ON CROSSING NUMBERS OF HYPERCUBES AND CUBE CONNECTED CYCLES

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Abstract. We prove tight bounds for crossing numbers of hypercube and cube connected cycles (CCC) graphs.

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1 Introduction

Recently the hypercube-like networks have received considerable attention in the field of parallel computing due to its high potential for system availability and parallel execution of algorithms (see e.g. [4]). This motivates to investigation of various, from this point of view important, properties of the n-dimensional hypercube graph Q_n and its bounded degree alternatives: Cube Connected Cycles (CCC), Butterfly and de Bruijn graphs. In this paper we concentrate on the crossing number of Q_n and CCC_n.

The crossing number $\mathrm{cr}(G)$ of a graph G is defined as the least number of crossings of its edges when G is drawn in a plane. In practice, crossing numbers appear in the fabrication of VLSI circuits. The crossing number of a graph corresponding to the VLSI circuit has strong influence on the area of the layout as well as on the number of wire - contact cuts that should be minimized. Leighton [7] pointed out that crossing numbers provide a good area lower bound argument in VLSI complexity theory. For a long time the only known results on $\mathrm{cr}(Q_n)$ has been $\mathrm{cr}(Q_3) = 0, \mathrm{cr}(Q_4) = 8$, $\mathrm{cr}(Q_5) \leq 56$ [3] and a conjecture of Erdös and Guy [2]:

$$\mathrm{cr}(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-1}.$$

Recently Madej [9] has derived that

$$\alpha 2^{n+0.215\log^2 n} < \mathrm{cr}(Q_n) < \frac{1}{6}4^n - (n^2+1)2^{n-3},$$

for some positive constant α.

In this paper we essentially improve the Madej's lower bound and prove tight bounds on $\mathrm{cr}(CCC_n)$. In fact we show that:

$$\mathrm{cr}(Q_n) > \frac{1}{20}4^n - (n^2+1)2^{n-1},$$

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$$\frac{1}{20}4^n - (9n+1)2^{n-1} < \mathrm{cr}(CCC_n) < \frac{1}{20}4^n + 3n^2 2^{n-3}.$$

To prove the lower bounds we apply the method proposed by Leighton [7]. Having prepared the final version of this paper we learnt that recently Shahrokhi and Székely [11, 12] had substantially developed the method. As a direct concequence they obtained several new lower bounds on crossing numbers for symmetric graphs including hypercubes and cube connected cycles. However, they have not concentrated themselves upon finding the best possible constants.

Our lower bounds on $\mathrm{cr}(Q_n)$ and $\mathrm{cr}(CCC_n)$ give immediately alternative proofs that VLSI layouts of hypercube and CCC networks occupy the area $A = \Omega(4^n)$. Previous proofs are in [1, 8]. Optimal layouts are proposed in [1, 10].

2 Bounds on $\mathrm{cr}(Q_n)$ and $\mathrm{cr}(CCC_n)$

First we introduce the Leighton's method [7]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An embedding of G_1 in G_2 is a couple of mappings (ϕ, ψ) satisfying

$$\phi : V_1 \to V_2 \quad \text{is an injection}$$

$$\psi : E_1 \to \{\text{set of all paths in } G_2\},$$

such that if $(u, v) \in E_1$ then $\psi((u, v))$ is a path between $\phi(u)$ and $\phi(v)$. For any $e \in E_2$ define

$$\mathrm{cg}_e(\phi, \psi) = \mid \{f \in E_1 \ : \ e \in \psi(f)\} \mid$$

and

$$\mathrm{cg}(\phi, \psi) = \max_{e \in E_2} \{\mathrm{cg}_e(\phi, \psi)\}.$$

The value $\mathrm{cg}(\phi, \psi)$ is called congestion.

Lemma 2.1 [7] *Let (ϕ, ψ) be an embedding of G_1 in G_2 with congestion $\mathrm{cg}(\phi, \psi)$. Let $\Delta(G_2)$ denote the maximal degree of G_2. Then*

$$\mathrm{cr}(G_2) \geq \frac{\mathrm{cr}(G_1)}{\mathrm{cg}^2(\phi, \psi)} - \frac{\mid V_2 \mid}{2} \Delta^2(G_2). \tag{1}$$

2.1 Hypercube

The n-dimensional hypercube graph Q_n has 2^n vertices each of which is a binary sequence $a_{n-1}...a_2a_1a_0$ of length n, where $a_i = 0$ or 1. Two of its vertices are adjacent whenewer their sequences differ in exactly one place. Thus each vertex has degree n. We say that an edge uv lies in the $i-$th dimension, $i = 1, 2, ...n$, if u and v differ in the i-th place.

