Efficiency of Competitive Search under Asymmetric Information*

Veronica Guerrieri

University of Chicago

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Abstract

In this paper, I study the efficiency properties of competitive search equilibria in economies with informational asymmetries. Employers and workers are both risk-neutral and ex-ante homogeneous. I characterize an equilibrium where employers post contracts and workers direct their search towards them. When a match is formed, the disutility of labor is drawn randomly and observed privately by the worker. An employment contract is an incentive-compatible mechanism that satisfies a participation constraint on the worker’s side. I first show that in a static setting the competitive search equilibrium is constrained efficient, that is, it cannot be Pareto improved by a Social Planner subject to the same informational and participation constraints faced by the decentralized economy. I then show that in a dynamic setting, on the contrary, the equilibrium can be constrained inefficient. The crucial difference between the static and the dynamic environment is that the worker’s outside option is exogenously given in the former, while in the latter it is endogenously determined as the equilibrium continuation utility of unemployed workers. Inefficiency arises because the worker’s outside option affects the ex-ante cost of information revelation, generating a novel externality which is not internalized by competitive search.

1 Introduction

Whether decentralized labor markets achieve the efficient allocation of resources is a central economic question. Back in 1968, Friedman highlights how, in a Walrasian world, a market economy

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cannot be kept away from the unemployment level that "would be ground out by the Walrasian system of general equilibrium equations, provided there is imbedded in them the actual structural characteristics of the labor and commodity markets." In labor markets, trade occurs bilaterally and is typically costly, due to physical distance, coordination, time-consuming search, and informational imperfections. This imposes a natural departure from the Walrasian paradigm, but not necessarily from the property of efficiency. Does the price mechanism still achieve the socially optimal allocation of resources? To what extent the equilibrium level of unemployment is efficient? In this paper I address this question, once I model explicitly the informational asymmetry that seem to be a crucial element of employment relationships. In this context, I show that there is a role for the government to welfare improve upon the market equilibrium.

Search theory typically models labor market imperfections under the heading of matching frictions, by assuming an aggregate matching technology.\(^1\) The conventional model, built on Diamond (1982), Mortensen (1982a, 1982b) and Pissarides (1984, 1985), combines random matching with a wage determination process based on Nash bargaining. In this context, the equilibrium level of unemployment is generically inefficient. Decentralized markets do not internalize the search externality generated by the matching frictions.\(^2\) However, going back to the Walrasian spirit, a new generation of search models, Shimer (1996), Moen (1997) and Acemoglu and Shimer (1999a), introduces a novel notion of competition in environments with trading frictions, referred to as competitive search. In competitive search models firms post wages and workers direct their search towards them. In this environment, decentralized markets internalize the search externality and the resulting equilibrium is efficient. The efficiency property of competitive search has been proven robust in several contexts and sheds light on the power of the price mechanism to induce firms to open the optimal quantity of vacancies.

In this paper, I introduce asymmetric information on the top of the standard matching frictions. In particular, I assume that the surplus produced by a worker-firm match depends on idiosyncratic features that are private information of the worker. The problem for a firm is not only to meet a worker, but also to find out the profitability of their match. Employment contracts are designed optimally in order to extract this information and to induce workers to participate in the productive relationship. My main result is that, in this environment, the equilibrium is generically constrained inefficient. This result is particularly interesting given that my model retains the Walrasian spirit of competitive search and allows for general employment contracts under asymmetric information. This suggests that the inefficiency is not trivially driven neither by the presence of search externality or by suboptimality in private contracting.

I show that, in my model, a new type of externality arises, generated by the interaction of asymmetric information and firms’ competition driven by the matching environment. The ability

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\(^1\)See Rogerson, Shimer and Wright (2005) for a broad survey of search theoretic models of the labor markets.

of a firm to extract information depends on the worker’s outside option. The outside option, in turn, is determined by the contracts offered by other firms in the future, who do not internalize their effect on incentives of contracts today. Because of that, the equilibrium fails to be generically constrained efficient. Most macroeconomic models study the design of a grand mechanism at the social level and ask whether the resulting allocation can be decentralized. Prescott and Townsend (1984) show that, in the presence of private information, competitive markets are constrained efficient, once one allows for general private contracts. In my paper, the externality is driven by the competition among posted mechanisms. The matching environment is a natural way to create competition among exclusive contracts. In this sense, the nature of this externality is similar to the one explored by Golosov and Tsyvinski (2006) in the context of social insurance programs.

I consider an economy where employers and workers are both risk-neutral and \textit{ex-ante} homogeneous. Employers post contracts and workers direct their search towards them. When a match is formed, the disutility of labor is drawn randomly and observed privately by the worker. An employment contract is an incentive compatible mechanism that satisfies a participation constraint on the worker’s side. This can be interpreted as the result of lack of commitment. A worker cannot be forced to work, he can always quit and join the ranks of the unemployed.\footnote{This corresponds to the typical \textit{at will} employment contracts enforced in the United States.}

In the first part of the analysis, I characterize the competitive search equilibrium. Not surprisingly, the equilibrium does not achieve the ex-post efficient allocation of resources. A trade-off emerges between the two margins of job creation: efficient creation at the hiring stage has to be sacrificed in order to induce vacancy creation \textit{ex-ante}. Then, I turn to the interesting notion of efficiency: \textit{constrained efficiency}. I define a social planner who faces \textit{the same frictions} of the competitive economy. The social planner controls the matching process by deciding how many vacancies to post at the beginning of each period and allocates consumption among employed and unemployed workers. He does not observe the match-specific disutility of the workers and has to induce them to reveal it. Moreover, workers can always quit and enjoy private utility from leisure. To make the analysis tractable I assume that workers who quit cannot be distinguished from all the other unemployed workers.

