Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. 01, 1995, pp. 27–29.
Submitted: July 19, 1995. Published: September 18, 1995.

A LOGICAL ALTERNATIVE TO THE EXISTING POSITIONAL NUMBER SYSTEM

Robert R. Forslund

Abstract. This article introduces an alternative positional number system. The advantages of this alternative system over the existing one are discussed, and an illustration of the use of the system to re-interpret apparent errors in ancient archaeological documents is presented.

(1991) A.M.S. (MOS) Subject Classification Codes. 11, 01.

Key Words and Phrases. positional number system, archaeology, zero, placeholder.

Introduction

To simplify this article, the history of positional number systems will not be discussed, and interested readers can find this information in the references listed. In addition, the discussion is restricted to the set of positive integers; however, the extension to decimal and negative values, if necessary, is easily accomplished.

The existing positive integer number system in a selected base (b), is represented by the following mathematical notation :

$$N = \sum (a_j * b^j) \text{ where } 0 \leq a_j \leq (b-1),\ b > 1, \text{ and the sum is over } j$$

with $0 \leq j \leq (n-1)$, and a_j are the n digits of the positive integer N in base b.

For example:

$$10203 = 1 * 10^4 + 0 * 10^3 + 2 * 10^2 + 0 * 10^1 + 3$$

Traditionally we have selected base : $b = 10$, as in the example, and commonly use other bases such as binary ($b = 2$), octal ($b = 8$) and hexadecimal ($b = 16$). Ancient civilizations introduced other bases such as 60 used by the Babylonians [1], and 18 and 20 used by the Mayans [1].

The alternative system

There is a problem with the existing number system. Base 1 is illogical since the digits are equal to zero, and therefore we must select base, $b > 1$ when defining the system. In addition, the digit integers starting at one are concrete concepts, while the concept of zero (nothingness) is abstract. Therefore, I sought a system that did not require the use of zero as a placeholder. As you will see shortly, there is such a system, and the concept of a zero placeholder adds unnecessary complexity.

As a result, I reasoned as follows. First of all, and most importantly, why not use digits from 1 to b rather than 0 to $(b-1)$? In base 10, this simply means introducing a 10th symbol for the digit "ten". Any

Ministry of Natural Resources Centre for Northern Forest Ecosystem Research c/o Lakehead University, 955 Oliver Road, Thunder Bay, Ontario, Canada P7B 5E1
E-mail Address: mjohnsto@sky.Lakehead.ca

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

symbol will do, however the symbol "**A**" borrowed from the hexadecimal system provides some familiarity. This is the only change to the notation used in the introduction for N : viz : $1 \leq a_j \leq b$ and $b \geq 1$. This alternative system has no need for a placeholder (zero) as I illustrate below. I will use bolded symbols for digits in the alternative system to distinguish between the two systems.

In the alternative system (base **A**), the number 10 is now represented by the digit **A**. The number 20 now becomes : $\mathbf{1A} = \mathbf{1} * \mathbf{A} + \mathbf{A}$, and 90 is **8A**. The number 100 now becomes : $\mathbf{9A} = \mathbf{9} * \mathbf{A} + \mathbf{A}$, and $1000 = \mathbf{99A} = \mathbf{9} * \mathbf{A^2} + \mathbf{9} * \mathbf{A} + \mathbf{A} = 900 + 90 + 10$, and so on. Notice that, with the exception of integers that have digits equal to zero in the existing system, all other numbers remain the same as in the alternative system as long as the digits lie between 1 and $(b-1)$ inclusive. Therefore the smaller the base, the more the differences will be between the two systems. Base 2 for example will remain the same as the existing system only when all digits are 1. An example in base 2 is : $21 = 10101$ in the existing system, and **1221** in the alternative system. In comparison, the number 31 in both systems remains at **11111** in base 2.

