On the Design of Delay-Tolerant Distributed Space-Time Codes With Minimum Length

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Abstract—The construction of distributed space-time codes for asynchronous relays is considered. A novel algebraic structure is proposed and shown to achieve full diversity for arbitrary number of relays, arbitrary input alphabets, and arbitrary delay profiles among the relays. Unlike previously proposed delay tolerant schemes, the new design has minimum length which translates into smaller decoding complexity at the same transmission rate. Full-rate and full-diversity are achieved by the new designs with or without the use of guard intervals between successive transmissions. Simulation results confirm the mathematical analysis of the proposed codes.

I. INTRODUCTION

Reliable data transmission over wireless channels can be achieved by diversity techniques. Among them, spatial diversity using multiple transmit and receive antennas has gained much interest in the last decade (e.g., [1], [2], [3] and references therein). However, in some cases such as mobile handsets and sensor networks, transmitters and receivers are restricted in size and power consumption. Hence, it may not be feasible or cost effective to equip them with additional RF hardware. In these cases, cooperative diversity allows the source to use nearby nodes as virtual antennas (e.g., [4]). In other words, the nearby transmitters act as relay nodes for the original transmitter.

Cooperative diversity opened a new research area for designing distributed space-time codes which provide specific diversity and coding gain. Although these codes follow the well-known rank and determinant criteria [1], the lack of a common timing reference can affect the structure of the code matrix and result in a rank deficient space-time code. Hence, the previously designed fully diverse space-time codes are no longer valid for cooperative diversity unless they are delay tolerant. In other words, the design methodology should provide a desired diversity gain not only when the transmitters are synchronous, but also when they transmit their symbols with arbitrary relative delays.

In [5], Mei, et al. considered the effect of synchronization error for two cooperative relays when the Alamouti scheme [6] is used to transmit information symbols over a block fading channel. The authors showed that the synchronization error turns a frequency-flat channel into a frequency-selective one with correlated channel coefficients. They proposed to use either Time Reverse Space-Time Coding (TR-STC) or Orthogonal Frequency Division Multiplex (OFDM) signaling to combat synchronization error between relay nodes. The proposed schemes achieve full-diversity for two relays at the expense of a rate loss resulted from either adding cyclic prefix to OFDM symbols or inserting guard intervals between blocks of symbols in the TR-STC technique.

In [7], Xia, et al. proposed a MIMO-OFDM scheme to use in cooperative networks which achieves a diversity gain equal to the number of active relays provided that the length of the added cyclic prefix is not smaller than the maximum length of the frequency selective channels plus the asynchronous delays.

Several distributed space-time codes have been designed for wireless relay networks [8], [9], [10], [11], [12], [13]. Inspired by El-Gamal-Hammons stacking construction [14], Li and Xia [8] considered the design of binary space-time trellis codes (STTC) that are full-diversity and delay tolerant for any number of cooperating relays. The authors extended their proposed structures to multilevel space-time codes for higher order constellations using the generalized stacking construction [15]. In [9] and [10], Shang and Xia extended the proposed structures of [8] to full-diversity delay tolerant STTCs of minimum delay length. In [11] and [12], Hammons showed that codes obtained from generalization of the construction of [15] from PAM and QAM to AM-PSK constellations preserve the diversity gain despite the timing offset among the relay nodes. He also showed that certain binary space-time block codes (STBCs) derived from the stacking construction [14] are delay tolerant. In

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[16], he proposed a delay tolerant transmission scheme of STBCs of short length to be used in asynchronous cooperative communication.

In [13], Damen and Hammons extended the class of threaded algebraic space-time (TAST) codes [3] to the case of delay tolerant codes for cooperative diversity. Their proposed codes are based on delay tolerant threaded structures of length growing exponentially with the number of relays. Different threads are separated by different algebraic or transcendental numbers which guarantee that the difference of every two distinct codewords is full rank. Although these codes provide full-rate and full-diversity gain simultaneously, they are not minimum delay length and are no longer delay tolerant if one deletes one column of the codeword matrix.

In this paper, we introduce a novel delay tolerant algebraic structure with minimum delay length. Our framework can also be considered as an extension of the TAST framework [3]. Similarly to [13], we consider the scenario of decode-and-forward single-carrier asynchronous relay transmission, where in the first time interval, the source node transmits to $M$ relay nodes. The source communicates with the relay nodes using common time slots and frequency bandwidth for their signal transmission, they cooperatively implement a space-time code. In this paper, we consider the design and performance evaluation of distributed space-time codes used by the relays. This can be considered as a MIMO system model with $M$ transmit antennas corresponding to $M$ relays and $N$ receive antennas at the destination. At time instant $t$, the received signal vector $\bar{r}_t \in \mathbb{C}^{N \times 1}$ is modeled as

