Collocation method for linear and nonlinear Fredholm and Volterra integral equations

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A collocation procedure is developed for the linear and nonlinear Fredholm and Volterra integral equations, using the globally defined B-spline and auxiliary basis functions. The solution is collocated by cubic B-spline and then the integral equation is approximated by the 5-points Gauss–Turán quadrature formula with respect to the Legendre weight function. Combination of these two approaches is the main idea of this paper to reduce the cost of computation and complexity. The error analysis of proposed numerical method is studied theoretically. Numerical results are given to illustrate the efficiency of the proposed method. The results are compared with the results obtained by other methods to verify that this method is accurate and efficient.

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1. Introduction

In this article we will develop an approximation based on B-spline to obtain numerical solution of the following Fredholm integral equation:

\[ y(t) = f(t) + \int_{a}^{b} k(t, x, y(x)) \, dx, \quad t \in [a, b], \]

and the Volterra integral equation:

\[ y(t) = f(t) + \int_{a}^{t} k(t, x, y(x)) \, dx, \quad a \leq t, x \leq b, \]

where \( k \) is continuous on \([a, b]\) and satisfies a uniform Lipschitz, \( f(t) \) is known function and \( y(t) \) is an unknown function. The existence and uniqueness solutions of these type or special forms of Eqs. \( (1.1) \) and \( (1.2) \) have been studied by \([1–12]\). The existence of continuous solutions of \((1.1)\) and \((1.2)\) have been studied when \( k \) be bounded w.r.t. \( x \) in \([4]\). Rejto and Taboada in \([6]\), show unique solvability of \((1.2)\) in weighted spaces. In \([11]\), unique solvability of \((1.1)\) using the contraction mapping is discussed where \( k \) and \( f(t) \) are continuous. Most numerical methods in the literature have focused on the cases where the functions to be approximated are differentiable on the whole interval \([a, b]\), and do not work well when the functions behave badly at the endpoints (see, for example, \([3, Example \, 4.2.5]\)). Haar wavelets are also applied for solving these problems in \([13,14]\). The sinc-collocation method for Fredholm and Volterra integral equations are used in \([16–21]\). Using a global approximation to the solution

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of a linear Fredholm integral equation of the second kind is constructed by means of the cubic spline quadrature in [23–28]. The series expansion is used for unknown function and kernel of linear and nonlinear integral equations of the second kind in [30]. The HPM and ADM methods have been used in [32–34]. The Harmonic wavelet method for Fredholm type integral equations of the second kind is used in [39]. The application of wavelet-basis is used for solution of the Fredholm type integral equations in [40]. The Bernstein polynomials are used for numerical solution of integral equations in [41]. In [42], semi orthogonal B-spline wavelet collocation method is used for the system of Fredholm integral equations. A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type is used in [43]. In [44], B-spline solution of boundary value problems of fractional order based is used.

In this article we will use cubic B-spline collocation to approximate the unknown functions in Eqs. (1.1) and (1.2) then the 5-points Gauss–Turán quadrature formula with respect to the Legendre weight is used to approximate the linear and nonlinear Fredholm and Volterra integral equations of second kind. The layout of this paper is as follows. In Section 2 the basic definitions, assumptions and preliminaries of the cubic B-spline collocation method are stated. The quadrature formulae of Gauss–Turán properties are discussed in Section 3. Nonlinear Fredholm and Volterra integral equations are considered in Section 4. In Section 5, the error analysis of our new approach is described. Finally, Section 6 contains the numerical experiments.

