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Stochastic Sparse-grid Collocation Algorithm (SSCA) for Periodic Steady-State Analysis of Nonlinear System with Process Variations

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Outline

- Introduction
- Problem Definition
- Stochastic Sparse-grid Collocation Algorithm (**SSCA**)
  - Stochastic Collocation Algorithm
  - Sparse-grid Technique
- Numerical Results
- Conclusion

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Introduction

- Impact of process variations on RF/Mixed-Signal system → Be amplified exponentially
  - Variation aware steady-state simulation is quite demanding
- Taylor series Methods
  - CORE/PMOR for linear system and TPWL-PMOR for nonlinear system
  - Limitations:
    - Same expansion order → Random process parameters and frequency parameter have different range
    - No consideration of randomness of process variations → No convergent expansion and can only handle small variations

Introduction

- Stochastic Spectral Algorithms
  - Stochastic Galerkin Algorithm for Linear system
    - **A much complicated coupled system for nonlinear circuit → Unacceptable computational complexity**
  - Stochastic Collocation Algorithm for Delay Modeling
    - **Selection of collocation points**
      - **Direct Tensor Product Scheme**
        - **Complexity increases exponentially with the dimension of random variable space**
      - **Efficient Collocation Method (ECM) → Heuristic method**
        - **"Rank deficient problem"**
        - **"Runge Phenomenon"**

Introduction

- Stochastic Spectral Algorithms
  - Stochastic Galerkin Algorithm for Linear system
    - **A much complicated coupled system for nonlinear circuit → Unacceptable computational complexity**
  - Stochastic Collocation Algorithm for Delay Modeling
    - **Selection of collocation points**

> - **Stochastic Collocation Method** → Steady-State analysis of nonlinear system with process variations
> - **Sparse-grid technique** → decrease the computational complexity while maintain the acceptable accuracy
>
> **Stochastic Sparse-grid Collocation Algorithm (SSCA)**

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Problem Definition

Steady-State Analysis

Deterministic Nonlinear System

$$\begin{cases} \dfrac{dx(t)}{dt} = f\left(x,t,u(t)\right) \\ x(t+T) = x(t) \end{cases}$$

$x(t) = \left[x_1(t),\cdots,x_P(t)\right]^T \in R^P$ → **state variables**

$f(\cdot) = \left[f_1(\cdot),\cdots,f_P(\cdot)\right]^T$ → **nonlinear vector function**

$u(t)$ → **input signal with frequency** $f_0 = \frac{1}{T}$

$$x(t) = \sum_{l=1}^{L} X_l \cdot \psi_l(t)$$

$\psi_l(t)$
- Fourier function for Harmonic Balance Method
- Wavelet basis for Wavelet Balance Method

Nonlinear System with Process Variations

$$\begin{cases} \dfrac{dx(t,\vec{\xi})}{dt} = f\left(x,t,\vec{\xi},u(t)\right) \\ x(t+T,\vec{\xi}) = x(t,\vec{\xi}) \end{cases}$$

$\vec{\xi} = [\xi_1,\xi_2,\cdots,\xi_M]^T$: → a series of interested independent Gaussian random variables in M-dimensional random space

$p(\vec{\xi})$ → probability density function

Stochastic Sparse-grid Collocation Algorithm

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Stochastic Collocation Algorithm

$$\begin{cases} \dfrac{dx\left(t,\vec{\xi}\right)}{dt} = f\left(x,t,\vec{\xi},u(t)\right) \\ x\left(t+T,\vec{\xi}\right) = x\left(t,\vec{\xi}\right) \end{cases}$$

$\vec{\xi}$: Gaussian random variables in M-dimensional random space with pdf $p(\vec{\xi})$

