Analytical Solutions to Feedback Systems on the Special Orthogonal Group \( SO(n) \)

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Abstract—This paper provides analytical solutions to the closed-loop kinematics of two almost globally exponentially stabilizing attitude control laws on the special orthogonal group \( SO(n) \). By studying the general case we give a uniform treatment to the cases of \( SO(2) \) and \( SO(3) \), which are the most interesting dimensions for application purposes. Working directly with rotation matrices in the case of \( SO(3) \) allows us to avoid certain complications which may arise when using local and global many-to-one parameterizations. The analytical solutions provide insight into the transient behaviour of the system and are of theoretical value since they can be used to prove almost global attractiveness of the identity matrix. The practical usefulness of analytical solutions in problems of continuous time actuation subject to piece-wise unavailable or discrete time sensing are illustrated by numerical examples.

I. INTRODUCTION

Rigid-body attitude stabilization has long been and still remains an active area of research within the field of applied nonlinear control due to its importance in diverse engineering disciplines such as aerospace and robotics. The nonlinear kinematic and dynamic equations as well as the global topology of the rotation group \( SO(3) \) contribute to the difficulty of the problem. The choice of parameterization used to represent \( SO(3) \) is a key aspect of any attitude control law [1], in particular it may limit the performance of a control algorithm [2] or yield counter-intuitive results when relating the global behaviour of a system in the parameter space to that on \( SO(3) \) [3, 4]. A well-known result state it to be impossible to globally stabilize an equilibrium on \( SO(3) \) by means of a continuous time-invariant feedback [3]. It is however possible to achieve almost global asymptotical stability through continuous time-invariant feedback [2], [5], almost semi-global stability [6], or global stability by means of a hybrid control approach [7].

Analytical solutions to the closed-loop state equations provide insight into the transient dynamics of a system and may be of practical value to a control engineer as discussed in §V of this paper. The literature on this subject in the context of attitude control may roughly be divided into two categories. Firstly, there are works for which the closed form solutions emerge as part of the control design, e.g. using exact linearization [8] or optimal control techniques such as Pontryagin’s maximum principle [9]. Secondly, there are studies of rigid-body dynamics such as Euler’s equations of motion that provide solutions under rather specific assumptions which may serve to illuminate certain behaviours of an aerospace vehicle [10], [11], [12].

The contributions of this paper can be summarized in the following points: (i) we provide analytical solutions to the closed-loop kinematics of two feedback systems, (ii) we generalize previous results on attitude control on \( SO(3) \) to the case of \( SO(n) \). Our main result is contribution (i) which none of the previously referenced works aimed to achieve although the works [8], [9] are related in this respect. We also provide a possible application of contribution (i) to address the problem of continuous time actuation subject to piece-wise unavailable or discrete time sensing whereas contribution (ii) is mainly of theoretical interest.

Contribution (i) consists of the interpretation of the closed form solutions as trajectories of a rigid body attitude and of additional results that characterize the behaviour of the underlying system, but not in solving systems of quadratic differential equations. The two equations that are studied in this paper are special cases of the well-known continuous time Riccati differential equation which can be solved using the adjoint equations [13]. However, the solutions of importance in linear-quadratic optimal control are symmetric matrices that are used to define linear feedback laws whereas our interest lies with solutions belonging to the special orthogonal group \( SO(n) \). These matrices are nonsymmetric in general, a fact that is utilized in the control laws we study.

Contribution (ii) is related to works that generalize aspects of rigid-body kinematics and dynamics to the \( n \)-dimensional case. This includes attitude stabilization [14] and attitude synchronization [15] on \( SO(n) \), but also results in mechanics such as generalized equations of motion [16]. There have not, to the best of our knowledge, been published any papers that provide solutions to the closed-loop kinematics of attitude stabilizing feedback laws on \( SO(n) \), or even solutions on \( SO(3) \) to the two systems solved here. A solution to a third system on \( SO(3) \) is given in our previous paper [17]. Unlike this paper however, the solutions of [17] are inconveniently expressed using notation on the level of matrix elements.

