**Strong Equality of MAJORITY Domination Parameters**

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**Abstract:** We study the concept of strong equality of majority domination parameters. Let \( P_1 \) and \( P_2 \) be properties of vertex subsets of a graph, and assume that every subset of \( V(G) \) with property \( P_1 \) also has property \( P_2 \). Let \( \psi_1(G) \) and \( \psi_2(G) \), respectively, denote the minimum cardinalities of sets with properties \( P_1 \) and \( P_2 \), respectively. Then \( \psi_1(G) \leq \psi_2(G) \). If \( \psi_1(G) = \psi_2(G) \) and every \( \psi_1(G) \)-set is also a \( \psi_2(G) \)-set, then we say \( \psi_1(G) \) strongly equals \( \psi_2(G) \), written \( \psi_1(G) \equiv \psi_2(G) \). We provide a constructive characterization of the trees \( T \) such that \( \gamma_M(T) \equiv i_M(T) \), where \( \gamma_M(T) \) and \( i_M(T) \) are majority domination and independent majority domination numbers, respectively.

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**I. Introduction:**

By a graph \( G \), we mean a finite, simple and undirected. Let \( G \) be a graph with \( p \) vertices and \( q \) edges. For a vertex \( v \in V(G) \), the open neighborhood of \( v \), \( N_G(v) \) is the set of vertices adjacent to \( v \) and the closed neighborhood \( N_G[v] = N_G(v) \cup \{v\} \). Other graph theoretic terminology not defined here can be found in [6].

In [6], a set \( S \subseteq V \) of vertices in a graph \( G=(V,E) \) is a dominating set if every vertex \( v \in V \) is either an element of \( S \) or is adjacent to an element of \( S \). A dominating set \( S \) is called a minimal dominating set if no proper subset of \( S \) is a dominating set. The minimum cardinality of a minimal dominating set is called the domination number \( \gamma(G) \) and the maximum cardinality of a minimal dominating set is called the upper domination number \( \Gamma(G) \) in a graph \( G \). A set \( S \subseteq V \) of vertices in a graph \( G \) is called an independent set if no two vertices in \( S \) are adjacent. An independent set \( S \) is called a maximal independent set if any vertex set properly containing \( S \) is not independent. The minimum cardinality of a maximal independent set is called the lower independence number and independent domination number and the maximum cardinality of a maximal independent set is called the independence number in a graph \( G \) and it is denoted by \( \beta(G) \) and \( \beta_v(G) \) respectively.

**Definition 1.1[3]:**

A subset \( S \subseteq V(G) \) of vertices in a graph \( G \) is called majority dominating set if at least half of the vertices of \( V(G) \) are either in \( S \) or adjacent to the vertices of \( S \). i.e., \( |N[S]| \geq \left\lfloor \frac{p}{2} \right\rfloor \). A majority dominating set \( S \) is minimal if no proper subset of \( S \) is a majority dominating set of \( G \). The majority domination number \( \gamma_M(G) \) of a graph \( G \) is the minimum cardinality of a minimal majority dominating set in \( G \). The upper majority domination number \( \Gamma_M(G) \) is the maximum cardinality of a minimal majority dominating set of a graph \( G \). This parameter has been studied by Swaminathan V and JoselineManora J.
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**Definition 1.2[2]:**

A set $S$ of vertices of a graph $G$ is said to be a majority independent set if it induces a totally disconnected subgraph with $|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|P[v, S]| \geq |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. If any vertex set $S'$ properly containing $S$ is not majority independent then $S$ is called maximal majority independent set. The maximum cardinality of a maximal majority independent set of $G$ is called majority independence number of $G$ and it is denoted by $\beta_M(G)$. A $M$-set is a maximum cardinality of a maximal majority independent set of $G$. This parameter is introduced by Swaminathan. V and JoselineManora. J.