I first show that in a static setting the competitive search equilibrium is constrained efficient. Competition among firms induces them to design contracts in order to attract workers’ job applications and make them correctly internalize the search externality. By contrast, once I turn to the dynamic setting, the competitive search equilibrium is generically constrained inefficient. The crucial difference between the static and the dynamic environments is that the worker’s outside option is exogenously given in the former, while in the latter it is endogenously determined as the continuation utility of unemployed workers. When informational asymmetry is present and workers must be induced to participate, the workers’ outside option affects the \textit{ex-ante} cost of information revelation. This generates a novel externality which is not internalized by dynamic competitive search. Firms who post contracts at time $t+1$ do not take into account the informational cost they
impose on contracts designed by other firms at time \( t \), by affecting the workers’ outside option. This externality can be the source of constrained inefficiency. The social planner takes into account the impact that the continuation utility of unemployed workers has on current contracts, and can improve upon the equilibrium allocation.

The main result of the paper is that, under asymmetric information and the workers’ participation constraint, the competitive search equilibrium is constrained inefficient whenever the economy is away from the steady state. In particular, the direction of the inefficiency depends on the equilibrium dynamics of the unemployment rate. Consider a competitive equilibrium where the initial unemployment rate is above the steady state level. This means that the mass of potential matches is higher today relative to tomorrow and, hence, the cost of information revelation is relatively higher today. It follows that the planner would like to reduce job creation tomorrow, in order to reduce the continuation utility of unemployed workers today and achieve higher job creation today. The opposite happens when the unemployment rate is rising. In particular, I show that a simple time-varying tax on job creation can restore the constrained efficient allocation of resources.

This paper is related to a growing literature on asymmetric information in search environments. In particular, Shimer and Wright (2004) and Moen and Rosen (2005) analyze labor markets where trading frictions interact with asymmetric information, using competitive search. However, they do not explore efficiency and do not analyze the transitional dynamics of the equilibrium. Shimer and Wright (2004) analyze a static economy where the employer has some private information about the match and the worker a private effort choice. They show that under mild regularity assumptions contracts take a simple form with at most two wages. Moen and Rosen (2005) study the steady state competitive search equilibrium with private information on the workers’ side. They show that cross subsidization between workers and firms can restore the full information allocation. This result is similar to the one I derive for the case of transferable endowment.\(^4\) However, they analyze only the steady state equilibrium, focusing on the impact of asymmetric information on the responsiveness of the unemployment rate to productivity shocks.

Another related paper is Faig and Jerez (2004) who propose a theory of commerce, where buyers have private information about their willingness to pay for a product. They also show that the static model is constrained efficient, if the social planner cannot transfer utility across agents. However, they point out that another source of inefficiency can be the non-linearity of the production function. They calibrate a dynamic version of their model embodied in a neoclassical framework where the existence of capital induces a non-linear production function. They show that the welfare losses of competitive search are negligible. In a similar spirit, Wolinsky (2005) analyzes the efficiency properties of a sequential procurement model where a small buyer cannot commit to a mechanism and finds inefficient equilibria. However, in his model the inefficiency arises because of contracting restrictions. The fact that the seller’s effort is not contractible distorts the buyer’s search intensity.

\(^4\)See subsection 3.2.
In my paper, private contracts are unrestricted and the equilibrium inefficiency comes from a general equilibrium effect.

Finally, from a methodological standpoint my paper is related to the literature on mechanism design with asymmetric information, e.g. Mirrlees (1971), Myerson (1981), Myerson and Satterthwaite (1981), Laffont and Maskin (1980).

The paper is organized as follows. In Section 2, I introduce the static environment of the economy, defines and characterizes the competitive search equilibrium. In Section 2.2, I analyze the efficiency properties of the static economy. In Section 3, I describe the dynamic environment, define and characterize the dynamic competitive search equilibrium. In Section 4, I describe the dynamic welfare properties of the model and derives the main result that competitive search, away from the steady state, is constrained inefficient. I also show that the constrained efficient allocation can be implemented with a simple time-varying tax on job creation. Section 5 explores the alternative environment where unemployment benefit is transferable and shows that the full information allocation can be achieved. Section 6 concludes. Finally, the Appendix contains all the proofs.

2 Static Economy

The crucial ingredient of my model is the interaction of informational asymmetry and trading frictions, when there is a participation constraint on the worker’s side. In this section, I start by introducing the static version of a decentralized economy facing these features. I define and characterize the competitive search equilibrium for this economy.

Environment. The economy is populated by a continuum of measure 1 of workers and a large continuum of employers. Both workers and employers are risk-neutral and ex-ante homogeneous. Workers can search freely, while employers need to pay an entry cost $k$ to post a vacancy. Each worker wants to match an employer and each employer with an open vacancy wants to match only one worker. When a match is formed, the disutility of labor $\theta$ is drawn randomly from the cumulative distribution function $F(.)$, with support $\Theta \equiv [\underline{\theta}, \overline{\theta}]$, and is observed privately by the worker. I assume that the cumulative distribution function $F(.)$ is differentiable, with $f(.)$ denoting the associated density function, and that it satisfies the monotone hazard rate condition, that is, $d[F(\theta)/f(\theta)]/d\theta > 0$. The net surplus of the match is given by $y - \theta$, where $y$ represents the amount of output generated by a productive match. The value of $y$ is common to all the matches and is given exogenously.\textsuperscript{6}

\textsuperscript{5}The value $\theta$ can also be interpreted as the cost of effort that the worker has to exert to make the match productive, which depends on the specificity of the match.

