_{\mathcal{D}})^2) \\ &\quad \times \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})).\end{aligned}$$

By Prop. 5.5,

$$M_{\mathcal{C}'} \leq \frac{l}{2}(M_{\mathcal{C}} + 1)E^2_{\mathcal{C}} \leq l M_{\mathcal{C}} E^2_{\mathcal{C}},$$

if $E_{\mathcal{C}} \neq 1$, and $M_{\mathcal{C}'} \leq \frac{l}{2}$ otherwise. Therefore,

$$\max(M_{\mathcal{C}'}, M_{\mathcal{D}'}) \leq l \max(M_{\mathcal{C}}, M_{\mathcal{D}}) \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2,$$

if $E_{\mathcal{C}} \neq 1$ or $E_{\mathcal{D}} \neq 1$, and

$$\max(M_{\mathcal{C}'}, M_{\mathcal{D}'}) \leq \frac{l}{2},$$

if $E_{\mathcal{C}} = E_{\mathcal{D}} = 1$. In both cases we conclude that (11) is valid. It follows

$$w^{(i)}_{\mathcal{C}}(L/K) = w^{(i)}_{\mathcal{C}'}(K'L/K') = w^{(i)}_{D'}(K'L/K') = w^{(i)}_{\mathcal{D}}(L/K)$$

for any i.

It remains to consider the case when $R_{K'/K,\mathcal{X}}$ consists of two components. Consider $K_1 = K(t_1)$ and $K_2 = K(t_1, u_1)$, where $t_1^l = t$, $u_1^l = u$. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the normalizations of $\mathcal{X}$ in K_1 and K_2 respectively. Then K_2 is a tame extension of K', whence Theorem 4.1 is true for $\mathcal{X}_2$ and K_2L/K_2. Applying the already considered case twice, we conclude that Theorem 4.1 holds for $\mathcal{X}_1$ and K_1L/K_1, as well as for $\mathcal{X}$ and L/K.

3 Let now $l = p$.

Let K'/K be given by Artin–Schreier equation

$$x^p - x = a,$$

$a \in K$. An argument as in the case $l \neq p$ shows that we may replace $\mathcal{X}$ with its monoidal transformation. After a suitable sequence of monoidal transformations, we may assume that $R \cup \operatorname{div} a$ is a normal crossing divisor, where $\operatorname{div} a$ is the divisor of a. It follows that in the neighborhood of any point P we have $a = \varepsilon t^i u^j$ in $\mathcal{O}_{\mathcal{X},P}$, where $\varepsilon \in \mathcal{O}^*_{\mathcal{X},P}$, and t, u are local parameters at P. Moreover, L/K is unramified with respect to any prime $\mathfrak{p}$ of $\mathcal{O}_{\mathcal{X},P}$ unless $\mathfrak{p} = (t)$ or $\mathfrak{p} = (u)$. Choosing a particular Artin-Schreier equation, we may require that either i or j is prime to p. Changing the respective local parameter, we may assume $\varepsilon = 1$.

In view of 7.1, we may assume that all the components of $R \cup \operatorname{div} a$ pass through some $P \in \mathcal{X}$, and $\mathcal{X} = \operatorname{Spec} A$ is a sufficiently small affine neighborhood of P. Taking $\mathcal{X}$ small, we may also assume that the divisor of t (resp., u) is a prime divisor F_t (resp., F_u).

Thus, we have

$$\operatorname{div} a = iF_t + jF_u.$$

We may also assume that at least one of i, j is negative. (Otherwise, K'/K is unramified, and we argue as in the case $l \neq p$.) We shall assume $i < 0$. If $j > 0$, we blow up at P and obtain a scheme with

$$\operatorname{div} a = iF_1 + (i + j)E + jF_2,$$

where F_1 is the strict transform of F_t, E is the exceptional divisor, F_2 is the strict transform of F_u. Note that $|i + j| < \max(|i|, |j|)$. If $i + j > 0$ (resp., $i + j < 0$), blow up at $F_1 \cap E$ (resp., at $E \cap F_2$) and so on. We obtain a scheme with

$$\operatorname{div} a = n_0E_0 + \cdots + n_sE_s,$$

where E_α does not meet E_β unless $\alpha = \beta \pm 1$, $n_0, \dots, n_{r-1} < 0$, $n_r = 0$, $n_{r+1}, \dots, n_s > 0$ for some r. Locally we have the situation as before, with $j \leq 0$.