NOMENCLATURE

Matrices and vectors are written using bold upper and lower case letters respectively. The standard basis for \( \mathbb{R}^n \) is denoted by \( \{e_1, \ldots, e_n\} \). The following square matrices of dimension \( n \) are used throughout the paper

\[
E_1 = I, \quad E_2 = I - e_ne_n^T, \quad (1)
\]
where the introduction of $E_1$ allows for a more compact exposition. Let $[A, B]$ denote the commutator of $A, B \in \mathbb{R}^{n \times n}$. The transpose of a matrix is written $A^\top$ and the conjugate transpose $A^\ast$. Let $\langle A, B \rangle = \text{tr}(A^\top B)$ denote the inner product between $A, B \in \mathbb{R}^{n \times n}$. The set of skew symmetric matrices are denoted by $\mathfrak{s}o(n)$ and the set of rotation matrices by $\text{SO}(n)$. The set of eigenvalues of a matrix $A$ is written $\sigma(A)$ and is referred to as the spectrum of $A$. The unit circle and the closed unit disc of $z \in \mathbb{C}$ with $|z| = 1$ and $|z| \leq 1$ respectively are written $\mathbb{S}$ and $\mathbb{D}$.

II. PROBLEM STATEMENT

The attitude kinematics of a two or three dimensional rigid body in an inertial frame of reference are given by $\dot{R} = \Omega R$, where $R \in \text{SO}(n)$ and $\Omega \in \mathfrak{s}o(n)$ for $n \in \{2, 3\}$. The body fixed frame control version is $\dot{R} = R \Omega'$, where $\Omega' = R^\top \Omega R$. The inertial and body fixed rotations are related by an orthogonal change of coordinates. Note that these two systems are equivalent in the case of $[\Omega, R] = 0$, a condition that is satisfied by one of the two control laws studied in this paper.

In a more abstract setting, we may consider the problem of formulating a feedback law $\Omega_c(R)$ that continuously rotate a positively oriented coordinate frame of $n$ orthonormal vectors to a setpoint which we without any loss of generality take to be the standard basis of $\mathbb{R}^n$. The generalized kinematics of rotation matrices are

$$\dot{R} = R \Omega_c,$$  

(2)

where $R \in \text{SO}(n)$, $\Omega_c \in \mathfrak{s}o(n)$ for any $n \in \mathbb{N}$ [16]. Equation (2) may be interpreted as saying that $R$ can be actuated in any direction along its tangent plane at the identity $\mathfrak{s}o(n)$. Note that SO$(n)$ is invariant under the kinematics (2), i.e., any solution $R(t)$ that satisfies $R(0) = R_0 \in \text{SO}(n)$ remains a rotation matrix for all $t \in \mathbb{R}^+$. The aim of this paper is to study two generalized attitude control laws on $\text{SO}(n)$, to find the exact solution to their closed-loop system kinematics, to use these solutions to establish almost global exponential stability of the identity matrix, and to provide a practical application of these results.

For $n \in \{2, 3\}$, this problem is closely related to the kinematic attitude stabilization problem and the problem of tracking a $C^1$ reference trajectory in $\text{SO}(n)$. In higher dimensions this is mainly a problem of theoretical interest.

III. CONTROL LAWS

Algorithm 1 (See e.g. [18]): The input matrix and the resulting closed loop system are respectively given by

$$\Omega_1 = R^\top \Omega - R, \quad \dot{R} = I - R^2.$$  

(3)

Remark 1: Algorithm 1 belongs to a class of well-known control laws for attitude stabilization on $\text{SO}(3)$ that uses kinematic level actuation. These algorithms include the geodesic control [18] and a number of feedback laws in the visual servo literature where kinematic level attitude control is often employed [19], [20].

Algorithm 2: The angular velocity matrix and the resulting closed loop system are respectively given by

$$\Omega_2 = E_2 R^\top - R E_2, \quad \dot{R} = E_2 - R E_2 R.$$  

(4)

where $E_1$ is defined by (1).

Remark 2: The diagonal zero of $E_1$ in Algorithm 2 could be placed anywhere along the diagonal but we put it in the bottom-right corner for notational convenience. Another such convenience is the exclusion of any gain parameter $\alpha \in \mathbb{R}^+$ in (3) and (4). Gain parameters can be included in the state equations and their solutions by scaling time.