Homogeneous Chaos Expansion

$$x_d\left(t,\vec{\xi}\right) = \sum_{l=1}^{L} X_l\left(\vec{\xi}\right)\psi_l(t)$$

$$= \sum_{l=1}^{L}\sum_{n=1}^{N} c_{nl}H_n\left(\vec{\xi}\right)\psi_l(t)$$

$$X_l(\vec{\xi}) = \sum_{n=1}^{N} c_{nl}H_n(\vec{\xi})$$

$H_n\left(\vec{\xi}\right)$ **Hermiter polynomials**

- **Inner product definition:** $\langle f(\vec{\xi}), g(\vec{\xi})\rangle = \int_{\vec{\xi}} f(\vec{\xi})g(\vec{\xi})p(\vec{\xi})d\vec{\xi}$
- **Orthogonal bases**
- **Exponential convergence Rate**

Probability distribution functior

Any second order Gaussian process can be approximated by a series of Hermite polynomials in the norm sense

Stochastic Collocation Algorithm

$$\begin{cases} \dfrac{dx(t,\vec{\xi})}{dt} = f\left(x,t,\vec{\xi},u(t)\right) \\ x\left(t+T,\vec{\xi}\right) = x\left(t,\vec{\xi}\right) \end{cases}$$

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Homogeneous Chaos Expansion

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$$X_l(\vec{\xi}) = \sum_{n=1}^{N} c_{nl} H_n(\vec{\xi})$$

$$\left\langle \boldsymbol{E}_n\left(\vec{\xi}\right), \delta\left(\vec{\xi}-\vec{\xi}_j\right)\right\rangle = 0 \qquad \boldsymbol{j} = 1,2,\cdots,\boldsymbol{J}$$

Residue → $E_n(\vec{\xi}) = \dfrac{dx_d(t,\vec{\xi})}{dt} - f(x_d(t,\vec{\xi}),t,\vec{\xi},u(t))$

The original function is exactly satisfied at some collocation points in process variable space

Stochastic Collocation Algorithm

- Step1. Select a series of collocation points
- Step2. Calculation of steady-state response at each collocation point
- Step3. Computation of Steady-state behavior for nonlinear system with process variations

Stochastic Collocation Algorithm

- Step1. Select a series of collocation points
  - Direct Tensor Product Scheme
  - Efficient Collocation Method (ECM)