$$\bar{r}_t = \mathbf{H}_t \bar{s}_t + \bar{n}_t,$$

where $\bar{s}_t \in \mathbb{C}^{M \times 1}$ is the modulated signal vector transmitted during the $t$-th symbol interval. $\mathbf{H}_t \in \mathbb{C}^{N \times M}$ and $\bar{n}_t \in \mathbb{C}^{N \times 1}$ denote the channel matrix and the additive white Gaussian noise vector, respectively. The channel is assumed to be quasi-static, i.e., the channel matrix $\mathbf{H}_t$ is constant over a codeword interval but is independent from codeword to codeword. The modulated signal vectors $\bar{s}_t, t = 1, \ldots, T$ span the space-time codeword $\mathbf{S}$ of size $M \times T$ over $T$ symbol intervals. The transmitted symbols are generated from a finite constellation. Let $\mathcal{A}$ denotes a two-dimensional constellation chosen from $\mathbb{Z}[i]$ or $\mathbb{Z}[j]$, and let $\mathbb{F} = \mathbb{Q}(i)$, or $\mathbb{Q}(j)$, denotes the field of complex Gaussian rational numbers, or complex Eisenstein rational numbers, respectively. Let $\mathbb{F}(\theta)$ be an extension field of degree $[\mathbb{F}(\theta) : \mathbb{F}]$ over $\mathbb{F}$. The input alphabet for our construction is given by

$$\Omega = \{x = \sum_{n=0}^{P-1} u_n \theta^n : u_n \in \mathcal{A}\},$$

where integer $P \leq [\mathbb{F}(\theta) : \mathbb{F}]$. Each transmitted symbol $s_{m,t}$ is from $\Omega$, or more generally, from its image $f(\Omega)$ under some specified one-to-one mapping $f : \Omega \to \mathbb{C}$.

### B. Delay tolerance of space-time codes

Let $\mathcal{S}$ be a STBC with codewords of size $M \times T$. Assume $\mathbf{S}_1$ and $\mathbf{S}_2$ are two distinct codewords of $\mathcal{S}$. The diversity order of $\mathcal{S}$ is the minimum rank of the difference matrix $\mathbf{S}_1 - \mathbf{S}_2$ over all pairs of distinct codewords in $\mathcal{S}$ (rank criterion) [1].

\footnote{For simplicity of notation, we restrict ourselves to $\mathbb{Z}[i]$ or $\mathbb{Z}[j]$; however, as in [13], the input alphabet $\mathcal{A}$ can be chosen from any cyclotomic ring $\mathbb{Z}[\exp(2\pi i/n)]$, with $n$ integer.}
We assume that the fractional delays are absorbed in multi path, so asynchronous delays are integer factors of the symbol interval. Delays are unknown at the relays, but are known at the destination. Here, we consider the transmission of either one single codeword or consecutive codewords with enough guard interval between them such that the earlier and the latter transmitted codewords do not interfere with the codeword under consideration.

The space-time code \( S \) is \( \tau \)-delay tolerant for the asynchronous cooperative diversity scenario if for all distinct codewords \( S_1, S_2 \in S \), the difference matrix \( S_1 - S_2 \) retains full-rank even though the rows of the codewords are transmitted with arbitrary delays of duration at most \( \tau \) symbols. To make this definition more precise, suppose that the \( M \times T \) modulated codeword matrix \( S \in S \), containing the rows \( \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_M \), is transmitted by \( M \) relays with delay profile \( \Delta = (\delta_1, \delta_2, \ldots, \delta_M) \). Here, \( \delta_i \) denotes the relative delay of the signal received from the \( i \)-th relay with respect to the earliest received relay signal. This is equivalent to receive the \( M \times (T + \delta_{\text{max}}) \) matrix

\[
S\Delta = \begin{pmatrix}
\delta_1 & \delta_{\text{max}} - \delta_1 \\
\delta_2 & \delta_{\text{max}} - \delta_2 \\
\vdots & \vdots \\
\delta_M & \delta_{\text{max}} - \delta_M
\end{pmatrix},
\]

where \( \delta_i, i = 1, \ldots, M \) and \( \delta_{\text{max}} \) denote an all-zero vector of length \( \delta_i \) and the maximum of the relative delays, respectively. Now, a space-time code \( S \) is called \( \tau \)-delay tolerant if for all delay profiles \( \Delta \) with \( \delta_{\text{max}}(\Delta) \leq \tau \), the effective space-time code \( S\Delta \) achieves spatial diversity as great as that of \( S \). A space-time code is fully delay tolerant if it is delay tolerant for any positive integer \( \tau \).