\textsuperscript{6}When the asymmetric information is on the side of the employer, the analysis is similar. The equilibrium allocation still exhibits less trade than in the full information case. However, now firms appropriate ex-post the informational rents required by incentive compatibility.
At the beginning of the period employers can open a vacancy at a cost \( k \) which entitles them to post an employment contract \( C \in \mathbb{C} \), where \( \mathbb{C} \) is the set of ad interim incentive compatible and individually rational mechanisms. As I describe below, a contract \( C : \Theta \mapsto [0, 1] \times \mathbb{R}_+ \) specifies the hiring probability and the wage for each matched worker reporting type \( \theta \in \Theta \). Therefore the strategy of a firm is a pair \( (\sigma, C) \in \{0, 1\} \times \mathbb{C} \) where \( \sigma \) denotes the decision of posting a vacancy and \( C \) is the posted contract. Next, each worker observes all the contracts posted and decides where to apply. He chooses a contract \( C \in \mathbb{C}^P \subset \mathbb{C} \), where \( \mathbb{C}^P \) denotes the set of contracts posted by active firms. After workers start to search for a specific contract, matching takes place and for each match the draw \( \theta \) is realized and is private information of the worker. The worker’s behavior is described by a map \( (s,a) : \Theta \mapsto \Theta \times \{0, 1\} \) that for each type \( \theta \) specifies a report \( \hat{\theta} = s(\theta) \) and a participation decision \( a(\theta) \). The worker can either implement the contract, that is choose \( a(\theta) = 1 \), or walk away, that is choose \( a(\theta) = 0 \). If he walks away he gets \( b \), a flow of non-transferable utility from leisure.

Trading frictions in the labor market are modeled through random matching and can be thought of as coordination frictions, as in Burdett, Shi and Wright (2001). Employers and workers know that their matching probabilities will depend on the contract that they respectively post and seek for. Each type of contract \( C \) is associated with a labor submarket, where a mass \( v(C) \) of employers posts contracts of type \( C \) and a mass \( u(C) \) of unemployed workers applies for jobs at firms offering that type of contract. I assume that each submarket is characterized by a constant returns to scale matching function \( m(v(C), u(C)) \) and by an associated “tightness” \( \gamma(C) = v(C)/u(C) \).\(^7\) Hence, for each contract \( C \), I can define the function \( \mu(\gamma) = m(\gamma, 1) \), which represents the probability of a worker applying for \( C \) meeting an employer posting it. On the other hand, \( \mu(\gamma)/\gamma \) denotes the probability of a firm posting \( C \) meeting a worker applying for it.

**Assumption A1.** The function \( \mu(\gamma) : [0, \infty) \rightarrow [0, 1] \) satisfies the following conditions:

\[(i) \quad \mu(\gamma) \leq \min\{\gamma, 1\};^8\]
\[(ii) \quad \text{for any } \gamma \text{ such that } \mu(\gamma) < \min\{\gamma, 1\}, \mu(\gamma) \text{ is twice differentiable with } \mu'(\gamma) > 0 \text{ and } \mu''(\gamma) < 0.\]

This assumption allows me to consider matching functions that either are everywhere differentiable or have one or two kinks. The standard matching functions considered in the literature are covered by one of these two categories: the first one includes the exponential case, while the properly *modified* linear and Cobb Douglas cases fall into the second one.

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^7In order to simplify the notation, from now on I am going to drop the dependence of \( u, v \) and \( \gamma \) on the contract \( C \), whenever it does not cause any confusion.

^8With discrete time, this condition ensures that both \( \mu(\gamma) \) and \( \mu(\gamma)/\gamma \) are proper probabilities.

^9From now on I define the *modified* version of a function \( \hat{\mu}(\gamma) \), the function \( \mu(\gamma) = \min\{\hat{\mu}(\gamma), \gamma, 1\} \).
In a decentralized economy the consumption of employed workers is given by the wage. Moreover, the consumption of unemployed workers must be equal to the value of leisure $b$, where unemployed workers are both unmatched workers and matched workers who have not been hired. Assume that $y > b + \theta$ in order to make the problem interesting.\footnote{Notice that if $y < b + \theta$ then even with full information the equilibrium would be characterized by zero trade.}

**Employment Contracts.** Without loss of generality, by invoking the Revelation Principle, I can restrict attention to direct revelation mechanisms, corresponding to a mapping $C : \Theta \mapsto [0, 1] \times \mathbb{R}_+$, specifying for each matched worker who reports type $\theta$, the hiring probability $e(\theta) \in [0, 1]$ and the wage $\omega(\theta) \in \mathbb{R}_+$. The contract must be incentive compatible and individually rational, that is, it has to ensure that the worker reveals truthfully his type and chooses to participate in the employment relationship after the draw has been realized. Individual rationality can be interpreted as a “no-commitment constraint” on the worker’s side and can be justified by the typical at will employment contracts widespread in the United States.\footnote{A part from a realistic justification, the assumption of lack of commitment on the worker’s side is necessary to make the role of asymmetric information interesting, once we assume linear utility.} Instead, firms can fully commit to the posted contract.