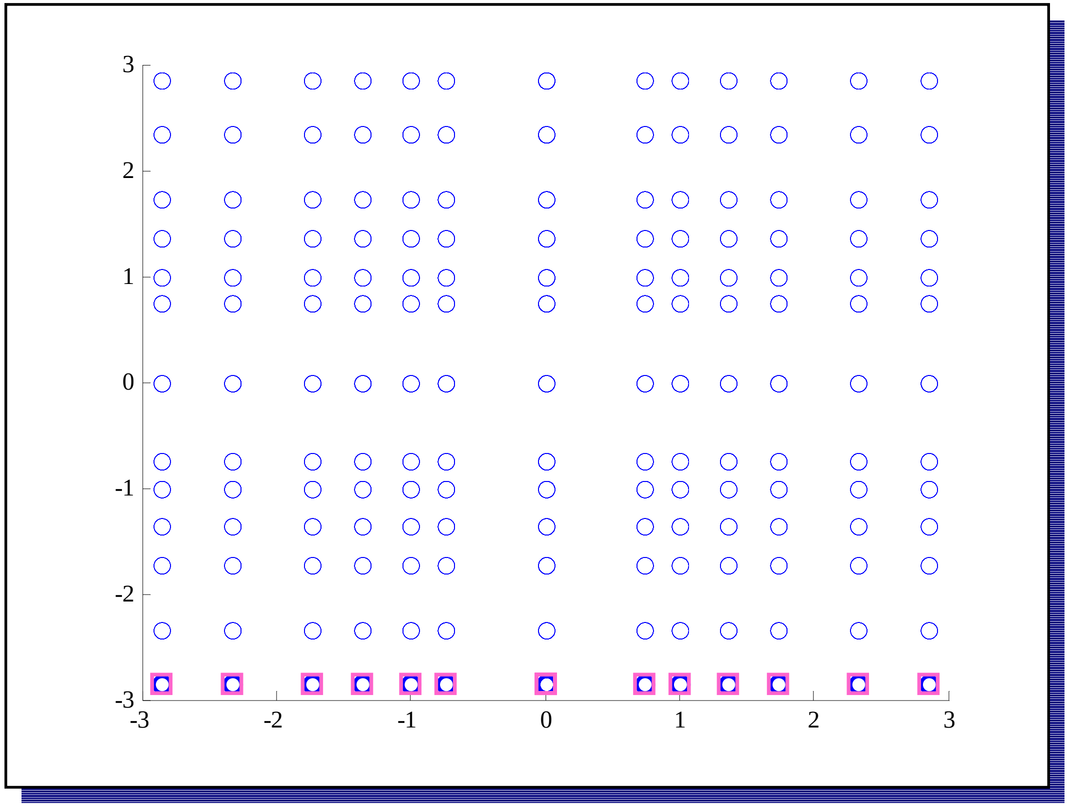
Stochastic Collocation Algorithm

- Step1. Select a series of collocation points
  - Direct Tensor Product Scheme:

□: 1-dimension
o: 2-dimension

Q-order
M-dimension
Points Number:

$$P_t = (Q+1)^M$$



Stochastic Collocation Algorithm

- Step1. Select a series of collocation points
  - Direct Tensor Product Scheme:
  - Efficient Collocation Method (ECM):
    - A simple heuristic approach
    - The number of selected collocation points is the same as the number of the applied Hermite polynomial basis

Stochastic Collocation Algorithm

- Step1. Select a series of collocation points
- Step2. Calculation of steady-state response at each collocation point

$$\frac{dx\left(t,\vec{\xi}_k\right)}{dt} = f\left(x\left(t,\vec{\xi}_k\right),t,\vec{\xi}_k,u(t)\right)$$

A truncated series of Fourier basis (HBM) or wavelet basis (WBH)

$$x_d\left(t,\vec{\xi}_k\right) = \sum_{l=1}^{L} X_l\left(\vec{\xi}_k\right)\psi_l(t)$$

Stochastic Collocation Algorithm

- Step3. Computation of Steady-state behavior for nonlinear system with process variations
  - Direct Tensor Product Scheme
    - Weighted Least Square Method

$$x_d\left(t,\vec{\xi}\right)=\sum_{l=1}^{L}X_l\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$=\sum_{l=1}^{L}\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$X_l\left(\vec{\xi}\right)=\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)$$

$\Rightarrow$

$$X_{1l}=\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}_1\right)$$

$$X_{2l}=\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}_2\right)$$

$$\cdots\cdots\cdots$$

$$X_{P_tl}=\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}_{P_t}\right)$$

Stochastic Collocation Algorithm

Step2 : Computation of Steady-state behavior for

- **The number of collocation points increases exponentially.**
- **The CPU cost of step2 and step3 will increase exponentially.**

  - Weighted Least Square Method

$$x_d\left(t,\vec{\xi}\right)=\sum_{l=1}^{L}X_l\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$=\sum_{l=1}^{L}\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)\psi_l\left(t\right)$$

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Stochastic Collocation Algorithm

- Step3. Computation of Steady-state behavior for nonlinear system with process variations
  - Efficient Collocation Method (ECM)

$$x_d\left(t,\vec{\xi}\right)=\sum_{l=1}^{L}X_l\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$=\sum_{l=1}^{L}\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$X_l\left(\vec{\xi}\right)=\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)$$

$$\Rightarrow$$

$$H\times C=X$$

$$H=\begin{bmatrix} H_1\left(\vec{\xi}_1\right) & \cdots & H_N\left(\vec{\xi}_1\right) \\ \vdots & \vdots & \vdots \\ H_1\left(\vec{\xi}_N\right) & \cdots & H_N\left(\vec{\xi}_N\right) \end{bmatrix}$$