Advantages of the alternative system

Base 1 is now permissible and completely logical and is simply a tally system. For example, $7 =$ **1111111** (base 1) in the alternative system. Secondly, for number theorists, the re-arrangement of digits in the existing system causes digits to be lost. For example the reversal of 12300 in the existing system is 321 and two zero digits are "lost". In the alternative system this does not occur. For example, $12300 =$ **1229A**, which is **A9221** upon reversal. As this simple example illustrates, this alternative system lends itself to further research in number theory. In addition, in the existing system, powers of the base are identified by the number of zeros in the number - for example $10000 = 10^4$. Powers are similar in the alternative system and are identified as in this example by $\mathbf{999A} = \mathbf{A^4}$. More generally, in base b, powers $= \ldots(\mathbf{b-1})(\mathbf{b-1})\ldots(\mathbf{b-1})\mathbf{b}$. Notice that fewer digits are often needed to represent the numbers - this too is an advantage, and one of the main reasons positional notation was first introduced. Also, in base 10 for example, instead of having 9 numbers with one digit, 90 numbers with 2 digits, and so on, the alternative system now has the first 10 numbers having a single digit, the next 100 numbers having 2 digits, and so on. This too is simpler since the number of integers with n digits in base b, is b^n rather than $b^n - b^{(n-1)}$ in the existing system.

Application of the alternative system to archaeology

It appears that this alternative system would have been a slightly more logical first step in the development of a number system in ancient times. One possible use of this system might be in the re-interpretation of ancient calculations found in archaeological documents. History tends to identify the introduction of the zero placeholder into the number system, as an advancement in mathematics. However, the alternative system, with its inherent simplicity, illustrates that the introduction of zero may have unnecessarily complicated the use of integers, and numbers in general.

Perhaps by re-interpreting ancient mathematical documents using this alternative system, mathematical archaeologists will reveal that some of the ancient calculations, previously thought to be in error, really were correct. I will illustrate using the commonly used digit symbols in base ten, realizing that ancient documents used other symbols for digits. The ancients may have used 10 symbols (if base 10 was used); however, the two symbols "10" may really have meant **1A** - which is twenty (not ten). In other words, it may be that the zero symbol represented the digit ten - not zero. To detect such discrepancies, ancient documents must contain calculations such as addition. For example : $10 + 11 = 31$ - this appears to be incorrect. However, if the symbol "0" actually represents the digit ten (**A**) and not zero, then the equation is correct and equivalent to $\mathbf{1A} + \mathbf{11} = \mathbf{31} = \mathbf{20} + \mathbf{11}$ in the existing system. Another example : $101 + 9 = 100$. Once again, this appears incorrect, however, if "0" symbol is re-interpreted as "**A**", then this is: $\mathbf{1A1} + \mathbf{9} = \mathbf{1AA}$, which is $201 + 9 = 210$ in the existing system. In base 10, these apparent errors would only exist when the symbol for ten (and interpreted as zero) occurred in the digits of numbers. Larger bases such as 60 used by the Mesopotamians [1], would be even more void of such "errors", and therefore it would be harder to determine if the alternative number system was used. I leave it to archaeologists specializing in the interpretation of these ancient documents to examine this usage of the alternative positional number system.

In summary, the alternative system appears to be a slightly more logical system relative to the existing one. As illustrated, it has immediate application to the field of archaeology. Readers should have little problem finding other fields for the application of this system especially in the theory of numbers. If this article motivates others to explore the possibilities of this alternative system, then it has succeeded.

References

1. Boyer, C.B. and Merzbach, U.T. : A History of Mathematics (Second Edition), John Wiley & Sons Inc., New York, 1991.
2. Hellemans, A. and Bunch, B. : The Timetables of Science - A Chronology of the Most Important People and Events in the History of Science, Simon & Schuster Inc., New York, 1991.
3. Swetz, F.J. : From Five Fingers to Infinity, Open Court, Chicago and La Salle Illinois, 1995.

Acknowledgements

I wish to thank Dr. Mark Johnston and Mr. Luke Dalla Bona at the Centre for Northern Forest Ecosystem Research for providing me with access to the use of electronic communication via the Internet and for their encouragement and comments regarding this paper.