Algorithm 1 and 2 both stabilize the equilibrium $R = I$ almost globally on $\text{SO}(n)$ while making $R$ move in the steepest descent direction of a Lyapunov function candidate. Each of the two candidates measure the Euclidean distance between $m \in \{n - 1, n\}$ orthogonal vectors and the vectors $\{e_1, \ldots, e_m\} \in \mathbb{R}^n$. Define the Lyapunov function candidates $V_k$, $k \in \{1, 2\}$, by

$$V_k = \sum_{i=1}^{n-k+1} \|R e_i - e_i\|^2 = \sum_{i=1}^{n-k+1} e_i^\top (R - I)^\top (R - I) e_i$$

$$= \text{tr}((R - I)^\top (R - I) E_k)$$

$$= 2(n-k+1) - \text{tr}((R + R^\top) E_k).$$  

(5)

Consider the case of a yet to be specified control signal $\Omega$. Then (2) gives

$$\dot{V}_k = - \text{tr}((\Omega R + R^\top \Omega^\top) E_k) = - \text{tr}(RE_k \Omega - E_k R^\top \Omega)$$

$$= - \text{tr}(RE_k - E_k R^\top \Omega)$$

$$= - \text{tr}((E_k R^\top - R E_k)^\top \Omega) = - \langle \Omega_k, \Omega \rangle.$$  

(6)

Equation (6) shows that the locally minimizing control laws, which we denote $\Omega = U_k$, are aligned with $\Omega_k$ as defined by (3), (4). They can be determined up to scalar factors by

$$U_k = \Omega_k + W_k,$$

where $W_k$ satisfies

$$\langle \Omega_k, W_k \rangle = 0, \quad W_k^\top = -W_k.$$

In the case of Algorithm 1 and 2 we have $W_k = 0$ whereby the optimality condition is satisfied.

Note that one may not use Algorithm 1 to make $R$ move in the steepest descent direction of $V_2$ since $\Omega_1 = \Omega_2 + e_n e_n^\top R^\top - R e_n e_n^\top$ and

$$\langle \Omega_2, e_n e_n^\top R^\top - R e_n e_n^\top \rangle = -2 \sum_{i=1}^{n-1} R_{ni} R_{in},$$

which is nonzero in general. Vice versa,

$$\langle \Omega_1, R e_n e_n^\top R^\top - R e_n e_n^\top \rangle = 2 \left( \sum_{i=1}^{n} R_{ni} R_{in} - 1 \right),$$

which is nonzero in general. It can be shown that $R_{nn} = 1$, $R_{ni} = R_{in} = 0$, $i \in \{1, \ldots, n - 1\}$ is an invariant set of Algorithm 2. In this case Algorithm 2 acts as Algorithm 1 on the first $n - 1$ columns and rows of $R$ which then forms an element of $\text{SO}(n - 1)$.
A control law that locally makes the states move towards an equilibrium in the steepest decent direction does not necessarily make them move along a geodesic curve in the global space. Algorithm 1 does however provide a geodesic control law in the case of SO(3) since it is aligned with the negative matrix logarithm [18].

IV. MAIN RESULTS

Let us first establish existence and uniqueness of the exact solutions. This will allow us to draw conclusions regarding the control performance from the closed form solutions.

Proposition 1: The equations (3) and (4) have unique solutions that belong to SO(n) for all \( t \in \mathbb{R}^+ \).

Proof: See the Appendix.

We introduce the notation

\[
N(n) = \{ \mathbf{R} \in SO(n) \mid -1 \in \sigma(\mathbf{R}) \}. \tag{7}
\]

Note that \( N(n) \) is a null set, i.e., a set of measure zero in SO(n). In this paper we show that \( \mathbf{R} = \mathbf{I} \) is an almost globally exponentially stable equilibrium of the closed-loop systems generated by Algorithm 1 and 2. The region of attraction of \( \mathbf{R} = \mathbf{I} \) is \( SO(n) \wedge N(n) \), i.e., all of \( SO(n) \) except the null set \( N(n) \).