$$C=\left[c_{1l} \quad \cdots \quad c_{Nl}\right]^T$$

$$X=\left[X_{1l} \quad \cdots \quad X_{Nl}\right]^T$$

Stochastic Collocation Algorithm

Rank Deficient Problem: the rank of matrix H may be smaller than its size, and ECM may fail to converge for high order expansion

$$x_d\left(t,\vec{\xi}\right)=\sum_{l=1}^{L}X_l\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$=\sum_{l=1}^{L}\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)\psi_l\left(t\right)$$

$$X_l\left(\vec{\xi}\right)=\sum_{n=1}^{N}c_{nl}H_n\left(\vec{\xi}\right)$$

$$\Rightarrow$$

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$$C=\left[c_{1l} \quad \cdots \quad c_{Nl}\right]^T$$

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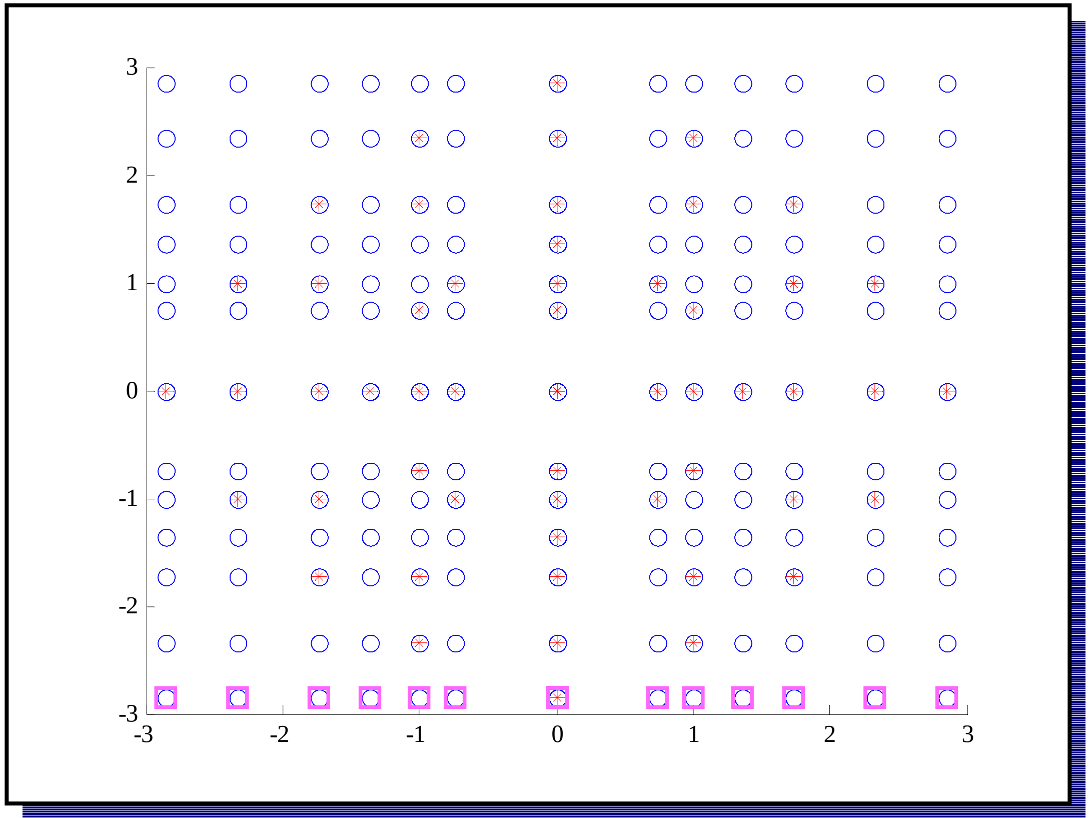
Sparse Grid Technique