**Definition 1.3[1]:**

A majority dominating set $D$ of a graph $G=(V,E)$ is called an independent majority dominating (IMD) set if the induced subgraph $\langle D \rangle$ has no edges. The minimum cardinality of a maximal majority independent set is called lower majority independent set of $G$, denoted by $i_M(G)$. If the degree of a vertex $v$ satisfies $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, then the vertex $v \in V(G)$ is called a majority dominating vertex of $G$.

**II. Strong equality of Majority domination Parameters.**

**Definition 2.1[5]:**

Let $P_1$ and $P_2$ be properties of vertex subsets of a graph, and assume that every subset of $V(G)$ with property $P_2$ also has property $P_1$. Let $\psi_1(G)$ and $\psi_2(G)$, respectively, denote the minimum cardinalities of sets with properties $P_1$ and $P_2$, respectively. Then $\psi_1(G) \leq \psi_2(G)$ and every $\psi_1(G)$-set is also a $\psi_2(G)$-set, then we say $\psi_1(G)$ strongly equals $\psi_2(G)$, written $\psi_1(G) \equiv \psi_2(G)$.

**Definition 2.2:**

Let $G$ be any graph with $p$ vertices. Let $\gamma_M(G)$ and $i_M(G)$ be the majority domination number and independent majority domination number of a graph $G$. Then $\gamma_M(G)$ and $i_M(G)$ are strongly equal for $G$ if $\gamma_M(G) = i_M(G)$ and every $\gamma_M(G)$-set is an $i_M(G)$-set. It is denoted by $\gamma_M(G) \equiv i_M(G)$.

**Example 2.3:**

Take $j = 2$, $p = 22$, $j = 44$. $D = \{u \_{1,1}, u \_{2,1}, u \_{1,2}, u \_{2,1}, u \_{1,3}, u \_{3,1}, u \_{3,2}, u \_{2,1}\}$. $\gamma_M(G_j) = |D| = 6$. Since all vertices in $D$ are independent, $i_M(G_j) = |D| = 6$. $\gamma_M(G_j) = i_M(G_j) = 3j$, $j = 2$. Where as $\gamma(G) = i(G) = 8j = 16$, $j = 2$. 

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Observations 2.4:

1. \( \gamma_M(G_j) < \frac{\gamma(G_j)}{2} \Rightarrow 3 < \frac{8j}{2} = 4j \), where \( G_j \) is in Fig (i).

2. When \( j = 1 \), \( \gamma_M(G_j) = 3 = i_M(G_j) \).

   When \( j = 2 \), \( \gamma_M(G_j) = 6 = i_M(G_j) \).

   When \( j = 3 \), \( \gamma_M(G_j) = 9 = i_M(G_j) \).

   In general, for \( G_j \), \( \gamma_M(G_j) \equiv i_M(G_j) = 3j, \quad j = 1, 2, ... \)

We can extend this graph by applying values to \( j = 2, 3, 4, ..., \) Then we obtain \( \gamma_M(G_j) = 3j = i_M(G_j) \).

Also, every \( \gamma_M \)-set is an \( i_M \)-set of \( G_j \). Hence \( \gamma_M(G_j) \equiv i_M(G_j) \).

Example 2.5:

The graph \( G \) is obtained from disjoint copies of \( p_5 \) by joining a central vertex of one \( p_5 \) to the central vertices of the remaining graphs \( p_5 \).
When \( k = 5 \), \( p = 25 \). \( D_1 = \{w_2, w_3, w_4, w_5\} \). \( D_1 \) dominates \( \left\lfloor \frac{p}{2} \right\rfloor = 13 \) vertices. \( \therefore \gamma_M(G) = |D_1| = 4 \).

\[ \gamma_M(G) = 4 \]

\( \therefore \) This \( \gamma_M \)-set \( D_1 \) is not an \( i_M \)-set of \( G \). \( D_2 = \{w_2, w_3, v_3, v_5\} \Rightarrow i_M(G) = |D_2| = 4 \). But \( D_2 \) is a \( \gamma_M \)-set which is also an \( i_M \)-set. Here, \( D_1 \) and \( D_2 \) are minimal majority dominating set for \( G \). \( \gamma_M(G) = i_M(G) = 4 \). Hence every \( \gamma_M \)-set is not an \( i_M \)-set for \( G \). In general, for any value of \( k \), if \( \gamma_M(G) \) is not \( \equiv i_M(G) \), if \( G = T_{5k}, k = 5 \). Ingeneral, for any value of \( k \), if \( \gamma_M(G) \) is not \( \equiv i_M(G) \), if \( G = T_{5k}, k = 5 \).