We may assume that $(p, i) = 1$. (Change the roles of t and u if necessary. Note that we have already excluded the case $p|i$, $p|j$.) Next, making a tame base change of type $t = t_1^\alpha$, $u = u_1^\beta$, where $\alpha < p$, $\beta < p$, we may assume that $i \equiv 1 \mod p$, and either $j \equiv 0 \mod p$ or $j \equiv -1 \mod p$. In the latter case, blow up the point P. We get a scheme with

$$\operatorname{div} a = iF_1 + (i+j)E + jF_2,$$

where F_1 (resp., F_2) is the strict transform of F_t (resp., F_u), E is the exceptional divisor. We have $i + j \equiv 0 \mod p$.

Thus, it remains to consider the case

$$x^p - x = t^{-mp+1}u^{-np},$$

where $m > 0$ and $n \geq 0$. Put $t_1 = t^m u^n x$. We have

$$t_1^p - t^{m(p-1)}u^{n(p-1)}t_1 = t,$$

whence t_1 is integral over A, and $A[t_1]$ is regular. Therefore, $\mathcal{X}_1 = \operatorname{Spec} A[t_1]$. We may also assume that the divisor of t_1 (resp., of u) on $\mathcal{X}_1$ is a prime divisor F_{t_1} (resp., F_u).

Denote by $f : \mathcal{X}' \to \mathcal{X}$ the normalization morphism. Then

$$f^*R \supset R_{K'L/K', \mathcal{X}'}.$$

Let Q be a point on R, and let $Q' \in \mathcal{X}'$ be a point above Q. Assume that F_t passes through Q. If there is another component of R through Q, let $u_Q = 0$ be its local equation (we have $Q = P$ in this case); otherwise choose any $u_Q \in A$ such that t, u_Q are local parameters at Q. It follows that t_1, u_Q are local parameters at Q'.

Since Theorem 4.1 holds for $\mathcal{X}'$ and $K'L/K'$, there exist some integers $\Delta_{\mathcal{X}'}(K'L/K', i)$, $i = 1, 2, \dots$ such that if $\mathcal{C}', \mathcal{D}' \in \mathbf{U}_{\mathcal{X}'}$ are primitive arcs with $P_{\mathcal{C}'} = P_{\mathcal{D}'}$ and

$$\begin{aligned}(\mathcal{C}'.\mathcal{D}') \geq &((\mathcal{C}'.F_{t_1}) + (\mathcal{C}'.F_u) + \max(M_{\mathcal{C}'}, M_{\mathcal{D}'})) \\ &\times \Delta_{\mathcal{X}'}(K'L/K', \max E_{\mathcal{C}}, E_{\mathcal{D}}),\end{aligned} \tag{12}$$

then $w_{\mathcal{C}'}^{(i)}(K'L/K') = w_{\mathcal{D}'}^{(i)}(K'L/K')$ for any i.

Now introduce

$$\Delta_{\mathcal{X}}^\circ(L/K, i) = p^2(i^2 + \mu i^2)\Delta_{\mathcal{X}'}(K'L/K', pi)$$

for all positive integers i, where $\mu = \max(m_1 + 1, \dots, m_s + 1)$, $m_1, \dots, m_s$ are associated with the extension K'/K as in Proposition 6.1.

Let $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ be primitive arcs such that

$$(\mathcal{C}.\mathcal{D}) \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}^\circ(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})). \tag{13}$$

Assume that the common closed point Q of $\mathcal{C}$ and $\mathcal{D}$ lies on F_t.