### C. Threaded code structure

Recall that a layer is a mapping that assigns a unique transmit antenna to be used at each individual time interval of a codeword [2]. It is called a thread if its domain and range include all possible transmit antennas and all possible symbol intervals, respectively. The proposed delay tolerant TAST codes in [13] are based on using three different threaded structures which are intrinsically delay tolerant: running use (RU), naive double use (NDU), and optimum double use (ODU). The latter has the smallest delay length. For \( M = 4 \), an ODU threading is given by

\[
1_{\text{ODU}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

As the rows of this matrix are linearly independent for every delay profile, it follows that the space-time code is delay tolerant. The structure can contain four different threads which are individually delay tolerant; however, when they join together in a matrix, it is not guaranteed that the resulted structure is full-diversity delay tolerant. This is guaranteed by multiplying all of the threads by different powers of an appropriate algebraic or a transcendental number \( \phi \). It was proved in [13] that for \( M \geq 5 \), the delay length of the code is \( T_{\text{ODU}}^M = 2M^2 - 2 + 1 \). Clearly, this code is not minimum delay length, and its length grows exponentially with the number of relay nodes. Unlike the ODU threaded structure, in this paper, we propose a minimum length delay-tolerant TAST-like code.

### III. Construction of new delay tolerant STBCs

In this section, we develop our new design of delay tolerant space-time codes with minimum delay length \( T = M \). The minimum rank of the difference between distinct codewords is the diversity order of a space-time code. One simple way to verify the rank of a matrix is to find the largest square submatrix with nonzero determinant. The determinant of an \( M \times M \) matrix \( T \) is given by

\[
\det(T) = \sum_{\sigma} \epsilon_\sigma t_{1\sigma(1)} \cdots t_{M\sigma(M)},
\]

where \( t_{jk}, j,k \in \{1, \ldots, M\} \) denote entries of \( T \), \( \sigma \) is a permutation of numbers \( \{1, \ldots, M\} \), and \( \epsilon_{\sigma} = \pm 1 \) is called signature of \( \sigma \) and depends on the number of exchanges in \( \sigma \).

First, we observe that randomly generated Gaussian space-time codes are delay tolerant with probability 1. As all the entries of the code matrix are chosen independently from a Gaussian distribution, the difference of distinct codewords does not have zero entry with probability one (except the zeros due to the delay of the rows). As a result, one can find an \( M \times M \) submatrix \( T \) such that the \( i \)-th column \( i = 1, \ldots, M \), has nonzero entry at the \( i \)-th row. In this case, \( T \) has at least one thread with all nonzero entries. According to (5), the determinant of \( T \) contains linear combination of the terms that is equal to the multiplication of the elements of the threads of \( T \). Since these terms are independent Gaussian random variables with at least one nonzero term corresponding to the main diagonal thread, \( \det(T) \) is nonzero with probability 1. Thus, this code which supports full spatial diversity is delay tolerant for every delay profile with probability 1.

Such a Gaussian code has two major drawbacks. First, as it does not have a specific structure, the decoding procedure is exponentially complex. Second, the codebook has to be saved in both transmitters and receiver places for encoding and decoding procedures which requires a
large amount of memory in both sides. The proposed scheme of this paper avoids these drawbacks while preserves the advantages of random Gaussian codes.

The general guidelines for the construction of our codes are described in the following. Let $C$ be a one dimensional full-diversity block code of length $MT$ suitable for SISO channels (i.e., it achieves full diversity, $MT$, when sent over an i.i.d. SISO fading channel [3]). Let $G$ be a fixed $M \times T$ matrix with elements as powers of some complex number $\phi$ on the unit circle. By parsing the entries of a codeword vector $\bar{c} \in C$ to an $M \times T$ matrix $C$ (whose entries are a rearrangement of those of $\bar{c}$), a space-time matrix codeword is formed as follows

$$S = C \odot G,$$

where $\odot$ denotes the component wise product. In this paper, Matrix $G$ is designed such that the resulted space-time code achieves full-diversity and is delay tolerant for arbitrary delay profile and arbitrary number of transmitter antennas (finding full diversity SISO codes is well documented in the literature, e.g., [3] and references therein). Two main families of codes are proposed in the sequel based on two different structures of matrix $G$ in (6). In the first family, $G = A_M$ is given by

$$A_M \triangleq \begin{pmatrix} \phi^{b_{1,1}} & \cdots & \phi^{b_{1,M}} \\ \phi^{b_{2,1}} & \cdots & \phi^{b_{2,M}} \\ \vdots & \ddots & \vdots \\ \phi^{b_{M,1}} & \cdots & \phi^{b_{M,M}} \end{pmatrix},$$

(7)

where $b_{i,j} \geq 0$, $i, j = 1, \ldots, M$ are positive integers, and $\phi$ is an appropriate number with properties detailed in the sequel. One can show that by an appropriate choice of the parameters $\phi$ and $b_{i,j}$'s followed by using proper one dimensional code $C$, the resulted space-time code is delay tolerant for every delay profile.