Observations 2.6:

1. If \( \gamma_M(G) = 1 \) then \( \gamma_M(G) \equiv i_M(G) \).
2. If \( G \) has a full degree vertex then \( \gamma_M(G) \equiv i_M(G) \).
3. For Corona graphs \( G \), \( \gamma_M(G) \equiv i_M(G) \), if \( G = \left( C_p \circ K_1 \right) \) and \( G = \left( P_p \circ K_1 \right) \).
4. \( \gamma_M(G) \equiv i_M(G) \) if \( G = \text{Caterpillar, with exactly one pendant.} \)
5. \( \gamma_M(G) \equiv i_M(G) \) if \( G = mK_2 \).
6. If \( G \) is Grotzsch graph, then \( \gamma_M(G) \equiv i_M(G) \).
7. For Tutte graph \( G \) with \( p = 46 \), \( q = 69 \). \( \gamma_M(G) = i_M(G) = 6 \).

8. For a Grinberg graph \( G \) with \( p = 46 \), \( q = 69 \), then \( \gamma_M(G) = i_M(G) \).
9. For a Petersen graph \( P \), \( \gamma_M(P) \) is not strongly equal to \( i_M(P) \).
10. For all Hajos graph \( H \) with \( p \) vertices,
    \[
    p = \frac{n(n+1)}{2}, n = 3, 4 \Rightarrow p = 6, 10 \text{, then } \gamma_M(H) \equiv i_M(H) \text{.}
    
    \text{But if } n = 5 \text{ and } p = 15 \text{, then } \gamma_M(H) \text{ is not strongly equal to } i_M(H) \text{.}

Proposition 2.7:

For the path \( P_p \) and cycle \( C_p \),

1. \( \gamma_M(P_{6k}) \equiv i_M(P_{6k}) \equiv \gamma_M(C_{6k}) \equiv i_M(C_{6k}) = k, k = 1, 2, 3, \ldots \)
2. \( \gamma_M(P_{6k+3}) \equiv i_M(P_{6k+3}) \equiv \gamma_M(C_{6k+3}) \equiv i_M(C_{6k+3}) = k + 1, k = 0, 1, 2, \ldots \)
3. \( \gamma_M(P_{6k+4}) \equiv i_M(P_{6k+4}) \equiv \gamma_M(C_{6k+4}) \equiv i_M(C_{6k+4}) = k + 1, k = 0, 1, 2, \ldots \)
4. \( \gamma_M(P_{6k+5}) = i_M(P_{6k+5}) = \gamma_M(C_{6k+5}) = i_M(C_{6k+5}) = k + 1, k = 0, 1, 2, \ldots \), but

5. \( \gamma_M(P_{6k+1}) \) is not dominated by \( i_M(P_{6k+1}) \).

6. \( \gamma_M(P_{6k+2}) \) is not dominated by \( i_M(P_{6k+2}) \).

Proposition 2.8[4]:

For any graph \( G \), \( \gamma_M(G) = 1 \) if and only if \( G \) has a majority dominating vertex.

Proposition 2.9:

\( \gamma_M(G) = i_M(G) = 1 \) if and only if \( G \) has a majority dominating vertex.

III. Trees \( T \) with \( \gamma_M(T) = i_M(T) \)

Our aim in this section is to give a constructive characterization for the trees \( T \) having \( \gamma_M(T) = i_M(T) \). For this purpose, we first prove two lemmas.