Theorem 1: The trajectories of the closed-loop system resulting from Algorithm 1 are given by

\[
\mathbf{R}(t) = (\tanh(t \mathbf{I} + \mathbf{R}_0)(\mathbf{I} + \tanh(t \mathbf{R}_0))^{-1}. \tag{8}
\]

The limit \( \lim_{t \to \infty} \mathbf{R}(t) \) is well-defined in the case of \( \mathbf{R}_0 \in N(n) \) and can be calculated using the spectral decomposition. Moreover, the equilibrium \( \mathbf{R} = \mathbf{I} \) is almost globally exponentially stable.

Proof: Proving that (8) solves (3) is most easily done by verification. Note that \( \mathbf{R}(0) = \mathbf{R}_0 \) so that the initial condition is satisfied. Rewrite the solution as

\[
\mathbf{R}(t) = (\sinh(t \mathbf{I} + \cosh(t \mathbf{R}_0))(\cosh(t \mathbf{I} + \sinh(t \mathbf{R}_0))^{-1}.
\]

Then

\[
\dot{\mathbf{R}}(t) = (\cosh(t \mathbf{I} + \sinh(t \mathbf{R}_0))(\cosh(t \mathbf{I} + \sinh(t \mathbf{R}_0))^{-1}
\]

\[
= (\sinh(t \mathbf{I} + \cosh(t \mathbf{R}_0))(\cosh(t \mathbf{I} + \sinh(t \mathbf{R}_0))^{-1}
\]

\[
= (\sinh(t \mathbf{I} + \cosh(t \mathbf{R}_0))(\cosh(t \mathbf{I} + \sinh(t \mathbf{R}_0))^{-1}
\]

\[
= \mathbf{I} - \mathbf{R}^2.
\]

The matrix \( \mathbf{R}_0 \) is normal. Let \( \mathbf{U}_0 \lambda_0 \mathbf{U}_0^* \) be the spectral decomposition of \( \mathbf{R}_0 \). Then \( \mathbf{R}(t) = \mathbf{U}_0(\tanh(t \mathbf{I} + \lambda_0))(\mathbf{I} + \tanh(t \lambda_0))^{-1} \mathbf{U}_0^* \) is the spectral decomposition of \( \mathbf{R}(t) \). Denote

\[
\Lambda(t) = (\tanh(t \mathbf{I} + \lambda_0)(\mathbf{I} + \tanh(t \lambda_0))^{-1}. \tag{9}
\]

Note that if \( \Lambda_{ii,0} = -1 \), then \( \Lambda_{ii}(t) = -1 \) for all \( t \in \mathbb{R}^+ \). This shows that \( \lim_{t \to \infty} \mathbf{R}(t) \) is well defined also in the case of \( \mathbf{R}_0 \in N(n) \). The limit is given by \( \lim_{t \to \infty} \mathbf{R}(t) = \lim_{t \to \infty} \mathbf{U}_0 \Lambda(t) \mathbf{U}_0^* \) where

\[
\lim_{t \to \infty} \Lambda_{ij}(t) = \begin{cases} 1 & \text{if } j = i \text{ and } \Lambda_{ii,0} \neq -1, \\ -1 & \text{if } j = i \text{ and } \Lambda_{ii,0} = -1, \\ 0 & \text{otherwise}. \end{cases} \tag{10}
\]

The equations \( 3 \) and \( 4 \) have unique solutions. This will allow us to draw conclusions regarding the control performance from the closed form solutions.

Remark 4: From the closed-loop system (3) it is clear that any symmetric rotation matrix is an equilibrium point. Since the identity is almost globally exponentially stable, all other symmetric matrices must be unstable. The proof of Theorem 1 characterize the complement to the region of attraction of \( \mathbf{I} \) as the set \( N(n) \) defined by (7). In SO(3),

\[
N(3) = \{ \mathbf{R} \in SO(3) \mid \mathbf{R}^T = \mathbf{R} \}/\{\mathbf{I}\}.
\]

In higher dimensions there are rotation matrices such as

\[
\mathbf{R} = \begin{bmatrix} \mathbf{R}' & 0 \\ 0 & ^T -\mathbf{I} \end{bmatrix},
\]

where \( \mathbf{R}' \in O(n-i) \) with \( \det \mathbf{R}' = (-1)^i \) and \( \mathbf{I} \) has dimension \( i \in \mathbb{N} \). These matrices need not be equilibria of (3) but they will retain the eigenvalue \(-1\) for all times as is evident from (9). Since the Lyapunov function candidate (5) has negative time derivative, these matrices will turn to symmetric matrices as time goes to infinity by LaSalle’s principle (this also follows from the eigendecomposition of \( \mathbf{R} \) and (10)).