Define $\Xi$ as a set which contains those $M$-tuples of $b_{i,j}$ that have no two entries with common $i$ index (i.e., not selected from the same row of $A_M$). $\Xi$ contains $M^M$ vectors of length $M$, $\Xi = \{b_1, b_2, \ldots, b_{M^M}\}$. Define $b_{\max}$ as

$$b_{\max} = \max_q \sum_{j=1}^M b_{q,j}, \quad b_{q,j} \in \Xi, q = 1, \ldots, M^M,$$

where $b_{q,j}$ is the $j$-th element of $b_q$.

At the beginning, for ease of understanding, we consider a case that $C$ is a repetition code over $\Omega$ and there are only three transmitting nodes, $M = 3$. Let $b_{i,j}$'s are defined as consecutive integer powers of 2. $A_3$ is given by

$$A_3 = \begin{pmatrix} 1 \\ \phi^4 \\ \phi^8 \\ \phi^{16} \\ \phi^{32} \\ \phi^{64} \end{pmatrix}. $$

(8)

As can be seen the summation of the elements of every 3-tuple in $\Xi$ is unique. Since $C$ is a repetition code, all differences between distinct codewords are multiples of $A_3$. Therefore, to verify the diversity order of this code for a delay profile $\Delta = \{\delta_1, \delta_2, \delta_3\}$, one only needs to find the largest square submatrix of $A_3^\Delta$ which is full rank. Assume $T$ is a $3 \times 3$ submatrix of $A_3^\Delta$ (columns of $T$ are chosen from columns of $A_3^\Delta$) with at least one thread with all nonzero entries. According to (5), the summation of $b_{i,j}$'s corresponding to this thread will appear as a power of $\phi$ in the determinant value. Since this value is unique, the corresponding term in the determinant value is not interfered by the other terms showed in (5), if $\phi$ is chosen such that the numbers $\{1, \phi, \ldots, \phi^{b_{\max}}\}$ are algebraically independent over the field that contains $\Omega$. Note that, by definition, the largest power of $\phi$ that may appear in the determinant value for all delay profiles is $b_{\max}$. Hence, the determinant value is not zero and the code achieves full diversity for every delay profile. Here, $b_{\max} = 146$ and the numbers $\{1, \phi, \ldots, \phi^{146}\}$ should be algebraically independent over $\mathbb{F}(\theta)$ in order to generate a full-diversity and delay tolerant code. Theorem 1 concludes the results.

**Theorem 1:** Let $S$ denote the space-time code in which the repetition code with codewords of length $M^2$ over alphabet $\Omega$ is used as the one dimensional SISO code in conjunction with matrix $A_M$. Then $S$ achieves full spatial diversity and is fully delay tolerant if the following conditions are satisfied:

1- $\phi$ is chosen as an algebraic or a transcendental number such that the numbers $\{1, \phi, \ldots, \phi^{b_{\max}}\}$ are algebraically independent over the field $\mathbb{F}(\theta)$ that contains $\Omega$.

2- The $b_{i,j}$ parameters are chosen such that the summation of the elements of every $b_q \in \Xi$ is unique.

**Proof:** Since all differences between distinct codewords in $S$ are multiples of $A_M$, it suffices to show that $A_3^\Delta$ is full rank for every arbitrary delay profile $\Delta = \{\delta_1, \delta_2, \ldots, \delta_M\}$. To verify the diversity order of the code, one needs to find the largest square submatrix $T$ of $A_3^\Delta$ which is full rank. $T$ is constructed by taking a column of $A_3^\Delta$ with a nonzero entry at the $i$-th ($i = 1, 2, \ldots, M$) row as the $i$-th column. Assume that construction of $T$ is started by choosing the first column. Since $A_M$ is an $M \times M$ matrix with all nonzero entries, at the $i$-th ($i = 1, \ldots, M$) step of the process, there exists at least $M - i + 1$ columns that contains a nonzero entry at the $i$-th row. Therefore, this form of column selection is feasible. As a result, $T$ is of size $M \times M$, and all the elements on its main diagonal thread are nonzero. If the sum of the parameters $b_{i,j}$ corresponding to this thread is $p$, we have

$$\det(T) = g(\phi) + \phi^p,$$

(9)
where \( g(\phi) \) is a polynomial of \( \phi \) with degree less than or equal to \( b_{\max} \). Since \( p \) is unique, \( g(\phi) \) does not contain any term in \( \phi^p \). As a result, if the numbers \( \{1, \phi, \ldots, \phi^{b_{\max}}\} \) are algebraically independent over \( \mathbb{F}(\theta) \), \( \det(T) \) is not zero and the code achieves full-diversity for every delay profile.

It is worth noting that the second condition of Theorem 1 is more than sufficient. It guarantees that every nonzero thread (a thread with all nonzero entries) of \( T \) will appear with a unique power of \( \phi \) in the determinant value. On the other hand, having only one nonzero thread of \( T \) with unique power of \( \phi \) in the determinant value is enough for a full diversity delay tolerant code. Therefore, there may exist some other schemes for assigning \( b_{ij} \) parameters that cause a full diversity delay tolerant code, while they do not follow the second condition of Theorem 1.