Lemma 3.1:

Let \( w \) be a vertex of a tree \( T_w \) such that every leaf of \( T_w \), except possibly for \( w \) itself, is at distance two from \( w \). Let \( S_w \) be the set of support vertices of \( T_w \). Let \( y \) be a pendant vertex of a non-trivial tree \( T_y \). Let \( T \) be obtained from \( T_w \cup T_y \) by adding the edge \( wy \). Then \( \gamma_M(T) = \gamma_M(T_y) + 1 \).

Proof: Let \( T = T_w \cup T_y \) and \( y \) be a pendant vertex of \( T \). Then \( \gamma_M(T_y) \) be a majority domination number of \( T_y \). Since \( w \) is a majority dominating vertex of \( T_w \), \( \gamma_M(T_y) \)-set can be extended to a majority dominating set of \( T \) by adding the vertex \( w \in T_w \). Hence \( \gamma_M(T) = \gamma_M(T_y) + 1 \).

Claim: \( \gamma_M(T) \geq \gamma_M(T_y) + 1 \). Let \( D \) be a \( \gamma_M \)-set of \( T \). Then \( D_y = D \cap V(T_y) \) and \( D_w = D \cap V(T_w) \).

Since \( T_w \) has a majority dominating vertex \( w \). \( |D_w| = \{w\} \).

Since \( D \) is a \( \gamma_M \)-set of \( T \), \( D_y \) is a majority dominating set of \( T_y \). Then \( \gamma_M(T_y) \leq |D_y| \leq |D - D_w| \Rightarrow \gamma_M(T_y) \leq |D| - 1 \Rightarrow \gamma_M(T) - 1 \Rightarrow \gamma_M(T) + 1 \leq \gamma_M(T) \).

Hence, \( \gamma_M(T) = \gamma_M(T_y) + 1 \). □

Lemma 3.2:

Let \( T_w, T_y \), and \( T \) be defined as in the statement of Lemma (3.1). Then \( \gamma_M(T) = i_M(T) \) if and only if \( \gamma_M(T_y) = i_M(T_y) \).

Proof: Suppose \( \gamma_M(T) = i_M(T) \) .........(1). Then \( \gamma_M(T_y) \)-set and \( D_y \cup \{w\} \) is a majority dominating set of \( T \) of cardinality \( \gamma_M(T_y) + 1 \). Then by lemma (3.1), \( \gamma_M(T) = \gamma_M(T_y) + 1 \).

Therefore \( D_y \cup \{w\} \) is a \( \gamma_M(T) \)-set and by (1), it is a \( i_M(T) \)-set. In particular, \( D_y \) is an independent majority dominating set of \( T_y \) and so, \( |D_y| = \gamma_M(T_y) \leq i_M(T_y) \leq |D_y| \). Hence \( |D_y| = i_M(T_y) \) and \( D_y \) is a \( i_M(T_y) \)-set. Thus, every \( \gamma_M(T_y) \)-set is an \( i_M(T_y) \)-set. \( \gamma_M(T_y) = i_M(T_y) \).

Conversely, Let \( \gamma_M(T_y) = i_M(T_y) \) .........(2). To prove \( \gamma_M(T) = i_M(T) \). Let \( D \) be a \( \gamma_M(T) \)-set and \( D_y = D \cap V(T_y) \) and \( D_w = D \cap V(T_w) \).

Suppose \( w \notin D \), then

\[ |D_w| = |S_w| \] and \( |D_y| = |D - D_w| = |D| - |S_w| \). Then \( |D_y| = \gamma_M(T) - |S_w| \Rightarrow \gamma_M(T) = \gamma_M(T_y) + |S_w| \), which is a contradiction to lemma (3.1), \( \gamma_M(T) = \gamma_M(T_y) + 1 \). Hence \( w \in D \). Then \( D_w = \{w\} \in T_w \), since \( w \) is a majority dominating vertex of \( T_w \). Since \( T \) has an edge \( wy \), \( w \) is the only vertex that dominates \( y \).