In order to proceed with the solution of (4) we introduce the following block partitioning of a matrix \( \mathbf{R} \in SO(n) \),

\[
\mathbf{R} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ ^c \mathbf{c} & d \end{bmatrix}, \tag{12}
\]

where \( \mathbf{A} \in \mathbb{R}^{n-1 \times n-1}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{n-1}, \) and \( d \in \mathbb{R} \).

Proposition 2: Let \( \mathbf{R} \in SO(n) \) be partitioned as in (12). All \( \lambda \in \sigma(\mathbf{A}) \) belong to \( \mathbb{D} \). Moreover, \(-1 \in \sigma(\mathbf{R})\) if and only if \(-1 \in \sigma(\mathbf{A})\).

Proof: See the Appendix.

To solve the closed-loop system resulting from control algorithm (4), we apply the block matrix partition (12) with...
\[ k = 1 \text{ to } R \text{ and write } \]
\[
\begin{bmatrix}
\dot{A} & \dot{b} \\
\dot{c}^T & \dot{d}
\end{bmatrix} = 
\begin{bmatrix}
I & 0 \\
0^T & 0
\end{bmatrix}
\begin{bmatrix}
A & b \\
c^T & d
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0^T & 0
\end{bmatrix}
\begin{bmatrix}
A & b \\
c^T & d
\end{bmatrix}
\]
\[ = 
\begin{bmatrix}
I - A^2 & -Ab \\
-c^T A & -c^T b
\end{bmatrix}.
\]

**Theorem 2:** The solutions to the closed-loop system (13) generated by Algorithm 2 are

\[
A(t) = (\tanh t + A_0)(I + \tanh t A_0)^{-1},
\]
\[
b(t) = \text{sech } t \left( I + \tanh t A_0 \right)^{-1} b_0,
\]
\[
c(t) = \text{sech } t \left( I + \tanh t A_0^T \right)^{-1} c_0,
\]
\[
d(t) = - \tanh t \left( c_0^T (I + \tanh t A_0) \right)^{-1} b_0 + d_0.
\]

Moreover, the equilibrium \( R = I \) is almost globally exponentially stable.

**Proof:** Any eigenvalue \( \lambda \) of \( A_0 \) satisfies \( \lambda \in \mathbb{D} \) by Proposition 2. The inverse in \( A(t) \) is hence well-defined for all \( t \in \mathbb{R} \) and the solution part of the proof of Theorem 1 can be modified so that it applies to \( A(t) \).

Note that \( b(t) \) satisfies \( b(0) = b_0 \). Rewrite the solutions as \( b(t) = (cosh t I + \text{sinh } t A_0)^{-1} b_0 \).

\[
\dot{b}(t) = - \left( \cosh t I + \text{sinh } t A_0 \right)^{-1} (\text{sinh } t I + \cosh t A_0) \cdot \left( \cosh t I + \text{sinh } t A_0 \right)^{-1} b_0 = -A(t)b(t).
\]

The same line of reasoning applies to \( c(t) \).

Note that \( d(t) \) satisfies \( d(0) = d_0 \). Rewrite the solutions as \( d(t) = - \text{sinh } t c_0^T \left( \cosh t I + \text{sinh } t A_0 \right)^{-1} b_0 + d_0. \)

\[
\dot{d}(t) = - \cosh t c_0^T \left( \cosh t I + \text{sinh } t A_0 \right)^{-1} b_0
\]
\[
+ c_0^T \sinh t \left( \cosh t I + \text{sinh } t A_0 \right)^{-1} b_0
\]
\[
+ \left( \sinh t I + \cosh t A_0 \right) \left( \sinh t I + \text{sinh } t A_0 \right)^{-1} b_0
\]
\[
= \text{c}_0^T (- \cosh^2 t I - \cosh t \sinh t A_0 + \sinh t \cosh t A_0
\]
\[
+ \sinh^2 t I) (\cosh t I + A_0 \sinh t) \left( c_0^T A_0 \sinh t \right)^{-1} b_0
\]
\[
= - c_0^T \left( \cosh t I + \text{sinh } t A_0 \right)^{-2} b_0 = -c(t)^T b(t).
\]