Similarly to the case $l \neq p$, we can construct liftings $\mathcal{C}'$ and $\mathcal{D}'$ of $\mathcal{C}$ and $\mathcal{D}$ such that

$$\begin{aligned}
n_{\mathcal{C}} n_{\mathcal{D}}(C_1.D') &\geq p(\mathcal{C}.\mathcal{D}) \\
&\geq p((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\
&\geq p((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))p^2(\max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 + \mu \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2) \\
&\quad \times \Delta_{\mathcal{X}'}(K'L/K', p\max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\
&\geq ((n_{\mathcal{C}}(\mathcal{C}'.R_{K'L/K',\mathcal{X}'}) + p\max(M_{\mathcal{C}}, M_{\mathcal{D}}))p^2 \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \\
&\qquad + p(\mathcal{C}.R)p^2\mu \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2) \\
&\quad \times \Delta_{\mathcal{X}'}(K'L/K', p\max(E_{\mathcal{C}}, E_{\mathcal{D}})) \\
&\geq (n_{\mathcal{C}}(\mathcal{C}'.R_{K'L/K',\mathcal{X}'}) + p\max(M_{\mathcal{C}}, M_{\mathcal{D}}))p^2 \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \\
&\qquad + p(\mathcal{C}.R)p^2\mu \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2) \\
&\quad \times \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})),
\end{aligned}$$

whence

$$\begin{aligned}
(\mathcal{C}'.\mathcal{D}') &\geq ((\mathcal{C}'.R_{K'L/K',\mathcal{X}'}) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))p(\max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \\
&\qquad + p(\mathcal{C}.R)\mu \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2) \\
&\quad \times \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})) \\
&\geq (p(\mathcal{C}'.R_{K'L/K',\mathcal{X}'}) + p\max(M_{\mathcal{C}}, M_{\mathcal{D}}) \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2 \\
&\qquad + p(\mathcal{C}.R_0(\mathcal{X}, K'/K)) \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2) \\
&\quad \times \Delta_{\mathcal{X}'}(K'L/K', \max(E_{\mathcal{C}'}, E_{\mathcal{D}'})).
\end{aligned}$$

By Prop. 6.3,

$$M_{\mathcal{C}'} < \frac{p}{2}(\mathcal{C}.R_0(\mathcal{X}, K'/K))E_{\mathcal{C}} + \frac{p}{2}M_{\mathcal{C}}E_{\mathcal{C}}^2.$$

It follows

$$\max(M_{\mathcal{C}'}, M_{\mathcal{D}'}) < \frac{p}{2}((\mathcal{C}.R_0(\mathcal{X}, K'/K)) + \max(M_{\mathcal{C}}, M_{\mathcal{D}})) \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2.$$

Indeed, this is trivial if both $\mathcal{C}$ and $\mathcal{D}$ are regular. In the opposite case, it follows from (13) that

$$(\mathcal{C}.\mathcal{D}) \geq (M_{\mathcal{C}} + 1) \max(E_{\mathcal{C}}, E_{\mathcal{D}})^2,$$

and Lemma 5.4 implies

$$(\mathcal{D}.R_0(\mathcal{X}, K'/K)) = (\mathcal{C}.R_0(\mathcal{X}, K'/K)) \cdot \frac{E_{\mathcal{D}}}{E_{\mathcal{C}}}.$$

From this, we obtain (11), whence $w_{\mathcal{C}'}^{(i)}(K'L/K') = w_{\mathcal{D}'}^{(i)}(K'L/K')$ for any i.

In the case $Q \notin F_t$ we obtain that $w_{\mathcal{C}'}^{(i)}(K'L/K') = w_{\mathcal{D}'}^{(i)}(K'L/K')$ in a similar way.

Next, by Corollary 6.1.1 we know that Theorem 4.1 is true for $\mathcal{X}$ and K'/K. This means that there exists a non-decreasing sequence of positive numbers $\Delta_{\mathcal{X}}(K'/K, i) \geq i^2$, $i = 1, 2, \ldots$ such that for any $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ with

$$(\mathcal{C}.\mathcal{D}) \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(K'/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})),$$

we have

$$w^{(1)}_{\mathcal{C}}(K'/K) = w^{(1)}_{\mathcal{D}}(K'/K).$$

Now let

$$\Delta_{\mathcal{X}}(L/K, i) = \max(\Delta^{\circ}_{\mathcal{X}}(L/K, i), \Delta_{\mathcal{X}}(K'/K, i)).$$