2. Cubic B-spline

We introduce the cubic B-spline space and basis functions to construct an interpolation \( s \) to be used in the formulation of the cubic B-spline collocation method. Let \( \pi = \{ a = t_0 < t_1 \leq \cdots < t_N = b \} \) be a uniform partition of the interval \([a, b]\) with step size \( h = \frac{b-a}{N} \). The cubic spline space is denoted by

\[
S_3(\pi) = \{ s \in C^2[a, b] : \text{s|}_{[t_i, t_{i+1}]} \in P_3, \quad i = 0, 1, \ldots, N \},
\]

where \( P_3 \) is the class of cubic polynomials. The construction of the cubic B-spline interpolate \( s \) to the analytical solution \( y \) for (1.1) and (1.2) can be performed with the help of the four additional knots such that \( t_{-2} < t_{-1} < t_0 \) and \( t_N < t_{N+1} < t_{N+2} \). Following [35] we can define a cubic B-spline \( s(t) \) of the form

\[
s(t) = \sum_{i=1}^{N+1} c_i \beta_i^3(t), \tag{2.1}
\]

where

\[
\beta_i^3(t) = \begin{cases} 
\frac{(t - t_{i-2})^3}{6h^3} & t \in [t_{i-2}, t_{i-1}] \\
\frac{h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3}{6h^3} & t \in [t_{i-1}, t_i] \\
\frac{h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^2 - 3(t_{i+1} - t)^3}{6h^3} & t \in [t_i, t_{i+1}] \\
(t_{i+2} - t)^3 & t \in [t_{i+1}, t_{i+2}] \\
0 & \text{otherwise},
\end{cases}
\]

satisfying the following interpolation conditions

\[ s(t_i) = y(t_i), \quad 0 \leq i \leq N, \]

and the end conditions

\[ c_1 s'(t_0) = y'(t_0), \quad s'(t_N) = y'(t_N), \]

or

\[ c_2 D^j s(t_0) = D^j s(t_N), \quad j = 1, 2, \]

or

\[ c_3 s''(t_0) = 0, \quad s''(t_N) = 0. \tag{2.3} \]

3. On quadrature formulae of Gauss–Turán

Let \( P_m \) be the set of all algebraic polynomials of degree at most \( m \). The Gauss–Turán quadrature formula in [36–38] is

\[
\int_a^b f(x) d\lambda(x) = \sum_{i=0}^{2s} \sum_{\nu=1}^{n} A_{i, \nu} f^{(i)}(\tau_\nu) + R_{n, 2s}(f), \tag{3.1}
\]

where \( n \in N, s \in N_0 \) and \( d\lambda(x) \) is a nonnegative measure on the interval \((a, b)\) which can be the real axis \( R \), with compact or infinite support for which all moments

\[ \mu_k = \int_a^b x^k d\lambda(x), \quad k = 0, 1, \ldots, \]
exist and are finite, moreover \( \mu_0 > 0 \), and \( A_{l,v} = \int_a^b l_{i,j}(x)\,d\lambda(x) \), \( i = 0, \ldots, 2s, v = 1, \ldots, n \) and \( l_{i,j}(x) \) are the fundamental polynomials of Hermite interpolation. The nodes \( \tau_v (v = 1, \ldots, n) \) in (3.1) are the zeros of monic polynomial \( \pi_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \), which minimizes the integral

\[
F(a_0, a_1, \ldots, a_{n-1}) = \int_a^b \left[ \pi_n(x) \right]^{2+2} d\lambda(x),
\]

then the formula (3.1) is exact for all polynomials of degree at most \( 2(s + 1)n - 1 \). that is, \( R_{n,2s}(f) = 0 \), \( \forall f \in P_{2(s+1)n-1} \). The condition (3.2) is equivalent with the following conditions:

\[
\int_a^b \left[ \pi_n(x) \right]^{2+1} x^k d\lambda(x) = 0, \quad k = 0, 1, \ldots, n - 1,
\]

let \( \pi_n^2(x) \) denoted by \( P_{n,s} \) and \( d\lambda(x) = w(x) \, dx \) on \([a, b]\). In this article, we use 5-points Gauss–Turán quadrature formula with respect to the Legendre weight function \( w(x) = 1 \) and \([-1, 1]\) with \( n = 5, s = 3 \), therefore we can approximate integrand as

\[
\int_{-1}^1 f(x) \, dx = \sum_{i=0}^5 \sum_{v=1} \left[ A_{i,v} \right] f^{(i)}(\tau_v) + R_{5,6}(f),
\]