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Since \( y \) is already dominated by \( w \in D_w \), \( D_y \) does not contain \( y \) in \( T_y \). But \( D_y \) is itself a majority dominating set of \( T_y \) of \( |D - D_w| \), i.e., \( |D_y| = |D - D_w| = |D| - 1 = \gamma_M(T) - 1 \). By lemma (3.1), \( |D_y| = \gamma_M(T_y) \), by (2), \( |D_y| \equiv i_{M}(T_y) \Rightarrow D_y \) is an independent majority dominating set of \( T_y \). Furthermore, \( D_w = \{w\} \) is also an independent majority dominating set of \( T_w \). Hence \( D \) is an \( i_M(T) \)-set. Thus every \( \gamma_M(T) \)-set is an \( i_M(T) \)-set. \( \therefore \gamma_M(T) \equiv i_M(T) \).

Next, a construction for characterization of the trees \( T \) for which \( \gamma_M(T) \equiv i_M(T) \) is provided by using the following operation.

**Operation - A:** Let \( w \) be a vertex of a tree \( T_w \) such that every leaf of \( T_w \) except possibly for \( w \) itself, is at distance two from \( w \). Let \( S_w \) be the set of support vertices of \( T_w \). Let \( y \) be a pendant vertex of a non-trivial tree \( T_y \). Let \( T \) be obtained from \( T_w \cup T_y \) by adding the edge \( wT \). Define the family as \( \mathcal{A}_1 = \{T/T = K_1 \text{ or } T \text{ is obtained from a non-trivial star by a finite sequence of operation } A\} \).

**Theorem 3.3:**

For any tree \( T \), \( \gamma_M(T) \equiv i_M(T) \) if and only if \( T \in \mathcal{A}_1 \).

**Proof:** Let \( T \in \mathcal{A}_1 \), if \( T = K_1 \) or if \( T \) is a non-trivial star, then \( \gamma_M(T) = i_M(T) = 1 \) and \( \gamma_M(T) \equiv i_M(T) \). On the other hand, if \( T \) is constructed from a non-trivial star by a finite sequence of at least one operation \( A \), then repeated applications of lemma (3.2), we get \( \gamma_M(T) \equiv i_M(T) \), since a star has majority domination number strongly equal to its independent majority domination number. Conversely, let \( \gamma_M(T) \equiv i_M(T) \). To prove \( T \in \mathcal{A}_1 \). By induction on the order \( p \) of a tree \( T \) for which \( \gamma_M(T) \equiv i_M(T) \). If \( T = K_1 \) or \( K_2 \), then \( T \in \mathcal{A}_1 \). If \( \text{diam } T = 2 \) then \( T \) is a non-trivial star and so \( T \in \mathcal{A}_1 \). When \( \text{diam } T = 3, 4, 5, 6 \) which satisfy \( \gamma_M(T) \equiv i_M(T) \) since \( \gamma_M(T) = 1 = i_M(T) \). Then \( T \in \mathcal{A}_1 \). Now, assume that \( \text{diam } T \geq 7 \) which satisfy \( \gamma_M(T) \equiv i_M(T) \). We now root the tree at a leaf \( r \) of maximum eccentricity \( \text{diam } T \). Let \( w \) be the vertex at distance \( \text{diam } T - 2 \) from \( r \) on a longest path starting at \( r \).

Let \( T_w \) be the subtree of \( T \) rooted at \( w \). Then the vertex cannot be adjacent to a leaf. If not, it will contradict our assumption that \( \gamma_M(T) \equiv i_M(T) \). Hence every leaf of \( T_w \), except possibly for \( w \) itself, is at distance two from \( w \). Let \( y \) denote the parent of \( w \) on \( T \) and let \( T_y \) denote the component of \( T - wT \) containing \( y \).

Since \( \text{diam } T \geq 7 \), \( T_y \) is a non-trivial tree. By lemma (3.2), if \( \gamma_M(T) \equiv i_M(T) \) then \( \gamma_M((T_y) \equiv i_M(T_y) \).