Theorem 2.1

$$\mathrm{cr}(Q_n) > \frac{1}{20}4^n - (n^2+1)2^{n-1}.$$

Proof: Let $2K_m$ denote the complete multigraph of m vertices, in which every two vertices are joined by two parallel edges. Set $G_1 = 2K_{2^n}$ and $G_2 = Q_n$. In what follows, we show, that there exists an embedding (ϕ, ψ) of $2K_{2^n}$ in Q_n with

$$\mathrm{cg}(\phi, \psi) \leq 2^n. \tag{2}$$

Kleitman's paper [6] implies

$$\mathrm{cr}(K_{2^n}) \geq \frac{2^n(2^n-1)(2^n-2)(2^n-3)}{80}. \tag{3}$$

According to Kainen [5] it holds

$$\mathrm{cr}(2K_{2^n}) = 4\mathrm{cr}(K_{2^n}). \tag{4}$$

Substituting (2), (3) and (4) into (1), we obtain the desired result.

Now we show an embedding satisfying (2). Let ϕ be any injection of the vertices of $2K_{2^n}$ in the vertices of Q_n. Further, we define the mapping ψ, i.e two paths between any pair of vertices of Q_n. Consider two arbitrary vertices u and v of Q_n. Let d be their distance. Then there exists the unique path of length d starting in u, traversing dimensions in ascending order and ending in v. Let the second path be the symmetrical one starting in v and ending in u. Let $e = xy$ be an arbitrary edge of Q_n lying in a dimension $i, 1 \leq i \leq n$. Now we count the number of edges of $2K_{2^n}$ whose images (paths) traverse the edge xy. Let A (B) be the maximal subcube of Q_n that contains x (y) and all its edges lie in dimensions $1, 2, ..., i-1 (i+1, i+2, ..., n)$. (If $i = 1$ or n then A or B is a single vertex, i.e. Q_0.) Similarly, let C (D) be the maximal subcube of Q_n that contains y (x) and all its edges lie in dimensions $1, 2, ..., i-1$ $(i+1, i+2, ..., n)$. From the construction of ψ it follows that any path (the image of an edge under ψ), which contains the edge xy, must start in A (or C) and end in B (or D). As A and C (B and D) contain 2^{i-1} (2^{n-i}) vertices we get

$$\mathrm{cg}_e(\phi, \psi) \leq 2^{i-1}2^{n-i} + 2^{i-1}2^{n-i} = 2^n$$

and consequently

$$\mathrm{cg}(\phi, \psi) \leq 2^n. \quad \Box$$

2.2 Cube Connected Cycles

We use the same method to prove the lower bound on $\mathrm{cr}(CCC_n)$. The graph CCC_n is defined as follows. The set of vertices consists of tuples $(i, j), i = 0, 1, 2, 3, ..., 2^n - 1, j = 0, 1, 2, ..., n-1$. Vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $i_1 = i_2$ and $\mid j_2 - j_1 \mid \bmod n = 1$ or $j_1 = j_2$ and the binary representations of i_1, i_2 differ only in the j_1-th place. Note that if we shrink the cycles $(i, 0)(i, 1)...(i, n-1)$, for $i = 0, 1, 2, ..., 2^n - 1$, the CCC_n degenerates to Q_n. Thus CCC_n can be obtained from Q_n by a reverse process i.e. by a suitable replacing of vertices of Q_n by cycles of length n. (See Fig. 1 for $n = 3$.)

Theorem 2.2

$$\frac{1}{20}4^n - (9n+1)2^{n-1} < \mathrm{cr}(CCC_n) < \frac{1}{6}4^n + 3n^2 2^{n-3}.$$

Proof:

The upper bound. Consider the crossing-free drawing of Q_n in the plane described in [9]. Around each vertex of Q_n, we find a small circle containing no crossings. In each circle, we replace the vertex by a cycle of length n in the following way. (See Fig. 2 for $n = 6$.) Let u be such a vertex. Let $v_1, v_2, ..., v_n$ be its neighbours such that uv_i lies in the $i-$th dimension, for $i = 1, 2, ..., n$. Let w_i be a new vertex placed on the edge uv_i in the circle, for $i = 1, 2, ..., n$. Finally, we delete the vertex u and insert new edges to form a cycle $w_1 w_2 w_3 ... w_n w_1$. We can do this in such a way that two edges of the cycle cross each other at most once. Thus we have constructed a plane drawing of CCC_n having $\leq \mathrm{cr}_0(Q_n) + \binom{n-1}{2} 2^n < \frac{1}{6} 4^n + 3n^2 2^{n-3}$ crossings.