While there may exist a lot of choices for the parameters \( b_{ij}, i, j = 1, \ldots, M \), those that result in a smaller value for \( b_{\max} \) are of interest. In this case, one can design the code with smaller degree extension of the base field. A simple choice is to use consecutive integer powers of 2.

The previous family of the codes assigns a specific power of \( \phi \) to every element of the codeword matrix resulting in a complex code structure which might be more than what is necessary to guarantee a full diversity delay tolerant code. The second family of the codes, where we call it the proposed code, is based on a simpler structure for matrix \( G \) and more relax conditions for \( b_{ij} \) parameters. Let \( G \triangleq B_M \), where \( B_M \) is defined as

\[
B_M \triangleq \begin{pmatrix}
1 & \phi^{b_{1,2}} & \phi^{b_{1,3}} & \cdots & \phi^{b_{1,M}} \\
1 & 1 & \phi^{b_{2,3}} & \cdots & \phi^{b_{2,M}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & \phi^{b_{M-1,M}} \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\]  \hspace{1cm} (10)

The space-time code \( S \) is constructed as explained in the general guidelines for the code construction. Here, \( \Xi \) is defined as a set which contains those \((M-1)\)-tuples of \( b_{ij} \) in \( B_M \) that have no two entries with common \( i \) index. Therefore, \( \Xi \) contains \((M-1)!\) vectors of length \( M-1 \), \( \Xi = \{b_1, b_2, \ldots, b_{(M-1)!}\} \). \( b_{\max} \) is defined similar to (8) over the new defined set \( \Xi \).

Theorem 2 characterizes sufficient conditions that make the resulted code full-diversity and delay tolerant when the one dimensional SISO code \( C \) is a repetition code over \( \Omega \).

**Theorem 2:** Let \( S \) denote the space-time code in which the repetition code with codewords of length \( M^2 \) over alphabet \( \Omega \) is used as the one dimensional SISO code in conjunction with matrix \( B_M \). Then \( S \) achieves full spatial diversity and is fully delay tolerant if the following conditions are satisfied:

1. \( \phi \) is chosen as an algebraic or a transcendental such that the numbers \( \{1, \phi, \ldots, \phi^{b_{\max}}\} \) are algebraically independent over the field \( \mathbb{F}(\theta) \) that contains \( \Omega \).

2. The parameters \( b_{ij} \geq 0, i = 1, \ldots, M-1, j = i+1, \ldots, M \), are chosen such that the summation of the entries of every \((M-1)\)-tuple in \( \Xi \) is a unique value.

**Proof:** Since the one-dimensional code \( C \) is a repetition code, it suffices to show that \( B_M^\phi \) is full rank for every arbitrary delay profiles \( \Delta = \{\delta_1, \delta_2, \ldots, \delta_M\} \). To verify the diversity order of the code, one needs to find the largest square submatrix of \( B_M^\phi \) which is full rank. \( T \) is constructed by choosing the \((M-i+1)\)-th column, \( i = 1, \ldots, M-1 \), such that it contains a power of \( \phi \) at the \((M-i)\)-th row. The first column is chosen such that it contains a nonzero element, 1, at the \( M \)-th row. To show that this form of column selection is feasible, assume that construction of \( T \) is carried out by choosing its columns in a reverse order, i.e., the \( M \)-th column is selected first, the \((M-1)\)-th column is selected second, and similarly up to the first column which is selected at last. Due to the original form of the matrix \( B_M \), at the \( i \)-th step of the process, there are \( l \) elements containing \( \phi \) factor at the \((M-i)\)-th row. Since \((i-1)\) columns have been chosen beforehand, there is at least one column containing \( \phi \) factor at this row. Thus, the aforementioned form of column selection is feasible for \( T \) for every delay profile. As can be seen, \( T \) is an \( M \times M \) matrix that has at least one thread \( l \) with all nonzero elements containing \((M-1)\) elements of powers of \( \phi \). It is given by

\[
T = \begin{pmatrix}
\star & \phi^{b_{1,2}} & \cdots & \star \\
\star & \star & \phi^{b_{2,3}} & \cdots & \star \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\star & \star & \cdots & \star & \phi^{b_{M-1,M}} \\
1 & 1 & \star & \cdots & \star
\end{pmatrix}
\]  \hspace{1cm} (11)

where \( \star \) can be zero due to the delay profile \( \Delta \) or any one of the entries of \( B_M \) which may or may not contain the factor \( \phi \). Let \( p \) be the summation of the powers of \( \phi \) existing in the thread \( l, p = b_{1,2} + b_{2,3} + \ldots + b_{M-1,M} \), then the determinant of \( T \) is

\[
\det(T) = g(\phi) + \phi^p,
\]  \hspace{1cm} (12)

where \( g(\phi) \) is a polynomial of \( \phi \) with degree less than or equal to \( b_{\max} \), and its coefficients belong to \([-1, 1]\). Since \( p \) is unique, \( g(\phi) \) does not contain any term in \( \phi^p \). Therefore, if the numbers \( \{1, \phi, \ldots, \phi^{b_{\max}}\} \) are algebraically independent over \( \mathbb{F}(\theta) \), \( \det(T) \) is not zero, and the constructed code achieves full-diversity for every delay profile.