The proof of almost global attractiveness of \( I \) is similar to that in the proof of Theorem 1. Note that the condition of \( -1 \notin \sigma(A) \) is equivalent to the condition \( -1 \notin \sigma(R) \) by Proposition 2. It is not entirely clear from (14) that \( \lim_{t \to \infty} d(t) = 1 \) when \( -1 \notin \sigma(A) \). To see this, note that Cramer’s rule gives

\[
R = (R^T)^{-1} = \frac{1}{\det R} \text{Adj}(R^T) = C,
\]
where \( C \) is the cofactor matrix of \( R \). Hence \( d = \det A \) which also holds at the limit.

Exponential stability of \( A \) follows from part of the linearization of (13) around \( R = I \) being

\[
\dot{A} = -2A, \quad \dot{b} = -b, \quad \dot{c} = -c,
\]
where \( \tilde{A} = R - I \). The linearization of \( \tilde{c} \) is critically stable, but the exponential rate of decay of \( \dot{b} \) and \( \dot{c} \) anyway imply an exponential convergence rate for \( d \) due to the constraints \( \|b\|^2 + d^2 = \|c\|^2 + d^2 = 1 \).

**Remark 5:** It is not, in general, possible to diagonalize \( A_0 \) when \( R_0 \in \mathbb{N}(n) \) as defined by (7). An example is given by

\[
R_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},
\]
where \( A_0 \) is on non-trivial Jordan normal form. In this case it is possible to calculate \( \lim_{t \to \infty} A(t) \) using the Jordan normal form since \( -1 \) is non-degenerate. Such an approach is however inadvisable from an application point of view due to numerical sensitivity of the Jordan normal form [21]. In the important cases of \( n \in \{2, 3\} \) all \( R \in \mathbb{N}(n) \) are symmetric and hence \( \lim_{t \to \infty} R(t) \) can always be calculated as done in Theorem 1.

It is interesting to note that the region of attraction of \( I \) of the closed systems (3) and (4) are identical due to the parameterization of \( A \) given by Proposition 2. The performance of Algorithm 1 and 2 is in this sense equal. Next we prove a result that shows that Algorithm 1 and 2 also are similar in preserving the \( \pm 1 \) eigenvalues of a rotation matrix throughout the evolution of the system.

**Proposition 3:** Let \( R \in \text{SO}(n) \) be a matrix that satisfies (4). Suppose that a \( \lambda(t) \) is a specific eigenvalue of \( R(t) \) and that \( \lambda_0 \in \{-1, 1\} \), then \( \lambda(t) \in \{-1, 1\} \) for all \( t \in \mathbb{R}^+ \).

**Proof:** See the Appendix.

**V. Applications**

Consider the problem of continuous time actuation subject to sensing that is either (i) piece-wise unavailable in time or (ii) discrete time. The relevance of this problem in the context of attitude stabilization may be motivated by cases where the attitude is calculated from images obtained by an image sensor for which (i) the reference used to obtain the attitude from the image is sometimes obscured or outside the image, or (ii) images are shot at a comparably slow rate. This type of problem is commonly addressed using piece-wise constant input signals [22], *i.e.* by applying a zero-order hold (ZOH). In this section we compare the ZOH approach to an approach based on the analytical solutions to the closed-loop kinematics. The question of comparing the two methods’ robustness to measurement noise is also interesting but outside the scope of this study.

More formally, let \( \{I_i\}_{i=1}^\infty \) be a sequence of disjoint intervals \( I_i = [a_i, b_i] \subset \mathbb{R}^+ \) with \( b_i \leq a_{i+1}, \forall i \). Denote \( I = \bigcup_{i \in \mathbb{N}} I_i \). We assume that an output \( Y(t) = R(t) \) is available for use in feedback control at times \( t \) such that \( t \in I \) and that it is unavailable when \( t \notin I \). In the case of discrete time sensing the states are sampled at each time instance of a sequence \( \{t_i\}_{i=1}^\infty \). This is simply the special case of \( \{I_i\}_{i=1}^\infty \) where \( a_i = b_i = t_i \) for all \( i \in \mathbb{N} \). In the following we assume that a sample is taken at time zero, *i.e.* \( a_1 = 0 \).