*: Sparse-grid technique
o: Direct Tensor product
□: **1-dimentional**



Q-order M-dimensional space

$$P_s \sim \frac{2^Q}{Q!} M^Q \sim 2^Q N$$

$$N = \binom{M+Q}{M}$$

Sparse Grid Technique

- Theorem: the following equation is exact for all M-variables polynomials of order at most 2Q+1 [1]

$$\int_{\vec{\xi}^s} f\left(\vec{\xi}\right) p\left(\vec{\xi}\right) d\vec{\xi} \approx \sum_{i=1}^{P_s} w_i^s f\left(\vec{\xi}_i^{\,s}\right)$$

Collocation points generated by Sparse-grid Technique

- Similar procedure with Direct Tensor Scheme
- Compared with Direct Tensor Product Scheme
  - Much smaller complexity
  - Similar accuracy
- Compared with Efficient Collocation Method
  - Much higher accuracy
  - No “Runge Phenomenon” and no “Rank deficient problem”

[1] D.B.Xiu and J.S.Hesthaven, High order collocation method for differential equations with random inputs. SIAM Journal of Sci. Comput., 27(3):1118-1139, 2005

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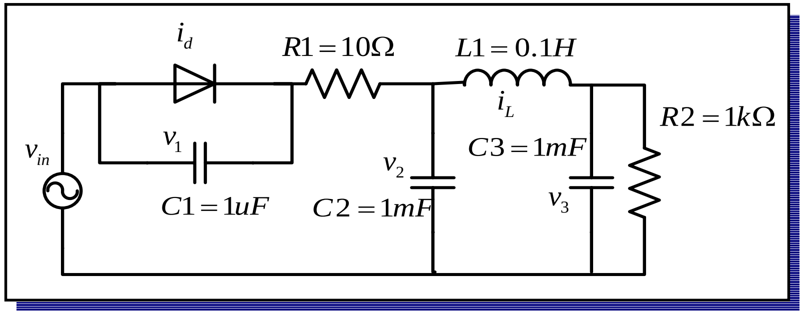
Numerical Results

- A DC power supply circuit



- The input is a sinusoidal voltage with frequency 100Hz and amplitude 10V
- Monte Carlo analysis with 10^4 sampling points by HSPICE is regarded as the standard result.

Any small variations of each circuit parameter will result in exponential variations for the state variable

System Equations

$$
\begin{cases}
i_d = i_s\left(e^{v_1/v_{th}} - 1\right) \\
C_1 \dfrac{dv_1}{dt} = \dfrac{v_{in} - v_1 - v_2}{R_1} - i_d \\
C_2 \dfrac{dv_2}{dt} = \dfrac{v_{in} - v_1 - v_2}{R_1} - i_L \\
C_3 \dfrac{dv_3}{dt} = i_L - \dfrac{v_3}{R_2} \\
L_1 \dfrac{di_L}{dt} = v_2 - v_3
\end{cases}
$$