There is an infinite number of schemes to select proper values for the parameters \( b_{ij} \); however, one simple
choice is to use geometric series of integer powers of 2 as
\[ b_{i,j} = 2^{\frac{1}{f_M f_{M-1}} M - i - j} \]
where \( i = 1, \ldots, M - 1, \quad j = i + 1, \ldots, M. \)

For example, for \( M = 3 \) we have,
\[ B_3 = \begin{pmatrix} 1 & \phi^4 & \phi^2 \\ 1 & 1 & \phi \\ 1 & 1 & 1 \end{pmatrix} \]

**Remark 1:** It is worth noting that the required condition for the numbers \( \{1, \phi, \ldots, \phi^{\phi_{\text{max}}}\} \) to be algebraically independent over the field \( \mathbb{F}(\theta) \) is just a sufficient condition. In some cases, having the numbers \( \{1, \phi, \ldots, \phi^{\phi_{\text{max}}}\} \) be algebraically independent over a field contained in \( \mathbb{F}(\theta) \) with smaller degree is enough to achieve full-diversity. Typically, \( \phi \) can be chosen as a cyclotomic number, i.e., the \( n \)-th root of unity \( \zeta_n = e^{2\pi i / n} \) (in this case, \( \phi \) is an algebraic number).

Theorem 1: For any full-diversity SISO code [17]. In general, one-to-one functions \( f_{i,j}, i, j = 1, \ldots, M \) of \( x \in \Omega \), \( f_{i,j} : \Omega \to \mathbb{C} \), are used instead of \( x \) at the \((i, j)\)-th element of the codeword matrix. Having \( \mathbb{C} \) as a general full diversity SISO code, the space-time code \( S \) is constructed as explained in the general guidelines of the code construction. The generalized codeword matrix of the proposed code is shown in (15). By following the same procedure of the original repetition code, one can prove that \( S \) is full-diversity and delay tolerant for every delay profile. The following theorem formulates the results.

**Theorem 3:** Let \( S \) denote the space-time code with codewords of the form \( S_M(x), x \in \Omega \) wherein a general full diversity code with codewords of length \( M^2 \) is used as the one dimensional SISO code in conjunction with matrix \( B_M \). Then \( S \) achieves full spatial diversity and is fully delay tolerant if the following conditions are satisfied:

1. \( \phi \) is chosen as an algebraic or a transcendental such that the numbers \( \{1, \phi, \ldots, \phi^{\phi_{\text{max}}}\} \) are algebraically independent over the field that contains \( f_{i,j}(x), i, j = 1, \ldots, M, x \in \Omega \).
2. The parameters \( b_{i,j} \geq 0, i = 1, \ldots, M - 1, j = i + 1, \ldots, M \), are chosen such that the summation of the entries of every \((M-1)\)-tuple in \( \mathbb{Z} \) is a unique value.

The proof is similar to the one of Theorem 2 and is omitted for brevity. \( b_{i,j} \) are chosen the same as before.

**Remark 2:** By assuming that the proposed code is designed to be used in a system containing \( M \) relay nodes, the full transmit diversity gain \( M \) is achieved provided that all the relays correctly decode the source messages. To this end, the relays are equipped with an error detection mechanism (e.g., cyclic redundancy check codes), only those relays that have not detected any error at the end of the first step will participate in the second step of the process. However, it is easy to check that the elements of any subset of the rows of the proposed structure are linearly independent for every delay profile. Therefore, the resulted space-time code provides the full diversity gain which is equal to the number of active relays.

**IV. DELAY TOLERANT SPACE-TIME CODES WITH OVERLAPPED CODEWORDS**

Thus far, we assumed that the relay nodes insert enough guard interval (i.e., filling symbols) between every two consecutive codewords to ensure that the constructed space-time code is received without any interference from either the next or the previous codewords. This assumption limits the transmission rate and confines the applications of the proposed codes to low bit rate scenarios. In this section, we consider the case that every relay transmits its symbols continuously without any delay guard between codewords.

In the problem under consideration, there exists \( k \geq 1 \) vectors of information symbols \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k \), where \( \bar{u}_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,M}) \), \( i = 1, \ldots, k \). Then \( x_i \) is the constellation point in \( \Omega \) corresponding to \( \bar{u}_i \) generated using (2). The matrices of the space-time codewords are constructed according to Theorem 2 or 3 and transmitted over the channel one codeword after another without any guard interval.