The lower bound. Denote by CCP_n (Cube Connected Paths) the graph which is obtained from CCC_n by removing edges $(i,0)(i,n-1)$, for $i = 0,1,2,3,...,2^n-1$. Observe that the graph CCP_n has a simple recursive structure. Clearly it holds

$$\mathrm{cr}(CCC_n) \geq \mathrm{cr}(CCP_n). \tag{5}$$

Set $G_1 = K_{2^n,2^n}, G_2 = CCP_n$. In what follows we construct an embedding (ϕ_n, ψ_n) of $K_{2^n,2^n}$ in CCP_n such that

$$\mathrm{cg}(\phi_n, \psi_n) = 2^n. \tag{6}$$

Once more the Kleitman's result [6] implies

$$\mathrm{cr}(K_{2^n,2^n}) \geq \frac{2^{2n-1}(2^n-1)(2^{n-1}-1)}{5} \tag{7}$$

Substituting (6) and (7) into (1) and noting (5) we obtain the desired result.

Assume $n \geq 2$. Let ϕ_n be an injection that maps the first (second) 2^n mutually nonadjacent vertices of $K_{2^n,2^n}$ in the set $\{(i,0) \mid i = 0,1,2,3,...,2^n-1\}$ ($\{(i,n-1) \mid i = 0,1,2,3,...,2^n-1\}$). We design ψ_n by induction. Let $n = 2$. The 16 paths between the vertices $\{(i,0) \mid 0 \leq i \leq 3\}$ and $\{(i,1) \mid 0 \leq i \leq 3\}$ are the following:

Suppose $k = 0$ or 2.

$(k,0)(k,1)$

$(k,0)(k+1,0)(k+1,1)$

$(k,0)(k,1)((k+2) \bmod 4, 1)$

$(k,0)(k+1,0)(k+1,1)((k+3) \bmod 4, 1)$

Suppose $k = 1$ or 3.

$(k,0)(k-1,0)(k-1,1)$

$(k,0)(k,1)$

$(k,0)(k-1,0)(k-1,1)((k+1) \bmod 4, 1)$

$(k,0)(k,1)((k+2) \bmod 4, 1)$

Clearly $\mathrm{cg}(\phi_2, \psi_2) = 4$.

Assume we have constructed (ϕ_{n-1}, ψ_{n-1}) such that $\mathrm{cg}(\phi_{n-1}, \psi_{n-1}) = 2^{n-1}$. Now we construct a path between every pair of vertices $(i_1, 0), (i_2, n-1)$ in CCP_n.

1. If $i_1, i_2 < 2^{n-1}$ or $i_1, i_2 \geq 2^{n-1}$ then we first form a path between $(i_1, 0)$ and $(i_2, n-2)$ using ψ_{n-1} and then prolong this path to $(i_2, n-1)$.

2. If $i_1 < 2^{n-1}$ and $i_2 \geq 2^{n-1}$ then we first form a path between $(i_1, 0)$ and $(i_2 - 2^{n-1}, n-2)$ using ψ_{n-1} and then prolong this path to $(i_2, n-1)$ through $(i_2 - 2^{n-1}, n-1)$. The case $i_1 \geq 2^{n-1}, i_2 < 2^{n-1}$ is symmetrical.

Consider an edge $e = (i_1, j_1)(i_2, j_2)$ of CCP_n. Without loss of generality assume that $i_1 \leq i_2, j_1 \leq j_2$. Distinguish three possibilities:

- Let $j_2 \leq n-2$. From the construction of ψ_n it follows that each path of the 1-st (2-nd) case traverses e at most $\mathrm{cg}_e(\phi_{n-1}, \psi_{n-1})$ times. Hence

$$\mathrm{cg}_e(\phi_n, \psi_n) \leq 2\mathrm{cg}_e(\phi_{n-1}, \psi_{n-1}) \leq 2^n.$$

- Let $e = (i_1, n-2)(i_1, n-1)$. Without loss of generality assume that $i_1 < 2^{n-1}$. The edge e is traversed by all paths that start in $(i, 0)$ and end in $(i_1, n-1)$ or $(i + 2^{n-1}, n-1)$, for $0 \leq i < 2^{n-1}$. Hence

$$\mathrm{cg}_e(\phi_n, \psi_n) \leq 2^{n-1} + 2^{n-1} = 2^n.$$

- Let $e = (i_1, n-1)(i_1 + 2^{n-1}, n-1), i_1 < 2^{n-1}$. Then e is traversed by all paths starting in $(i, 0)$, ending in $(i_1 + 2^{n-1}, n-1)$, for $i < 2^{n-1}$ and all paths starting in $(i, 0)$ and ending in $(i_1, n-1)$, for $i \geq 2^{n-1}$. Hence

$$\mathrm{cg}_e(\phi_n, \psi_n) \leq 2^{n-1} + 2^{n-1} = 2^n. \qquad \square$$

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