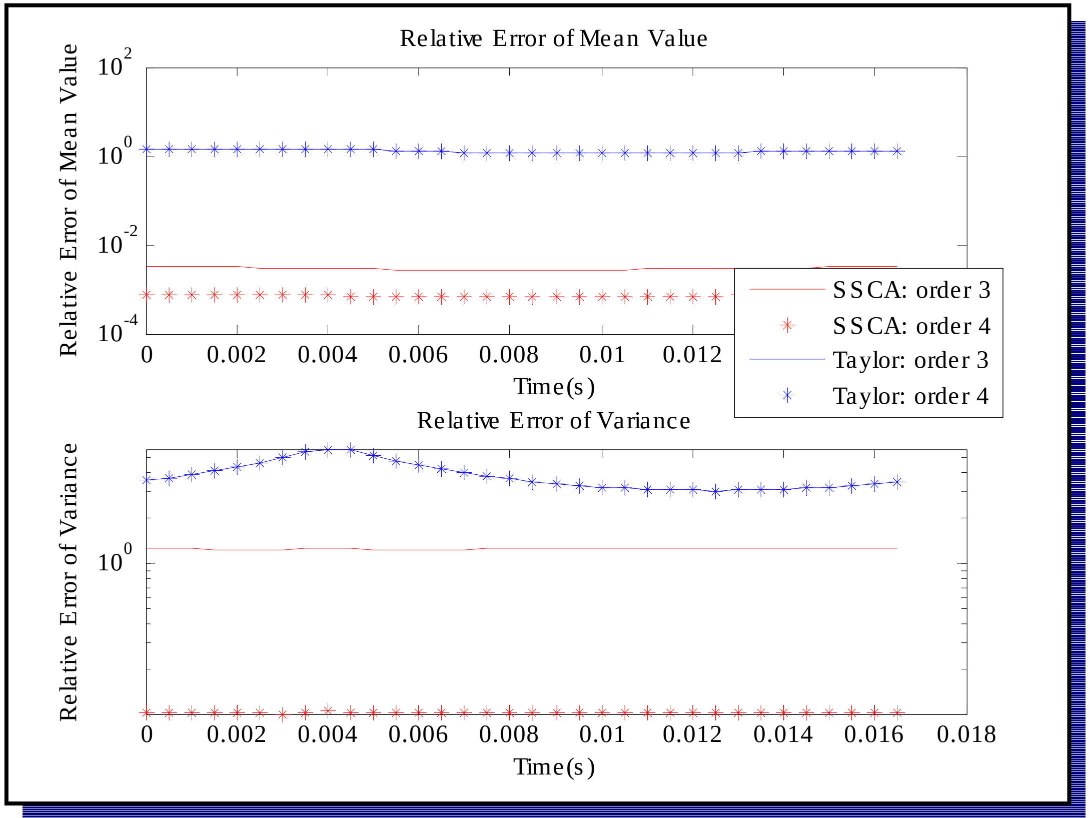
Testing signal

Numerical Results

- Comparison with Taylor Expansion

R1 and R2 : Random parameters with Gaussian variations <= 7%

Very high convergence rate of Homogeneous Chaos approximation for stochastic process