While it is difficult to prove (or disprove) that the overlapped code is still delay tolerant for every delay profile, we propose in the following a simple modification that guarantees delay-tolerance regardless of the use of guard intervals. We propose to use different algebraic or transcendental numbers for different blocks of the overlapped codeword. One can show that when the maximum possible delay between relay nodes is confined to the delay length of the code \( T \), one only needs to have two different numbers \( \phi_1, \phi_2 \) to make the resulted overlapped code delay tolerant. When \( k \) (number of codewords transmitted over the channel without guard interval) is large, the rate loss of the code due to the required guard interval at both end of the overlapped codeword becomes negligible.

As a simple example, let \( \bar{u}_1, \bar{u}_2 \) be two vectors of information symbols. \( x_1, x_2 \) are the corresponding constellation points in \( \Omega \), and \( S_1, S_2 \) are the corresponding repetition space-time codewords according to Theorem 2. For \( M = 3 \), the resulted overlapped codeword
SISO code and the numbers \( \delta \) corresponding code matrices constructed as explained in the following.

Theorem 4: Let \( u_i, u_2, \ldots, u_k, k \geq 1 \), be vectors of information symbols, where \( u_i = (u_i^1, u_i^2, \ldots, u_i^{MT}), i = 1, \ldots, k \). Let \( x_1, x_2, \ldots, x_k \) be the corresponding constellation points in \( \Omega \), and \( S_1, S_2, \ldots, S_k \) be the corresponding code matrices constructed as explained in Theorem 2. Assuming \( \delta_{\text{max}} \leq T \), the overlapped code \( S \) with codewords of the form \( S_{1,2,\ldots,k} = (S_1, S_2, \ldots, S_k) \) (constructed by concatenation of \( S_i, i = 1, \ldots, k \) without guard interval) achieves full-diversity and is delay tolerant for every delay profile if the following conditions are satisfied:

1- \( \phi_1 \) is an appropriate algebraic or a transcendental such that the numbers \( \{1, \phi_1, \ldots, \phi_{\text{max}}^{\phi_1}\} \) are algebraically independent over the field \( \mathbb{F}(\theta) \) that contains \( \Omega \).

2- \( \phi_2 \) is chosen such that the numbers \( \{1, \phi_2, \ldots, \phi_{\text{max}}^{\phi_2}\} \) are algebraically independent over the field that contains the elements of the first block. \( \phi_1, \phi_2 \) are alternately used for every other block.

The proof is similar to the proof of Theorem 2 and is omitted for brevity. As in the non-overlapped scenario, the results can be generalized from repetition codes to any full-diversity SISO code [3, 17].

V. EXAMPLES

In this section, we give some examples of the new designed codes proposed in Theorem 3. The construction of these codes is carried out by properly choosing the SISO code and the \( \phi \) number.

The required full-diversity SISO codes over fading channels can be constructed by applying full-diversity unitary transformations to input signals drawn from lattices or multidimensional constellations carved from a ring. In [18], Damen et al. provided an explicit construction of \( M \times M \) fully diverse unitary transformations \( U_M \) over the field that contains the elements of information symbols.

\[
U_M = W_M^\dagger D_\theta, \tag{17}
\]

where \( W_M \) is the \( M \times M \) discrete Fourier transform (DFT) matrix, \( W_M^\dagger \) is its conjugate transpose, and \( D_\theta = \text{diag}(1, \theta, \ldots, \theta^{M-1}) \), where \( \theta \) is a transcendental or an algebraic of suitable degree to guarantee the full-diversity of the rotation [18].

We consider design of the new proposed codes for \( M = 2, 3 \) active relays. The required full-diversity rotations are of size \( 4 \times 4 \) and \( 9 \times 9 \), respectively. To benefit from optimum rotations, information symbols are chosen from \( \mathbb{Z}[i] \), and \( \mathbb{Z}[j] \) for \( M = 2, 3 \), respectively. Two explicit examples of the new proposed codes are given below.

1. For \( M = N = T = 2 \), one has the delay tolerant STBC with codeword matrices of the form

\[
\begin{pmatrix}
  x_1 & \phi x_2 \\
  x_2 & x_4 \\
\end{pmatrix}
\]

\[
\text{where } \bar{x} = (x_1, x_2, \ldots, x_4)^T = \Theta \bar{u}, \bar{u} \text{ is a } 4 \times 1 \text{ vector of QAM symbols, and } \Theta = U_4 \text{ is the optimal } 4 \times 4 \text{ complex rotation according to (17). By setting } \phi = e^{2\pi i/3}, \text{ this code provides the rate of } 2 \text{ QAM symbols per channel use and achieves a transmit diversity of } 2 \text{ regardless of the delay profile among its rows.}
\]