where \( \pi_5(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \), by solving system (3.4) we can obtain \( a_i (i = 0, 1, 2, 3, 4) \) coefficients, on the other hand, we have

\[
\begin{align*}
\pi_{v+1}(x) &= (x - \alpha_v)\pi_v(x) - \beta_v \pi_{v-1}(x), & v = 0, 1, 2, 3, 4, \\
\pi_{-1}(x) &= 0, & \pi_0(x) &= 1,
\end{align*}
\]

where

\[
\begin{align*}
\alpha_v &= \alpha_v(3, 5) = \frac{\pi_{v+1}(x)\pi_v(x)}{\pi_v(x)\pi_{v-1}(x)}, \\
\beta_v &= \beta_v(3, 5) = \frac{\pi_{v-1}(x)\pi_v(x)}{\pi_{v-1}(x)\pi_{v-1}(x)}.
\end{align*}
\]

so we can obtain the zeros of monic polynomial \( \pi_5^{3,5}(x) \) of eigenvalue Jacobian matrix

\[
J_5 = \begin{pmatrix}
\alpha_0 & \sqrt{\beta_1} & 0 & 0 & 0 \\
\sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & 0 & 0 \\
0 & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & 0 \\
0 & 0 & \sqrt{\beta_3} & \alpha_3 & \sqrt{\beta_4} \\
0 & 0 & 0 & \sqrt{\beta_4} & \alpha_4
\end{pmatrix},
\]

and the values of \( \tau_v, \alpha_v, \beta_v \) which tabulated in Table (1).

Finally to determine \( A_{l,v} \), we use the following polynomial for approximation of function \( f(x) \)

\[
f_{k,v}(x) = (x - \tau_v)^k \Omega_v(x) = (x - \tau_v)^k \prod_{i \neq v} (x - \tau_i)^{2s+1},
\]

Table 1

<table>
<thead>
<tr>
<th>( v )</th>
<th>( \tau_v )</th>
<th>( \alpha_v )</th>
<th>( \beta_v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.93810699475975</td>
<td>-0.2041899952</td>
<td>3.83357763 \times 10^{-7}</td>
</tr>
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<td>0.2697108295</td>
</tr>
<tr>
<td>2</td>
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<td>0.0010583788</td>
<td>0.3046524435</td>
</tr>
<tr>
<td>3</td>
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<td>0.2173728849</td>
<td>0.2789709710</td>
</tr>
<tr>
<td>4</td>
<td>0.93810699475975</td>
<td>-0.6744354850</td>
<td>0.2554389765</td>
</tr>
</tbody>
</table>
where \( 0 \leq k \leq 2s, 1 \leq v \leq n \) and
\[
\Omega_v(x) = \left( \frac{\tau_v(x)}{x - \tau_v} \right)^{2s+1} = \prod_{i \neq v} (x - \tau_i)^{2s+1}, \quad v = 1, \ldots, n,
\]
since Eq. (3.1) is exact for all polynomials of degree at most \( 2(s + 1)n - 1 \), then accuracy degree \( f_{k,v} \) is
\[
deg f_{k,v} = (n - 1)(2s + 1) + k \leq (2s + 1)n - 1.
\]
Then Eq. (3.1) is exact for polynomials \( (3.5) \), that is, \( R(f_{k,v}) = 0 \), \( 0 \leq k \leq 2s, 1 \leq v \leq n \) then by replacing \( f_{k,v}(x) \) instead of \( f(x) \) in (3.1) we have
\[
\sum_{i=0}^{2s} \sum_{j=1}^{n} A_{i,j} f_{k,v}^{(i)}(\tau_j) = \int_{a}^{b} f_{k,v}(x)dx = \mu_{k,v},
\]
therefore for each \( v = j \), we get the linear system \( (2s + 1) \times (2s + 1) \), where \( A_{i,j} \) are unknowns \( i = 0, \ldots, 2s, v = 1, \ldots, n \).