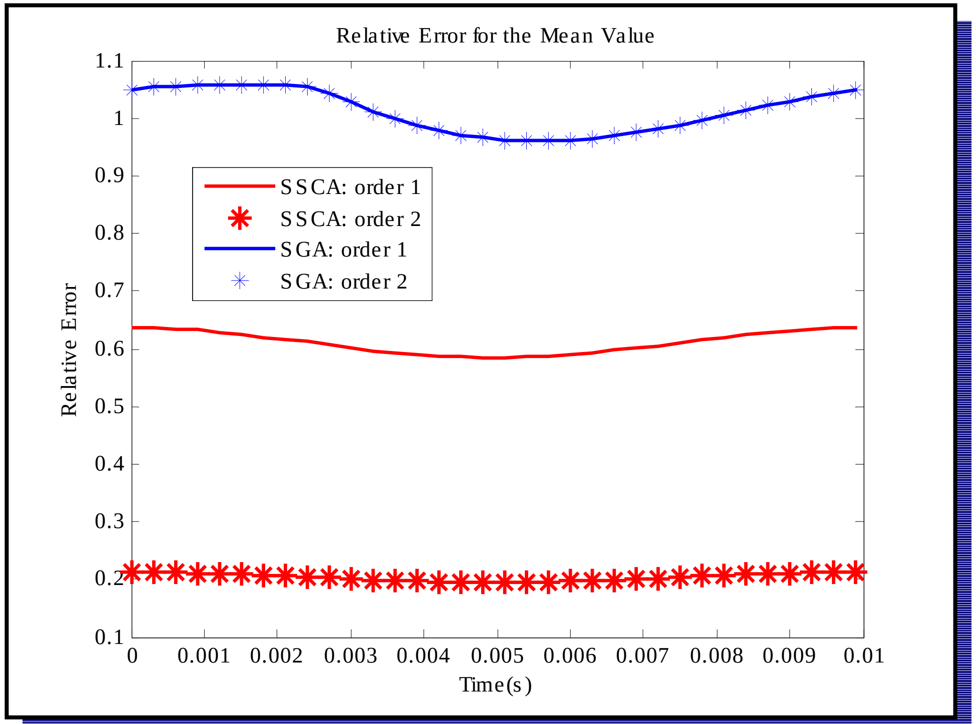


Numerical Results

- Comparison with Stochastic Galerkin Algorithm

i_s, v_{th}, R_1, R_2 : random variables with Gaussian variations <= 7%

Accuracy of SGA with Sparse-Grid Technique and SSCA are similar



Numerical Results

Comparison with ECM

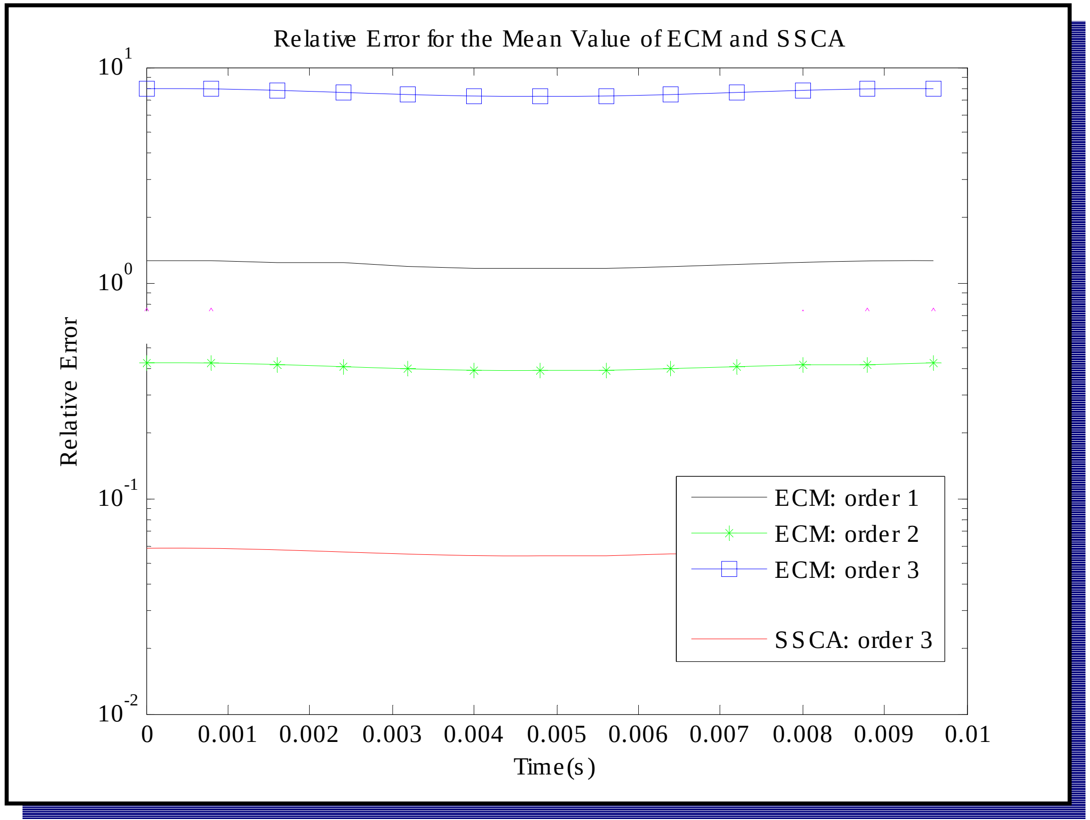
i_s, v_{th}, R_1, R_2 : random variables with Gaussian variations <= 7%

- SSCA
  much higher accuracy and convergence rate than ECM

- ECM
  ✓Rank deficient problem : order is higher than 3

  ✓Runge phenomenon: the relative error for order 2 is smaller than those for order 1 and order 3