2. For \( M = T = 3, N = 2 \), one has the delay tolerant STBC with codeword matrices of the form

\[
\begin{pmatrix}
  x_1 & \phi^4 x_4 & \phi^2 x_7 \\
  x_2 & x_5 & \phi x_8 \\
  x_3 & x_6 & x_9 \\
\end{pmatrix}
\]

\[
\text{where } \bar{x} = (x_1, x_2, \ldots, x_9)^T = \Theta \bar{u}, \bar{u} \text{ is a } 9 \times 1 \text{ vector of information symbols belong to the } 4\text{-array constellation in } \mathbb{Z}[j], \text{ and } \Theta = U_9 \text{ is the optimal } 9 \times 9 \text{ complex rotation according to (17). By setting } \phi = e^{2\pi i/7}, \text{ this code provides the rate of } 3 \text{ symbols per channel use and achieves a transmit diversity of } 3 \text{ regardless of the delay profile among its rows.}
VI. Simulation Results

In this section, we show some performance results of the proposed codes for \( M = 2 \) and 3 active relays. The decoding procedure is performed by sphere decoding [19]. We assume that the destination is equipped with two receive antennas. This results in an under determined system for \( M = 3 \) transmit antennas. In this case, we benefit from the minimum mean square error decision feedback equalizer (MMSE-DFE) as the preprocessing stage followed by the sphere decoding algorithm [19].

The figures show bit and codeword error rates as a function of \( E_b/N_0 \) in dB which is adjusted as follows.

\[
\frac{E_b}{N_0}_{db} = \frac{E_s}{N_0}_{db} - 10 \log_{10} R, \tag{20}
\]

where \( E_s \) is the average signal energy per receive antenna and \( R \) is the code rate in bit per channel use (bpcu).

Figures 1 and 2 show the bit error rate (BER) and the word error rate (WER) performances of the new proposed code with codewords of the form (18) for both cases of synchronous as well as asynchronous relays. At the latter case, the second row is shifted to the right by one symbol interval. The results are compared with the results of the Golden code [20] as well as the results of the delay tolerant 2 × 2 code (matrix \( D \)) proposed by Damen in [21]. Here, \( M = N = T = 2 \) and \( \phi = e^{2\pi i/3} \). Information symbols are chosen from the 4-QAM constellation to adjust the rate of \( \frac{8}{2+\delta_{\text{max}}} \) bpcu, where \( \delta_{\text{max}} = 0 \) when the relays are synchronous (see Fig. 1), and 1 when they are asynchronous (see Fig. 2). As can be seen, the proposed code outperforms the matrix \( D \) in both cases of synchronous as well as asynchronous relays. For example, at the BER of \( 10^{-4} \), the proposed code has about 0.6 dB gain over \( D \) when \( \delta_{\text{max}} = 0 \). When \( \delta_{\text{max}} = 1 \), the performances of the two codes are almost the same at low SNR regime; however, our code outperforms the other one at high SNR regime. This confirms that our proposed code achieves the diversity gain of 2 and is delay tolerant. In addition, the performance of the proposed code is quite close to the one of the Golden code when the relays are synchronous, and is much better than when they are asynchronous (the latter is not delay tolerant).

Figures 3, 4, show the BER and the WER performances of the proposed code with codewords \( f \) the form (19) when \( M = T = 3 \) and \( N = 2 \) with and without delay, respectively. At the latter case, the first and the second rows are shifted once and twice to the right. The 9 × 9 optimal rotation is applied to the vector of information symbols which are carved from a four point constellation of \( \mathbb{Z}[j] \). This code provides the rates 6, and 3.6 bpcu corresponding to \( \delta_{\text{max}} = 0, 2 \), respectively. The results are compared with the one of the delay tolerant TAST codes proposed by Damen and Hammons in [13], where the code matrix of size \( 3 \times 5 \) with two nonzero threads are used to transmit 4, and 8-QAM symbols over the channel. As can be seen, the proposed code performs much better than the equivalent delay tolerant TAST code at the same rate in both synchronous and asynchronous cases. For example, at the BER of \( 10^{-5} \), and \( \delta_{\text{max}} = 0 \) (\( R = 6 \) bpcu), it has 5 dB gain over Damen-Hammons code, and even more when \( \delta_{\text{max}} = 2 \) (\( R = 3.6 \) bpcu). These figures show that the proposed code at the rates 6 and 3.6 bpcu (corresponding to the synchronous and the asynchronous cases) performs almost the same as Damen-Hammons code which has smaller rates of 4 and 2.9 bpcu, respectively. The results confirm that \( B_3 \) is delay tolerant.

In another scenario, we assume that the proposed
In this paper we introduced a novel design for the construction of full-diversity full-rate delay tolerant distributed space-time codes with minimum temporal length for cooperative diversity in wireless networks. The new codes achieve full-diversity for arbitrary number of active relays, arbitrary delay profiles and arbitrary input alphabets. Due to their minimal temporal span, the new codes enjoy smaller decoding complexity compared to those proposed in [8] and [13] while they achieve the same diversity gain.

**REFERENCES**