4. Nonlinear Fredholm and Volterra integral equations

4.1. Nonlinear Fredholm integral equation

We consider nonlinear Fredholm integral equation
\[
y(t) = f(t) + \int_{a}^{b} k(t, x, y(x))dx, \quad t \in [a, b]. \tag{4.1}
\]
the solution of Eq. (4.1) can be replaced with cubic B-spline (2.1), then collocate Eq. (4.1) at collocation points \( t_i = a + ih, \ h = \frac{b-a}{N}, i = 0, 1, \ldots, N \), and we obtain
\[
s(t_i) = f(t_i) + \int_{a}^{b} k(t_i, x, s(x))dx, \quad i = 0, 1, \ldots, N. \tag{4.2}
\]
By partitioning the interval \([a, b]\) to \( N \) equal subintervals we obtain
\[
s(t_i) = f(t_i) + \sum_{p=0}^{N-1} \int_{t_p}^{t_{p+1}} k(t_i, x, s(x))dx, \quad i = 0, 1, \ldots, N. \tag{4.3}
\]
For using the Gauss-Turán formula we need to change each subinterval \([t_p, t_{p+1}]\) to the interval \([-1, 1]\). Then by the following change of variable, we have
\[
x = \frac{1}{2}[(t_{p+1} - t_p)u + (t_{p+1} + t_p)], \quad dx = \frac{t_{p+1} - t_p}{2}du = \frac{h}{2}du,
\]
and then
\[
\int_{t_p}^{t_{p+1}} k(t, x, s(x))dx = \frac{h}{2} \int_{-1}^{1} k\left( t, \frac{1}{2}[(t_{p+1} - t_p)u + (t_{p+1} + t_p)], s\left( \frac{1}{2}[(t_{p+1} - t_p)u + (t_{p+1} + t_p)] \right) \right)du.
\]
To approximate the integral Eq. (4.3), we can use the 5-points Gauss-Turán quadrature formula in the case \( n = 5 \) and \( s = 3 \), then we get the following nonlinear system:
\[
s(t_i) = f(t_i) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{v=1}^{5} \sum_{i=0}^{6} A_{i,v} k^{(i)}(t_i, \xi_{pv}, s(\xi_{pv})), \quad i = 0, 1, \ldots, N, \tag{4.4}
\]
where \( \xi_{pv} = \frac{1}{2}[(t_{p+1} - t_p)\tau_v + (t_{p+1} + t_p)] \) and we have the nodes \( \tau_v \) and coefficients \( A_{i,v} \) of previous section. We need two more equations to obtain the unique solution for the system (4.4), we impose the end conditions (2.3). Hence by associating Eq. (4.4) with end conditions (2.3), we have the following nonlinear system \((N + 3) \times (N + 3)\):
\[
\begin{align*}
\begin{cases}
  s(t_i) = f(t_i) + \frac{h}{2} \sum_{p=0}^{N-1} \sum_{v=1}^{5} \sum_{i=0}^{6} A_{i,v} k^{(i)}(t_i, \xi_{pv}, s(\xi_{pv})), & i = 0, 1, \ldots, N, \\
  D^j s(t_0) = D^j s(N), & j = 1, 2.
\end{cases}
\end{align*}
\tag{4.5}
\]
By solving the above nonlinear system , we determine the coefficients \( c_i, i = -1, \ldots, N + 1 \) by setting \( c_i \) in Eq. (2.1), we obtain the approximate solution for Eq. (4.1).
4.2. Nonlinear Volterra integral equation

Now we consider nonlinear Volterra integral equation

\[ y(t) = f(t) + \int_a^t k(t, x, y(x)) \, dx, \quad a \leq t \leq b. \]  \hspace{1cm} (4.6)

the solution of Eq. (4.6) can be replaced with cubic B-spline and so we collocate Eq. (4.6) at collocation points \( t_i = a + ih, \ h = \frac{t - a}{N}, \ i = 0, 1, \ldots, N \) then we obtain

\[ s(t_i) = f(t_i) + \int_a^{t_i} k(t_i, x, s(x)) \, dx, \quad i = 1, \ldots, N. \]  \hspace{1cm} (4.7)