1
2
3
4



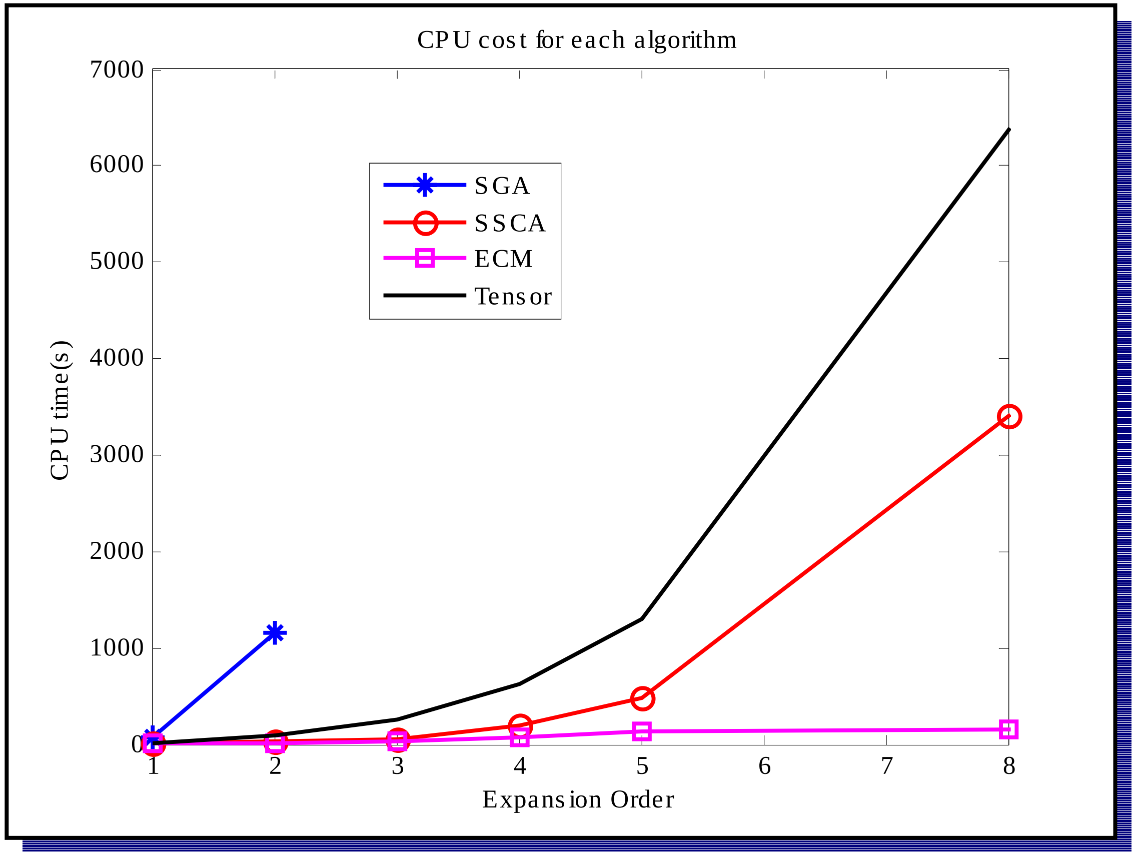
Numerical Results

Complexity

- Stochastic Galerkin Algorithm
  Out of memory with the order higher than 3

- SCA with Direct Tensor Scheme
  CPU cost increases exponentially

- SSCA
  Acceptable complexity



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Conclusion for SSCA

- Compared with the Taylor series methods
  - Homogeneous chaos expansion → random process variations
    - Exponential Convergence Rate
  - Fourier basis or Wavelet basis → time domain
- Compared with Stochastic Galerkin method
  - Much smaller complexity with similar accuracy
- Compared with Tensor Product Scheme
  - Much fewer collocation points
- Compared with Efficient Collocation Method
  - High convergence rate and high accuracy
  - No “Rank deficient problem” and no “Runge phenomenon”

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Thank you