Let $v(\theta, \hat{\theta})$ denote the *ad interim utility* for worker of type $\theta$ revealing $\hat{\theta}$, associated with a contract $C = \{e(\theta), \omega(\theta)\}_{\theta \in \Theta},$\footnote{In order to simplify the notation, I drop the dependence of $v(\cdot, \cdot)$ on the contract $C$, since it does not cause any confusion.} that is,

$$v(\theta, \hat{\theta}) \equiv \omega(\hat{\theta}) - e(\hat{\theta})\theta + [1 - e(\hat{\theta})]b. \tag{1}$$

An employment contract is *incentive-compatible* whenever it satisfies

$$v(\theta, \theta) \geq v(\theta, \hat{\theta}) \text{ for all } \theta, \hat{\theta} \in \Theta, \tag{IC}$$

and *individually rational* whenever

$$v(\theta, \theta) \geq b \text{ for all } \theta \in \Theta. \tag{IR}$$

Following a standard result in the mechanism design literature,\footnote{Among others, Mirlees (1971), Myerson (1981), Myerson and Satterthwaite (1981), Laffont and Maskin (1980).} I can reduce the dimensionality of the constraints. In particular, conditions IC and IR are equivalent to $e(\cdot)$ non-increasing and

$$v(\theta, \theta) = v(\bar{\theta}, \bar{\theta}) + \int_{\theta}^{\bar{\theta}} e(y) \, dy \text{ for all } \theta \in \Theta, \tag{IC’}$$

$$v(\bar{\theta}, \bar{\theta}) \geq b. \tag{IR’}$$

This allows me to separate the problem of finding an optimal allocation from the problem of finding
a wage schedule that implements it.

Define \( v(\theta, \bar{\theta}) - v(\bar{\theta}, \bar{\theta}) \) as the *informational rent* of a worker of type \( \theta \leq \bar{\theta} \), that is, the additional utility that such a worker must receive in order to reveal his own type. Condition IC’ ensures that no worker would gain by pretending to have a higher disutility from working than the realized one. Moreover, condition IR’ ensures that the worse type, and, by monotonicity, everybody else, does not expect an utility level lower than the one he could get by staying in autarky.

Finally, the large number of potential firms ensures free entry, imposing that the value of an open vacancy must be zero in equilibrium, that is,

\[
\frac{\mu(\gamma)}{\gamma} \int_0^\bar{\theta} [e(\theta) y - \omega(\theta)] dF(\theta) = k. \tag{2}
\]

### 2.1 Static Competitive Search Equilibrium

I now define the concept of competitive search equilibrium in this economy, I prove that it always exists, is unique and I characterize it.

**Definition 1** A static symmetric Competitive Search Equilibrium (CSE) is a set of incentive-compatible and individually rational contracts \( C^* \) together with a function \( \Gamma^*: C \rightarrow \mathbb{R}_+ \cup \infty \) and a utility level \( U^* \in \mathbb{R}_+ \) satisfying

(i) **employers’ profit maximization** and **free-entry**: \( \forall C \equiv [e(\theta), \omega(\theta)]_{\theta \in \Theta} \),

\[
\frac{\mu(\Gamma^*(C))}{\Gamma^*(C)} \int_0^\bar{\theta} [e(\theta) y - \omega(\theta)] dF(\theta) - k \leq 0
\]

subject to incentive compatibility IC and individual rationality IR, with equality if \( C \in C^* \);

(ii) **workers’ optimal job application**: \( \forall C \equiv [e(\theta), \omega(\theta)]_{\theta \in \Theta} \),

\[
U^* \geq \mu(\Gamma^*(C)) \int_0^\bar{\theta} [\omega(\theta) - e(\theta)(\theta + b)] dF(\theta) + b
\]

and \( \Gamma^*(C) \geq 0 \) with complementarity slackness, where \( U^* \) is given by

\[
U^* = \max_{C' \in C^*} \mu(\Gamma^*(C')) \int_0^\bar{\theta} [\omega'(\theta) - c'(\theta)(\theta + b)] dF(\theta) + b
\]

or \( U^* = b \) if \( C^* \) is empty.

In equilibrium, both firms and workers know which market tightness is associated with each contract, that is, they know the function \( \Gamma^*(C) \). Given that, profit maximization ensures that firms post the incentive compatible and individually rational contract that maximizes their profits,
anticipating the tightness associated even to contracts not offered in equilibrium. This ensures that there are no profitable deviations for the firm and free entry drives profits to zero. Moreover, optimal job application ensures that workers choose which type of contracts to look for, so as to maximize their ex-ante utility. In particular, notice that the tightness associated with contracts that are not optimal is zero, since firms will never post those contracts anticipating that they will not be able to attract workers.

It follows that the equilibrium unemployment rate of workers applying to firms posting a contract of type $C$ is given by

$$u(C) = 1 - \mu(\Gamma^*(C)) \int_{\theta} e(\theta) dF(\theta)$$

and is affected by both matching and informational frictions. In fact job creation depends not only on the equilibrium matching probability, through $\mu(\Gamma^*(C))$, but also on the equilibrium hiring decision, once the match is realized, through $\int_{\theta} e(\theta) dF(\theta)$.

Generalizing the standard result in the search literature,\textsuperscript{14} I can show that the symmetric competitive search equilibrium is such that the utility of an unemployed worker is maximized subject to the zero profit condition for the employer and the incentive and the participation constraint for the worker.

**Proposition 1** If $\{C^*, \Gamma^*, U^*\}$ is an equilibrium, then any $C^* \in C^*$ and $\gamma^* = \Gamma^*(C^*)$ solves

$$U = \max_{C, \gamma} \mu(\gamma) \int_{\theta} [\omega(\theta) - e(\theta)(\theta + b)] dF(\theta) + b$$

subject to $e(\theta) \in [0, 1]$, non-negative consumption, the incentive constraint $IC'$, together with the monotonicity of $e(.)$, the participation constraint $IR'$ and the free-entry condition (2). Conversely, if a pair $\{C^*, \gamma^*\}$ solves the program P1, then there exists an equilibrium $\{C^*, \Gamma^*, U^*\}$ such that $C^* \in C^*$ and $\gamma^* = \Gamma^*(C^*)$.