**Algorithm 3 (Zero-order hold, see e.g. [22]):** The zero-order hold algorithm is a time-varying feedback law \( \Omega(R, t), \)
given by
\[ \Omega(R, t) = \begin{cases} \Omega(Y(t)), & t \in I, \\ \Omega(Y(b_i)), & t \notin I, \end{cases} \]
where \( \Omega(R) \) is any attitude stabilizing kinematic control law, e.g. \( \Omega(R) = \Omega_k(R) \) with \( k \in \{1, 2\} \), and \( b_i = \max_{j \in \mathbb{N}} \{b_j \in I \mid b_j \leq t\} \).

The closed-loop system resulting from Algorithm 3 is a switched linear system due to the kinematics (2) being linear for a constant input. The ZOH approach typically requires short hold times, i.e. small intervals \((b_i, a_{i+1})\), to work effectively. The \( n \)th-order hold may have better performance, but it is still discontinuous [22]. In the absence of sensing, we instead propose the use of analytical solutions to predict future values of the states that are used for open-loop control in between sample times.

Let \( \Phi_k(R_0, t) \) denote the flow on \( SO(n) \), i.e. \( \Phi_k(R_0, t) = R(t) \) where \( R(t) \) is the solution at time \( t \) to \( \dot{R} = \Omega_k R \) with initial value \( R_0 \in SO(n) \) and \( \Omega_k, k \in \{1, 2\} \) is given by (3) or (4) respectively.

**Algorithm 4**: The flow algorithm is a time-varying feedback law \( \Omega_k(R, t) \) given by
\[ \Omega_k(R, t) = \begin{cases} \Omega_k(Y(t)), & t \in I, \\ \Omega_k(\Phi_k(Y(b(t)), t)), & t \notin I, \end{cases} \]
where \( \Omega_k, k \in \{1, 2\} \) and \( b(t) = \max_{j \in \mathbb{N}} \{b_j \in I \mid b_j \leq t\} \).

Algorithm 4 will generate the same system trajectory as the feedback would subject to continuous time sensing for all \( t \in \mathbb{R}^+ \). It is clear that the flow approach have advantages over the ZOH. Algorithm 4 may be applied as an open-loop controller based on a single measurement in which case the ZOH approach would fail. It is also reasonable to expect that Algorithm 3 will have problems with large hold times that are tolerable for Algorithm 4 (neither algorithm guarantees robustness under such circumstances but that is a somewhat different matter). Clearly, one could obtain the performance of Algorithm 4 without access to analytical solution by numerical quadrature of (2) but this also introduces discretization errors and a computational cost.

**VI. NUMERICAL EXAMPLES**

The closed-loop systems resulting from the use of Algorithm 1 and 2 was simulated for random initial conditions on \( SO(n) \) for various values of \( n \). The numerical solutions approached the identity asymptotically and the solutions to (3) and (4) also agreed with the analytical solutions given by Theorem 1 and 2 respectively up to the order of the error tolerances.

We simulated the zero-order hold, Algorithm 3, for random initial conditions on \( SO(3) \). The algorithm appears to work fine for sufficiently small hold times. For large hold times, it may however display high amplitude oscillations. These oscillations occur more often as the sample time and the gain parameters are increased (see Remark 2). Algorithm 3 behave somewhat differently depending on which of Algorithm 1 and 2 that is used as the underlying attitude stabilizer although it is difficult to discern any pattern.