By partitioning the interval \([a, t_j]\) to \(N\) equal subintervals we obtain

\[
\begin{aligned}
    s(t_i) &= f(t_i) + \sum_{p=0}^{i-1} \int_{t_p}^{t_{p+1}} k(t_i, x, s(x)) \, dx, \quad i = 1, \ldots, N, \\
    s(t_0) &= f(t_0).
\end{aligned}
\]  \hspace{1cm} (4.8)

To approximate the integral Eq. (4.8), we can use the 5-points Gauss–Turán quadrature formula in the case \(n = 5\) and \(s = 3\), then we have

\[
\begin{aligned}
    s(t_i) &= f(t_i) + \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} A_{l,v} k^{(l)}(t_i, \xi_{pu}, s(\xi_{pu})), \quad i = 1, \ldots, N, \\
    s(t_0) &= f(t_0). \\
\end{aligned}
\]  \hspace{1cm} (4.9)

We need 2 more equations to obtain the unique solution for system (4.9), we impose the end conditions (2.3). Hence by associating system (4.9) with end conditions (2.3), we have the following nonlinear system \((N + 3) \times (N + 3)\):

\[
\begin{aligned}
    s(t_i) &= f(t_i) + \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} A_{l,v} k^{(l)}(t_i, \xi_{pu}, s(\xi_{pu})), \quad i = 1, \ldots, N, \\
    s(t_0) &= f(t_0), \\
    D^j s(t_0) &= D^j s(t_N), \quad j = 1, 2.
\end{aligned}
\]  \hspace{1cm} (4.10)

By solving the above nonlinear system, we can determine the coefficients \(c_i, i = -1, \ldots, N + 1\), by setting \(c_i\) in (2.1), we obtain the approximate solution for Eq. (4.6).

5. Error analysis

We analyze the convergence of the Volterra integral equation. To obtain the error estimation of our approximation, first we recall the following definitions in [35–38].

**Definition 5.1.** The Gauss–Turán quadrature formula with multiple nodes,

\[\int_a^b f(x) \, d\lambda(x) = \sum_{i=0}^{2s} \sum_{v=1}^n A_{l,v} f^{(l)}(\tau_v) + R_{n,2s}(f),\]

is exact for all polynomials of degree at most \(2(s + 1)n - 1\), that is, \(R_{n,2s}(f) = 0\) for all \(f \in P_{2(s+1)n-1}\).

**Definition 5.2.** Let \(S(t)\) be the cubic B-spline interpolate to \(f \in C^4[a, b]\), then for all admissible \(h\), there is a number \(M_j < \infty\), independent of \(h\), such that

\[\|D^j(f - S)\|_2 \leq M_j \|f^{(4)}\|_2 h^{4-j-1/2}, \quad j = 0, \ldots, 3,\]

where \(M_j = \frac{2}{J^j}, \quad j = 0, \ldots, 3,\)

and \(D^j\) is the \(j\)-th derivative and if \(p = 4 - j - 1/2\) is the largest number for which such an inequality holds, then \(p\) is called the order of convergence of the method.
Theorem 5.1. The approximate method (4.9)

\[
\begin{align*}
    s(t_i) &= f(t_i) + \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} A_{l,v} k^{(l)}(t_i, \xi_{pv}, s(\xi_{pv})), & i = 1, \ldots, N, \\
    s(t_0) &= f(t_0),
\end{align*}
\]

(5.1)

for solution of the nonlinear Volterra integral Eq. (4.6) is converge and the error bounded is

\[
\| e_i \| \leq \frac{hL}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{l,v}| |e_{pv}|.
\]