This proposition shows how a CSE must solve Problem P1. The next proposition shows how it can be equivalently described by a tightness $\gamma$ and a hiring function $e(\theta)$ solving a simplified program P2 and by an associated wage function $\omega(\theta)$ which can be constructed such that the incentive and the participation constraints are satisfied.

**Lemma 1** Any function $[e(\theta)]_{\theta \in \Theta}$ and $\gamma$ which solve Problem P1 solves also

$$U = \max_{e(\cdot), \gamma} \mu(\gamma) \int_{\theta} e(\theta)[y - \theta - b] dF(\theta) + b - \gamma k$$

\textsuperscript{14}Moen (1997), Acemoglu and Shimer(1999a) analyze a competitive search equilibrium when information is complete. Shimer and Wright (2004) define a competitive search equilibrium with bilateral asymmetric information.
\[
\mu(\gamma) \int_{\bar{\theta}}^{\underline{\theta}} e(\theta) \left[ y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) \geq \gamma k \tag{3}
\]

and \(e(\cdot)\) non-increasing.

Furthermore, for any function \([e(\theta)]_{\theta \in \Theta}\) and \(\gamma\) solving problem P2, there exists a function \([\omega(\theta)]_{\theta \in \Theta}\) such that the contract \(C = [e(\theta), \omega(\theta)]_{\theta \in \Theta}\) and \(\gamma\) solve problem P1.

Free-entry implies that the entire surplus of the economy accrues to workers. Hence, the competitive search equilibrium maximizes the net surplus of the economy, subject to the constraint that the net output must cover both the \textit{ex-ante} cost of vacancy creation, that is, \(\gamma k\), and the average informational rents, that is, \(\int_{\bar{\theta}}^{\underline{\theta}} e(\theta) F'(\theta) d\theta\)
\footnote{From condition IC', using integration by parts, it follows that the average information rents are:}

Under full information, the maximization problem is unconstrained and the equilibrium coincides immediately with the social optimum.

Finally, the following Proposition establishes the existence and uniqueness of a CSE.

**Proposition 2** A Competitive Search Equilibrium exists and is unique.

The proof relies on the result of Proposition 1, that the existence of a solution to problem P2 is sufficient to prove the existence of a solution for problem P1.

**Equilibrium Characterization.** Lemma 1 allows me to characterize the competitive search equilibrium of the static economy in a simple way. Proposition 2 proves that Problem P2 has a unique solution and that the first order conditions are necessary and sufficient to characterize it. The analysis proceeds by focusing on the relaxed problem without the monotonicity assumption on \(e(\theta)\). Then, using pointwise maximization for \(e(\theta)\), I show that the trading area can be fully described by a cut-off value \(\hat{\theta}\) such that

\[
e(\theta) = \begin{cases} 
1 & \text{if } \theta \leq \hat{\theta} \\
0 & \text{if } \theta > \hat{\theta} 
\end{cases}
\]

implying that the optimal \(e(\theta)\) is in fact monotone. When the constraint is binding
\footnote{When the constraint is not binding, the informational problem is irrelevant and the competitive search equilibrium is standard.} and \(\mu(\gamma)\) is everywhere differentiable, the equilibrium can be characterized by an array \(\hat{\theta}, \gamma\) and \(\lambda\) satisfying the first-order conditions

\[
\hat{\theta} = y - b - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})}, \tag{4}
\]
\[ \mu'(\gamma) \int_{\hat{\theta}}^{\theta} \left[ y - \theta - b - \lambda \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k \]  

(5)

and the binding constraint (3). The variable \( \lambda \) represents a normalized version of the shadow value of the informational rents. I define \( \lambda \equiv \hat{\lambda}/(1 + \hat{\lambda}) \), where \( \hat{\lambda} \) is the Lagrangian multiplier attached to the constraint (3). From equation (4) it follows that the trading cut-off \( \hat{\theta} \) is decreasing in \( \lambda \), that is, as the constraint is tighter, the shadow value that workers have to receive in order to reveal their information increases and the equilibrium is characterized by less trade.\(^{17}\) When \( \lambda = 0 \), the constraint (3) is slack and \( \gamma \) is simply determined by (5). Then, the full information allocation is achieved. This is possible only when the \textit{ex-ante} cost \( k \) is zero. As shown in the next Lemma, incentive compatibility would drive employers to zero profits \textit{ex-post}, if the full information allocation would be implemented, contradicting the possibility of an equilibrium where they have to pay a positive cost \textit{ex-ante}.

**Lemma 2** If \( k > 0 \), then the solution to problem P2 requires \( \lambda > 0 \), where \( \lambda = \hat{\lambda}/(1 - \hat{\lambda}) \) and \( \hat{\lambda} \) is the Lagrangian multiplier attached to the constraint.

This argument highlights the main channel driving the misalignment between \textit{ex-ante} and \textit{ex-post} efficiency, which keeps the economy away from the full information allocation. \textit{Ex-post} allocative distortions are necessary to induce employers to open vacancies \textit{ex-ante} and make the economy productive. From now on, I focus on \( k > 0 \) so that the constraint is binding and the informational problem interesting.