Fig. 1 and 2 illustrate the case of oscillations induced by the ZOH (Algorithm 3). The peaks of the oscillations coincident with sample times, and it appears that the algorithm repeatedly overshoots the equilibrium when attempting to stabilize \( R_{11}, R_{12}, R_{21}, \) and \( R_{22} \) whereas the stabilization of \( R_{13}, R_{23}, R_{31}, R_{32}, \) and \( R_{33} \) is successful. The initial condition is approximately
\[
R_0 = \begin{bmatrix} -0.5854 & 0.5914 & -0.5546 \\ -0.5541 & -0.7912 & -0.2589 \\ -0.5919 & 0.1558 & 0.7908 \end{bmatrix}.
\]

**APPENDIX**

**Proof of Proposition 1**: The kinematics (2) constrain the solutions to lie in \( SO(n) \) for any initial condition on \( SO(n) \) by only allowing movement in the tangent space \( so(n) \). Recall that it will suffice to prove that \( \Omega_iR, i \in \{1, 2\} \),
is locally Lipschitz in $\mathbb{R}$ for all $R \in SO(n)$ to prove global existence and uniqueness of solutions to (2) due to $SO(n)$ being a compact and invariant subset of $\mathbb{R}^{n \times n}$ [23]. Furthermore, any linear combination or product of two functions that are Lipschitz on a domain is also Lipschitz on the same domain. It is clear that $\Omega_i R$, $i \in \{1, 2\}$, can be decomposed in this manner using functions that are Lipschitz on $SO(n)$.

**Proof of Proposition 2:** Take any $v \in \mathbb{R}^{n-1}$. The matrix $R$ being orthogonal gives
\[
\frac{\| R \begin{bmatrix} v \\ 0 \end{bmatrix} \|}{\| v \|} = \| A v + c^T \Sigma v \| = \| A v \|^2 + \| c^T v \|^2 = \| v \|^2,
\]
Hence $\| A v \| \leq \| v \|$ for all $v$, i.e. the spectrum of $A$ belongs to $\mathbb{D}^{n-1}$.

By supposing that $A v = -v$ we obtain $\| c^T v \| = 0$ whereby
\[
\frac{\| v \|}{\| v \|} = \frac{\| c^T \Sigma v \|}{\| v \|} = 0,
\]
i.e. $(-1, [v^T 0]^T)$ is an eigenpair of $R$.

Let $(-1, [u^T 1]^T)$ be an eigenpair of $R$. This is does not imply any loss of generality since if the last component of the eigenvector were 0 then $(-1, u)$ would be an eigenpair of $A$. Note that $R[a^T 1]^T = [a^T 1]^T$ implies that
\[
(I + A)u + b = 0, \quad c^T u + a + d = 0. \tag{16}
\]
Assume that $-1 \notin \sigma(A)$. Then $I + A$ is nonsingular and $u$ is uniquely determined. The system (16) hence has either one or no solutions. But $-1$ only occurs as an eigenvalue of a rotation matrix if it has algebraic multiplicity greater than or equal to two. Rotation matrices are normal, and the uniqueness of $u$ hence contradicts there being at least two linearly independent eigenvectors with eigenvalue $-1$.

**Proof of Proposition 3** We use the spectral decomposition $R = U \Lambda U^*$. The kinematics (4) become
\[
\dot{U} \Lambda U^* + U \dot{\Lambda} U^* + U A \dot{U} = E - R E R.
\]
This equation can be rewritten as
\[
\dot{A} = U^E \dot{U} - \Lambda U^E \Sigma A + [A, U^* \dot{U}],
\]
where we utilized that $\dot{U}^* U = -U^* \dot{U}$. Partition $U^*$ as $U^* = [V \ v]$. Then
\[
U^E \dot{U} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^* \\ 0 \end{bmatrix} = V^* V = I - vv^*.
\]
It follows that
\[
\dot{A} = I - vv^* - (I - vv^*)A + [A, U^* \dot{U}]
\]
\[
= I - A^2 + (A + I)vv^*(A - I) + [A, U^* \dot{U} - vv^*].
\]
Note that the above equation decompose into $n$ state equations concerning $A$ and $n(n-1)$ state equations concerning $\dot{U}$ due to the commutator having a zero diagonal. Hence
\[
\dot{A} = (I - Di(vv^*))(I - A^2),
\]
\[
[U^* \dot{U}, A] = (A + I)(vv^* - Di(vv^*)/(A - I)) \tag{17}
\]
where we define the $Di$ operator to nullify the non-diagonal elements of a matrix while leaving the diagonal intact. An inspection of (17) reveals that if $A_{i,0} = \pm 1$, then $A_{ii} = 0$, i.e. $A_{ii}(t) = \pm 1$ for all $t \in \mathbb{R}^*$.

**REFERENCES**


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