Take any $\mathcal{C}, \mathcal{D} \in \mathbf{U}_{\mathcal{X}}$ with

$$(\mathcal{C}.\mathcal{D}) \geq ((\mathcal{C}.R) + \max(M_{\mathcal{C}}, M_{\mathcal{D}}))\Delta_{\mathcal{X}}(L/K, \max(E_{\mathcal{C}}, E_{\mathcal{D}})).$$

Then for suitable liftings $\mathcal{C}'$, $\mathcal{D}'$ of $\mathcal{C}$, $\mathcal{D}$ we have

$$W_{\mathcal{C},K'L/K} = W_{\mathcal{C},K'/K} \circ W_{\mathcal{C}',K'L/K'} = W_{\mathcal{D},K'/K} \circ W_{\mathcal{D}',K'L/K'} = W_{\mathcal{D},K'L/K}.$$

Recall that $K'L/L$ is tame, whence

$$W_{\mathcal{C},L/K} = W_{\mathcal{C},K'L/K} = W_{\mathcal{D},K'L/K} = W_{\mathcal{D},L/K},$$

i. e., L/K has equal wild jumps at $\mathcal{C}$ and $\mathcal{D}$. $\square$

Applying Proposition to the case $L = K'$ and using Lemma 8.1, we see that Theorem 4.1 holds in full generality.

9 Towards equal ramification in the strong sense

Let $\mathcal{X}$ be as in the introduction, L/K a finite Galois extension of its function field, and $\mathcal{Y}$ the normalization of $\mathcal{X}$ in L. Assume that $G = \mathrm{Gal}(L/K)$ is a p-group with the following property: any cyclic subgroup of G is normal in G.

9.0.1 Remark. There are some non-abelian groups with this property, e. g., the groups $\langle x, y | x^l = y^l, x^{l^2} = e, xy = y^{l+1}x\rangle$ for any prime l; this was pointed out to me by V. P. Snaith.

9.1 Theorem. *There exists an effective divisor R_0 supported at $R = R_{L/K,\mathcal{X}}$ such that for any $P \in R$ and any $C, C' \in U_P$ the condition $(C.C')_P \geq (C.R_0)_P$ implies that L/K is equally ramified with respect to C and C' at P in the strong sense.*

Proof. In view of Corollary 4.1.1, for any Galois subextension of M/K there exists an effective divisor $R_{0,M/K}$ supported at $R_{M/K,\mathcal{X}} \subset R_{L/K,\mathcal{X}}$ such that $(C.C')_P \geq (C.R_{0,M/K})_P$ implies $W_{C,P,M/K} = W_{C',P,M/K}$. Now fix any R_0 supported at $R = R_{L/K,\mathcal{X}}$ such that $R_0 \geq R_{M/K,\mathcal{X}}$ for any M/K.

Let $C, C' \in U_P$, and $(C.C')_P \geq (C.R_0)_P$. Let C_L and C'_L be any irreducible curves on $\mathcal{Y}$ over C and C' respectively. Let v be the valuation on $k(C)$ that corresponds to P, and let v_L be any extension of v onto $k(C_L)$. Define v' and v'_L similarly. We have to prove that $G_j(L_{C_L,v_L}/K_{C,v}) = G_j(L_{C'_L,v'_L}/K_{C',v'})$ for any $j \geq -1$.

Choose any $g \in D_{C_L,v_L}$. It is sufficient to prove that $g \in D_{C'_L,v'_L}$ and

$$a_{C_L,v_L}(g) = a_{C'_L,v'_L}(g),$$

where

$$a_{C_L,v_L}(g) = \max\{j | g \in G_j(L_{C_L,v_L}/K_{C,v})\},$$

and $a_{C'_L,v'_L}(g)$ is defined similarly. Note that $D_{C_L,v_L} = G_0(L_{C_L,v_L}/K_{C,v})$ since the residue field of the valuation on $K_{C,v}$ is the algebraically closed field k. On the other hand, $L_{C_L,v_L}/K_{C,v}$ is a p-extension, whence $G_0(L_{C_L,v_L}/K_{C,v}) = G_1(L_{C_L,v_L}/K_{C,v})$. Therefore, $a_{C_L,v_L}(g)$ (as well as $a_{C'_L,v'_L}(g)$) is a positive integer.