Now, since \( T_y \) is a tree of order less than \( p \) satisfying \( \gamma_M(T_y) \equiv i_M(T_y) \), we can apply the induction hypothesis, to \( T_y \) to show that \( T_y \in \mathcal{A}_1 \). Since \( T \) is obtained from \( T_y \) by a operation \( A \), we have \( T \in \mathcal{A}_1 \). Hence the theorem. \( \square \)

**Theorem 3.4:** Let \( D_1 \) be the set of all \( \gamma_M \)-sets of \( G \). Then

(i). \( \gamma_M(G) \equiv i_M(G) \) if and only if induced subgraph \( \{D\} \) has only isolates, for every \( \gamma_M \)-set \( D \in D_1 \).

(ii). \( \gamma_M(G) \) is not \( \equiv i_M(G) \) if and only if the induced subgraph \( \{D\} \) is not totally disconnected for any \( \gamma_M \)-set \( D \in D_1 \).

**Proof:** Let \( D_j \) be the set of all \( \gamma_M \)-set \( D \) of a graph \( G \).

(i). Suppose \( \gamma_M(G) \equiv i_M(G) \). Then \( \gamma_M(G) \leq i_M(G) \) and every \( \gamma_M \)-set \( D \) of a graph \( G \) is an independent majority dominating set of \( G \). The induced subgraph \( \{D\} \) has only isolates for every \( \gamma_M \)-set.
$D \in D_i$. Conversely, for every $\gamma_M$-set $D$, the induced subgraph $\langle D \rangle$ has only isolates. Then $D$ is an independent set of $G \implies$ every $\gamma_M$-set $D$ is an $i_M$-set of $G$. \( \therefore i_M(G) \leq \gamma_M(G) \). For any graph $G$, $\gamma_M(G) \leq i_M(G)$. Hence $\gamma_M(G) \equiv i_M(G)$.

(ii). Suppose $\gamma_M(G)$ is not $\equiv i_M(G)$. Then for any graph $G$, $\gamma_M(G) \leq i_M(G)$ but not every $\gamma_M$-set $D$ is an $i_M$-set of $G$. Then the $\gamma_M$-set $D$ is not independent for any one $D \in D_i$. Hence, the induced subgraph $\langle D \rangle$ is not totally disconnected for any $D \in D_i$. Conversely, if $\langle D \rangle$ is not totally disconnected for at least one $D \in D_i$ then $D$ is not an independent $\gamma_M$-set. It does not satisfy the fact that every $\gamma_M$-set is an $i_M$-set of $G$. \( \therefore i_M(G) \leq \gamma_M(G) \) is not true. Thus, $\gamma_M(G)$ is not $\equiv i_M(G)$.

### IV. Strong Equality of $\gamma(G)$ and $\gamma_M(G)$ and of $i(G)$ and $i_M(G)$.

**Observations 4.1:**

1. For any graph $G$, $\gamma(M) \leq \gamma(G)$.
2. If $\gamma(G) = 1$ then $\gamma_M(G) = 1$.
3. If $G$ has a full degree vertex then every $\gamma_M$-set is a $\gamma$-set.

**Proposition 4.2:**

For any graph $G$, $\gamma(G) \equiv \gamma_M(G)$ if and only if $G$ has a full degree vertex.

**Proof:** Let $\gamma(G) \equiv \gamma_M(G)$. Suppose $G$ has no full degree vertex. Then $\gamma(G) \geq 2$. $G$ may have a majority dominating vertex $v$ with $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$. Then $\gamma_M(G) = 1$ but $r > 1$. Therefore every $\gamma_M$-set is not a $\gamma$-set $\implies \gamma(G)$ is not strongly equal to $\gamma_M(G)$, a contradiction. Hence $G$ has a full degree vertex. Conversely, if $G$ has a full degree vertex, then $r = 1$. Then $\gamma_M(G) = 1$. Since $\gamma(G) = \gamma_M(G) = 1$, every $\gamma_M$-set is also a $\gamma$-set. Hence $\gamma(G) \equiv \gamma_M(G)$.

**Proposition 4.3:**

For any graph $G$, $i(G) \equiv i_M(G)$ if and only if $G$ has a full degree vertex.

**References:**