From incentive compatibility it follows that the optimal wage schedule must take the following form:

\[ \omega(\theta) = \begin{cases} 
\omega(\hat{\theta}) + \hat{\theta} + b & \text{if } \theta \leq \hat{\theta} \\
\omega(\hat{\theta}) & \text{if } \theta > \hat{\theta} \end{cases} \]

Notice that when \( k > 0 \) and \( \lambda > 0 \), then \( \omega(\hat{\theta}) = 0 \). Then, a constant wage is paid only to workers who are effectively hired and is equal to the disutility of the marginal hired worker plus the outside option. In fact, if two workers with different types receive different wages, then the worse type would always pretend to be the best in order to get a higher compensation. Moreover, the marginal hired worker would have no incentive to lie if he is indifferent about being unemployed, that is, if he is compensated exactly for his disutility and for the working opportunity cost \( b \). It follows that the wage is increasing in the trading cut-off \( \hat{\theta} \). The more trade is generated, the higher the wage must be in order to induce the marginal hired worker to reveal his type.

**Remark 1** The static economy is equivalent to a reduced form economy where firms post a constant wage and workers apply for jobs.

\(^{17}\)Notice that, when \( \mu(\gamma) \) is not differentiable at some point, as I describe in the Appendix, equation (5) will be replaced by inequalities involving the left and right derivatives of \( \mu(\gamma) \) when the solution will be at the points of non differentiability.
2.2 Constrained Efficiency in a static economy

I now define the relevant social planning problem, that is, a planner subject to the same environmental constraints faced by the market. The planner controls the matching process by deciding how many vacancies to open at the beginning of the period. He does not observe the types of the matched workers and has to induce them to truthfully reveal their match-specific disutility. Moreover, he is also subject to a participation constraint on the side of the workers, who can decide not to produce and enjoy leisure. Under these environmental constraints, together with the resource constraint of the economy, the social planner decides how to allocate consumption among employed and unemployed workers.

An *allocation* is a pair of functions \( \{c(\theta), e(\theta)\}_{\theta \in \Theta} \) representing the consumption and the hiring probability for a matched worker who reports type \( \theta \), a value for consumption of unmatched workers \( C^U \), and a value \( \gamma \) denoting the tightness of the market.

As in the case of private contracts, also here, the Revelation Principle allows me to restrict attention to direct revelation mechanisms, without loss of generality. Following the analysis of the previous section, an allocation is *incentive-compatible* when, for all \( \theta \in \Theta \), \( e(\cdot) \) is non-increasing and

\[
v^{SP}(\theta, \theta) = v^{SP}(\hat{\theta}, \hat{\theta}) + \int_{\hat{\theta}}^{\theta} e(y) \, dy. \tag{6}
\]

where the utility for a worker of type \( \theta \) reporting type \( \hat{\theta} \), in the centralized economy, is given by

\[
v^{SP}(\theta, \hat{\theta}) = c(\hat{\theta}) - e(\hat{\theta}) \theta + [1 - e(\hat{\theta})] C^U.
\]

Moreover there is a *participation constraint* coming from a lack of commitment on the worker’s side together with the assumption that \( b \) cannot be destroyed or transferred and that negative consumption is not allowed. It requires that all workers who participate in the society, both if matched and unmatched, consume more than the private utility \( b \) that they can appropriate by not participating, that is,

\[
C^U \geq b \text{ and } v^{SP}(\theta, \theta) \geq b \text{ for all } \theta \in \Theta. \tag{7}
\]

Finally, the *resource constraint* for the static economy ensures that aggregate consumption is covered by aggregate net production, that is,

\[
\mu(\gamma) \int_{\hat{\theta}}^{\theta} c(\theta) dF(\theta) + (1 - \mu(\gamma)) C_u \leq \mu(\gamma) (y - b) \int_{\hat{\theta}}^{\theta} e(\theta) dF(\theta) + b - \gamma k. \tag{8}
\]

I can now define a feasible and a constrained efficient allocation. For simplicity, assume that all the workers are unemployed *ex-ante*. Given that employers get zero profits in expectation due to the free entry assumption, social welfare coincides with the *ex-ante* value of being unemployed.
Definition 2 A constrained efficient allocation maximizes workers’ ex-ante utility

\[ \mu(\gamma) \int_{\theta}^{\bar{\theta}} [c(\theta) - e(\theta) \, \theta] \, dF(\theta) + \left( 1 - \mu(\gamma) \int_{\theta}^{\bar{\theta}} e(\theta) \, dF(\theta) \right) \, C^U \]  

(9)

subject to feasibility, that is, (i) the incentive-compatibility constraint (6) together with the monotonicity of \( e(\cdot) \), (ii) the participation constraint (7) and (iii) the resource constraint (8).

The maximization problem defining a constrained efficient allocation can be expressed, after some algebra, in a very similar form to problem P2, by substituting constraint (3) with

\[ \int_{\theta}^{\bar{\theta}} e(\theta) \left[ y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] \, dF(\theta) \geq \left( \frac{1 - \mu(\gamma)}{\mu(\gamma)} \right) (C^U - b) + \frac{\gamma}{\mu(\gamma)} k, \]  

(10)

and imposing the additional constraint \( C^U \geq b \). In fact, the only difference with the market equilibrium is that the planner can potentially transfer resources to the unemployed workers. However, next proposition states that such a transfer is not desirable and that the competitive search equilibrium is constrained efficient in the static setting.

Proposition 3 Assume that \( b \) cannot be destroyed or transferred and that negative consumption is not allowed, then a static Competitive Search Equilibrium is constrained efficient.

The proof is straightforward. First, notice that \( C^U \) does not appear in the objective function so that the social planner cannot do worse by choosing \( C^U = b \). Then the planner problem becomes equivalent to problem P2 and competitive search is constrained efficient.

It is interesting to notice that, as in the decentralized equilibrium, the social optimum does not reach the full information allocation. The result is driven by the binding participation constraint for the workers. The argument shows clearly that the Planner could do better if he could reduce the consumption of unemployed workers, which does not affect the objective function, below the level of leisure \( b \). This suggests a simple exercise able to capture the mechanism that will lead to dynamic inefficiency. Suppose that \( b \) can be destroyed, what I refer to as money burning. I now show how money burning can generate a Pareto improvement.