Proof. We know that at \( t_i = a + ih, h = \frac{\xi_{pv}}{N}, \) \( i = 1, \ldots, N, \) the corresponding approximation method for nonlinear Volterra integral Eq. (4.6) is

\[
\begin{align*}
    s(t_i) &= f(t_i) + \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} A_{l,v} k^{(l)}(t_i, \xi_{pv}, s(\xi_{pv})), & i = 1, \ldots, N, \\
    s(t_0) &= f(t_0),
\end{align*}
\]

(5.2)

by discretizing (4.6) and approximating the integrand by the 5-points Gauss–Turán rules, we obtain

\[
\begin{align*}
    y(t_i) &= f(t_i) + \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} A_{l,v} k^{(l)}(t_i, \xi_{pv}, y(\xi_{pv})), & i = 1, \ldots, N.
\end{align*}
\]

(5.3)

and since error in approximating by the 5-points Gauss–Turan rule for polynomials of degree at most 2\((s + 1)n – 1 = 39\) is zero, then in the case of \( n = 5, s = 3, R_{5,25} = R_{5,6} = 0, \) and we are ignored of writing \( R_{5,6} \) in Eq. (5.3). By subtracting (5.3) from (5.2) and using interpolation conditions of cubic B-spline, we get

\[
\begin{align*}
    s(t_i) - y(t_i) &= \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} A_{l,v} [k^{(l)}(t_i, \xi_{pv}, s(\xi_{pv})) - k^{(l)}(t_i, \xi_{pv}, y(\xi_{pv}))],
\end{align*}
\]

so

\[
|s(t_i) - y(t_i)| \leq \frac{h}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{l,v}| |k^{(l)}(t_i, \xi_{pv}, s(\xi_{pv})) - k^{(l)}(t_i, \xi_{pv}, y(\xi_{pv}))|.
\]

We suppose that \( s(t_i) = s_i, \) \( y(t_i) = y_i, \) \( i = 1, \ldots, N \) and kernels \( k^{(l)}, l = 0, \ldots, 6 \) satisfy a Lipschitz condition in its third argument of the form

\[
|k^{(l)}(t, \xi, s) - k^{(l)}(t, \xi, y)| \leq L|s - y|,
\]

where \( L \) is independent of \( l, t, \xi, s \) and \( y. \) We get

\[
|s_i - y_i| \leq \frac{hL}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{l,v}| |s_{pv} - y_{pv}|.
\]

then we have

\[
|e_i| = \frac{hL}{2} \sum_{p=0}^{i-1} \sum_{v=1}^{5} \sum_{l=0}^{6} |A_{l,v}| |e_{pv}|,
\]

where \( e_i = s_i - y_i, \) \( i = 1, \ldots, N, e_{pv} = s_{pv} - y_{pv}. \) When \( h \to 0 \) then the above term is zero and also this term is due to interpolating \( y(t) \) by cubic B-spline. We get for a fixed \( i, \)

\[
|e_i| \to 0 \quad \text{as} \quad h \to 0.
\]

We may mention that the truncation error due to replacing the solution \( y(t) \) by cubic spline \( s(t) \) is \( O(h^4) \) and the truncation error due to approximation of the integrand by 5-points Gauss–Turan quadrature is \( O(h^{10}) \). So that the truncation error of the our approach can be given as the \( min(40, 4) \). Then we can conclude that the totally convergence rate of the our approach is of the order four. \( \square \)

6. Numerical examples

In order to test the applicability of the presented method, we consider five test examples linear and nonlinear Volterra and Fredholm integral equations with the end conditions, these examples have been solved with various values of \( N \). The absolute errors at the particular points are given to compare our solutions with the solutions obtained by \([13,15,21,22,26,29,31]\). Computed
Table 2
The error \( \|E\| \) in solution of Example 6.1 at particular points.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Our method ( N = 10 )</th>
<th>Our method ( N = 20 )</th>
<th>Our method ( N = 40 )</th>
<th>Method in [31]</th>
<th>Method in [26]</th>
</tr>
</thead>
<tbody>
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<td>2.91E–09</td>
<td>4.38E–04</td>
<td>1.36E–06</td>
</tr>
</tbody>
</table>