Let M be the fixed field of $\langle g\rangle$ inside L/K, $\mathcal{Z}$ the normalization of $\mathcal{X}$ in M, C_M (resp., C'_M) the image of C_L (resp., C'_L) on $\mathcal{Z}$, v_M (resp., v'_M) the restriction of v_L (resp., v'_L) onto the function field of C_M (resp., C'_M). By the definition of R_0 we have $W_{C,P,L/K} = W_{C',P,L/K}$ and $W_{C,P,M/K} = W_{C',P,M/K}$, whence

$$\begin{aligned} W_{L_{C_L,v_L}/M_{C_M,v_M}} &= W^{-1}_{M_{C_M,v_M}/K_{C,v}} \circ W_{L_{C_L,v_L}/K_{C,v}} \\ &= W^{-1}_{M_{C'_M,v'_M}/K_{C',v'}} \circ W_{L_{C'_L,v'_L}/K_{C',v'}} \\ &= W_{L_{C'_L,v'_L}/M_{C'_M,v'_M}}. \end{aligned} \tag{14}$$

Next,

$$G_j(L_{C'_L,v'_L}/M_{C'_M,v'_M}) = G_j(L_{C'_L,v'_L}/K_{C',v'}) \cap \mathrm{Gal}(L_{C'_L,v'_L}/M_{C'_M,v'_M}),$$

whence

$$a_{C_L,v_L}(g) = \max\{j | g \in G_j(L_{C_L,v_L}/M_{C_M,v_M})\}. \tag{15}$$

Let p^r be the order of g in G. The condition $g \in D_{C_L,v_L}$ is equivalent to $|G_1(L_{C_L,v_L}/M_{C_M,v_M})| = p^r$ and to $w^{(r)}(L_{C_L,v_L}/M_{C_M,v_M}) > 0$. Note that $|G_1(L_{C_L,v_L}/M_{C_M,v_M})|^{-1}$ is the minimal slope of the graph of $W_{L_{C_L,v_L}/M_{C_M,v_M}}$. We obtain from (14) that $g \in D_{C'_L,v'_L}$.

It is clear from (15) that

$$a_{C_L,v_L}(g) = w^{(r)}(L_{C_L,v_L}/M_{C_M,v_M}).$$

Since $w^{(r)}(L_{C_L,v_L}/M_{C_M,v_M})$ is just the abscissa of the end of the first segment in the graph of $W_{L_{C_L,v_L}/M_{C_M,v_M}}$, we conclude from (14) that $a_{C_L,v_L}(g) = a_{C'_L,v'_L}(g)$. □

9.1.1 Corollary. *Let $G = \mathrm{Gal}(L/K)$ be a p-group with the following property: any cyclic subgroup of G is normal in G. Then the answer to Question 2.5 is affirmative.*

Proof. See Remark 2.5.1. □

References

[B] N. Bourbaki, *Algèbre Commutative. Eléments de Math. 27,28,30,31*, Hermann, Paris, 1961–1965.

[Br] J.-L. Brylinski, *Théorie du corps de classes de Kato et revêtements abéliens de surfaces*, Ann. Inst. Fourier, Grenoble **33** (1983), 23–38.

[C] A. Campillo, *Algebroid curves in positive characteristic*, Springer-Verlag, Berlin etc, 1980 (Lecture Notes in Mathematics, vol. 813).

[D] P. Deligne, *Letter to L. Illusie of 28.11.76*, to appear in Proceedings of the conference "Ramification theory of arithmetic schemes" (Luminy, 1999).

[N] M. Nagata, *Local rings*, Interscience publ., N.Y. – London, 1962.

[S] J.-P. Serre, *Local fields*, Springer-Verlag, Berlin etc, 1979.

[Z] I. B. Zhukov, *Ramification of surfaces: Artin-Schreier extensions*, Contemporary mathematics **300** (2002), 211–220.

Department of Mathematics and Mechanics, St. Petersburg University
Bibliotechnaya pl., 2, Staryj Petergof
198904 St. Petersburg Russia

E-mail: `Igor.Zhukov@mail.ru, igor_zh@hotmail.com`