Let \( U(b) \) denote the ex-ante utility of unemployed workers as a function of the level of leisure \( b \) and suppose that it is differentiable\(^{18} \). A Pareto improvement is feasible when \( U'(b) < 0 \).

Proposition 4 Suppose \( b \) can be wasteful destroyed. Whenever \( 1 - \mu(\gamma) \, F(\hat{\theta}) < \lambda \), the competitive search equilibrium allocation can be Pareto improved by reducing \( b \).

\(^{18}\) It is easy to show that when \( \gamma \), \( \hat{\theta} \) and \( \lambda \) are uniquely defined, then \( U(b) \) is differentiable. In fact this is always the case in the rest of the analysis.
The proof comes straight from the Envelope condition which implies that \( \frac{dU}{db} < 0 \) iff \( 1 - \mu(\gamma) F(\hat{\theta}) < \lambda \).\(^{19}\) The expression \( 1 - \mu(\gamma) F(\hat{\theta}) - \lambda \) represents the effect of the workers’ outside option on welfare. There is a direct positive effect coming from the fact that, as the outside option is higher, workers who end up being unemployed will be better off. This force is summarized by \( 1 - \mu(\gamma) F(\hat{\theta}) \), which represents the \textit{ex-ante} probability of being unemployed at the end of the period. However, there is a negative indirect effect coming from the tightness of the informational constraint, represented by \( \lambda \). As the outside option increases, the shadow cost of revealing information is higher, since workers have a higher opportunity cost of remaining in the employment relationship. When \( 1 - \mu(\gamma) F(\hat{\theta}) < \lambda \) the indirect effect dominates and a Pareto improvement can be implemented by reducing \textit{ex-ante} the workers’ outside option.

In Guerrieri (2005) I show that, there exists an open set of the parameter space such that \( 1 - \mu(\gamma) F(\hat{\theta}) < \lambda \) at the competitive search equilibrium, for two general families of functions \( \mu(\gamma) \), which include the specifications commonly used in the search literature, that is, exponential and properly modified linear and Cobb-Douglas.

The result that money burning can be desirable suggests a source of dynamic constrained inefficiency. It can be socially optimal to reduce \textit{ex-ante} the workers’ outside option, which, here, is exogenously given by \( b \), but will be endogenous in the dynamic economy.

### 3 Dynamic economy.

This section introduces the dynamic environment of this economy leading to the main result of the paper: the dynamic competitive search equilibrium can be constrained inefficient. The crucial difference between the static and the dynamic environment is that the worker’s outside option is exogenously given in the former, while in the latter, it is endogenously determined as the equilibrium continuation utility of unemployed workers. Inefficiency arises because the worker’s outside option affects the \textit{ex-ante} cost of information revelation, generating an externality. The social planner can improve upon the decentralized economy by internalizing this \textit{informational externality}.

**Environment.** Consider an economy with infinite horizon and discrete time. Both workers and employers have linear preferences and discount factor \( \beta \). The search and production technologies are natural generalizations of the static setting, except that now \( y \) and \( \theta \) denote the expected value of output and disutility at the moment of the match. At the beginning of each period \( t \) employers can be either productive or not. Workers can be either employed or unemployed. Non-productive employers can open a vacancy at a cost \( k \) which entitles them to post an employment contract \( C_t \in \mathbb{C} \) where \( \mathbb{C} \) is the set of \textit{ad interim} incentive compatible and individually rational mechanisms.

\(^{19}\)The Envelope condition is \( \frac{dU}{db} = 1 - (1 - \lambda)^{-1} \mu(\gamma) F(\hat{\theta}) \) where \( \lambda = \hat{\lambda}/(1 + \hat{\lambda}) \) and \( \hat{\lambda} \) is the multiplier attached to constraint (10).
As I describe below, a contract $C_t : \Theta \mapsto [0, 1] \times \mathbb{R}_+$ specifies the hiring probability and the expected value of wages for each matched worker at time $t$, who reports type $\theta$. Therefore at each time $t$, a non-productive firm chooses a pair $(\sigma_t, C_t) \in \{0, 1\} \times C$ where $\sigma_t$ denotes the decision of posting a vacancy and $C_t$ the posted contract. Next, each unemployed worker observes all the contracts posted and decides where to apply. He chooses a contract $C_t \in C^P_t \subset C$, where $C^P_t$ denotes the set of contracts posted by active firms at time $t$. As in the static environment, each contract $C_t$ is associated to a specific $\gamma_t$ so that employers and workers know that their matching probabilities will depend on the contract that they respectively post and seek for. After workers start to search for a specific contract, matching takes place and, for each match, the draw is realized and is private information of the worker. The behavior of a worker who is matched at time $t$ is described by a map $(s_t, a_t) : \Theta \mapsto \Theta \times \{0, 1\}$ that for each type $\theta$ specifies a report $s(\theta) = \hat{\theta}_t$ and a participation decision $a_t(\theta)$. After he sees his type, the worker can either implement the contract, that is, choose $a_t(\theta) = 1$, or walk away, that is choose $a_t(\theta) = 0$. If he walks away, he stays in autarky for one period, gets a non-transferable utility from leisure $b$, enters an anonymous pool of unemployed workers and look for another match next period. If the worker is effectively hired, the parties are productive until separation, which happens according to a Poisson process with parameter $s$.