Table 3
The error \( \|E\| \) in solution of Example 6.2 at particular points.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Our method ( N = 10 )</th>
<th>Our method ( N = 20 )</th>
<th>Our method ( N = 40 )</th>
<th>Method in [29]</th>
</tr>
</thead>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>2.10E–07</td>
<td>2.54E–08</td>
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<td>9.25E–04</td>
</tr>
</tbody>
</table>

results are tabulated in Tables. The tables verified that our approach is more accurate. Programs are performed by Mathematica version 8 for all the numerical examples.

**Example 6.1.** Consider the following linear Volterra integral equation with exact solution \( y(t) = 1 – \sinh t \).

\[
y(t) = 1 - t - \frac{t^2}{2} + \int_0^t (t - x)y(x) \, dx, \quad t \in [0, 1].
\]

This example has been solved by our method with \( N = 10, 20, 40 \), the absolute errors at the particular grid points are tabulated in Table 2 and compared with the absolute errors obtained by [26,31]. This table verified that our results are considerable accurate in comparison.

**Example 6.2.** Consider the following nonlinear Volterra integral equation with exact solution \( y(t) = \cos t \).

\[
y(t) = 1 + \sin^2 t - \int_0^t 3 \sin (t - x)y^2(x) \, dx, \quad t \in [0, 1].
\]

The approximate solution is calculated for different values of \( N = 10, 20, 40 \), the absolute errors at the particular grid points are tabulated in Table 3 and compared with the absolute errors obtained by [29]. This table verified that our results are considerable accurate in comparison.

**Example 6.3.** Consider the following linear Fredholm integral equation with exact solution \( y(t) = e^{2t} \).

\[
y(t) = e^{2t} + \frac{1}{3} \int_0^t e^{2t-\frac{x}{2}}y(x) \, dx, \quad t \in [0, 1].
\]

We solved Example 6.3 for different values of \( N = 10, 20, 30 \), and compared our results with results in [13,22], the absolute errors at the particular grid points are tabulated in Table 4 which show that our results are more accurate in comparison with [13,22].

**Example 6.4.** Consider the following linear Volterra integral equation with exact solution \( y(t) = e^{-t^2} \).

\[
y(t) = e^{-t^2} + \frac{t}{2} (1 - e^{-t^2}) - \int_0^t xty(x) \, dx, \quad t \in [0, 1].
\]

The approximate solution is calculated for different values of \( N = 8, 18, 28, 38 \), the absolute errors at the particular grid points are tabulated in Table 5 and compared with the maximum absolute errors obtained by [21]. This table verified that our results are more accurate in comparison.
7. Conclusion

We solved this example with Example 6.5. Consider the following nonlinear Fredholm integral equation with exact solution

\[ y(t) = \sin(\pi t) + \frac{1}{2}(20 - \sqrt{39}) \cos(\pi t), \]

\[ y(t) = \sin(\pi t) + \frac{1}{2} \int_{0}^{1} \cos(\pi x) y^3(x) \, dx, \quad t \in [0, 1]. \]

We solved this example with \( N = 5, 10, 20 \), the absolute errors at the particular grid points are tabulated in Table 6 and compared with the absolute errors obtained by [15]. This table verified that our results are more accurate in comparison with [15].

7. Conclusion

In the present work, a new approach has been developed for solving linear and nonlinear Fredholm and Volterra integral equations by using the 5-points Gauss–Turán quadrature formula with respect to the Legendre weight function and collocating by cubic B-spline. These equations are converted to a system of linear or nonlinear algebraic equations. By solving we can determine the unknown coefficients which appear in representation of the spline base functions. We verified that the presented method can be applied with large number of \( N \). The presented method is stable because when \( N \) increases the error in the solution also decreases. This method tested on five examples. Our method by the suggested method is compared with the methods in [13,15,21,22,26,29,31]. Our results verified the accurate nature of our approach.

References