In a decentralized economy, the consumption of unemployed workers, that is, both unmatched workers and matched workers who have not been hired, is equal to the value of leisure $b$. Moreover, the expected value of consumption for a matched workers is given by the contracted expected value of wages. Linear preferences, together with the fact that types are fixed over time within a match, imply that the wage profile over the life of the relationship is irrelevant for the analysis.

**Employment Contracts and Bellman Values.** Invoking the Revelation Principle, without loss of generality, I can again restrict attention to incentive-compatible and individually rational direct revelation mechanisms, corresponding to a mapping $C_t : \Theta \mapsto [0, 1] \times \mathbb{R}_+$, specifying for each matched worker at the end of time $t$ who reports type $\theta$, the hiring probability $e_t(\theta) \in [0, 1]$ and the expected value of wages $\omega_t(\theta) \in \mathbb{R}_+$. Notice that I can restrict attention to the set $C$ of incentive compatible and individually rational mechanisms described above, due to the unemployed anonymity assumption. All the unemployed workers searching for a job cannot be distinguished, so that contracts cannot be conditioned on the past employment history.

Let define $V_t$ the continuation value of being employed net of wages and disutility at time $t$, which from now on I will refer to simply as the continuation utility of employed workers. It represents just the discounted expected value of being separated and becoming unemployed, that is,

$$V_t = \beta s U_{t+1} + \beta (1 - s) V_{t+1}. \tag{11}$$

\[^{20}\text{Notice that the problem is equivalent to one stated in terms of instantaneous output and disutility, } \tilde{y} \text{ and } \tilde{\theta} \text{ where they are both constant for the duration of the match. Then, simply } y = \tilde{y}(1 - \beta (1 - s))^{-1} \text{ and } \theta = \tilde{\theta}(1 - \beta (1 - s))^{-1}.\]
Moreover the continuation value of being unemployed at time \( t \) is given by

\[
U_t = b + \beta \mu (\gamma_t) \int_\theta [\omega_t (\theta) - e_t (\theta) (\theta - V_{t+1} + U_{t+1})] dF (\theta) + \beta U_{t+1}.
\]

(12)

The \textit{ad interim} utility of a worker of type \( \theta \), who reports type \( \hat{\theta} \) at the end of time \( t \), is given by

\[
v_t(\theta, \hat{\theta}) = \omega_t (\hat{\theta}) - e_t (\hat{\theta}) (\theta - V_{t+1}) + [1 - e_t (\hat{\theta})] U_{t+1}
\]

(13)

The large number of potential firms ensures free entry and implies that the value of an open vacancy will be zero at each time, that is,

\[
\beta \frac{\mu (\gamma_t)}{\gamma_t} \int_\theta [e_t (\theta) y - \omega_t (\theta)] dF (\theta) = k.
\]

(14)

A natural generalization of the static analysis, gives that a contract \( \mathcal{C}_t \) is \textit{incentive-compatible} and \textit{individually rational} whenever \( e_t (\cdot) \) is non-increasing and the following conditions hold:

\[
v_t (\theta, \theta) = v_t (\bar{\theta}, \bar{\theta}) + \int_\theta e_t (y) dy \text{ for all } \theta \in \Theta
\]

\text{(IC')}

and

\[
v_t (\bar{\theta}, \bar{\theta}) \geq U_{t-1}.
\]

(15)

Following the static analysis, the informational rents for a worker of type \( \theta \) who meets a firm at time \( t \) are \( v_t (\theta, \theta) - v_t (\bar{\theta}, \bar{\theta}) \), as defined by equation IC’.

### 3.1 Dynamic Competitive Search Equilibrium

In this section, I define the dynamic version of the Competitive Search Equilibrium. Generalizing the static definition, a dynamic Competitive Search Equilibrium, in sequential terms, is a sequence of sets of incentive-compatible and individually rational contracts \( \{ \mathcal{C}^*_t \} \) and a sequence of tightness functions \( \{ \Gamma^*_t \} \), where \( \Gamma^*_t : \mathcal{C}^*_t \rightarrow \mathbb{R}_+ \cup \infty \), such that, at any \( t \) employers maximize profits and workers apply optimally for jobs, taking as given the future sequence of sets of contracts, \( \{ \mathcal{C}_{t+1} \} , \{ \mathcal{C}_{t+2} \} , ... \), and tightness functions, \( \{ \Gamma^*_t \} , \{ \Gamma^*_{t+1} \} , ... \).

In order to simplify the analytical treatment, I introduce an equivalent definition of the dynamic competitive search equilibrium in recursive terms. The first thing to notice is that the pair of continuation utilities for unemployed and employed workers, \( U \) and \( V \), are a sufficient statistic for future sets of \( \mathcal{C}'s \) and \( \Gamma'\)s. This allows me to describe the dynamic competitive search equilibrium in a recursive way, as stated by the following Definition.

**Definition 3** A \textit{dynamic Competitive Search Equilibrium (CSE)} is a sequence of sets of incentive-
compatible and individually rational contracts \( \{ C_t^* \} \), a sequence of functions \( \{ \Gamma_t^* \} \), where \( \Gamma_t^*: C_t^* \mapsto \mathbb{R}_+ \cup \infty \), and a sequence of pairs of continuation utility levels \( \{ U_t^*, V_t^* \} \), where \( (U_t^*, V_t^*) \in \mathbb{R}^2_+ \) for any \( t \), satisfying

(i) **employers’ profit maximization** and **free-entry** at each time \( t \): \( \forall C_t \equiv [e_t(\theta), \omega_t(\theta)]_{\theta \in \Theta} \),

\[
\frac{\mu(\Gamma_t^*(C_t))}{\Gamma_t^*(C_t)} \beta \int_{\Theta} [e_t(\theta) y - \omega_t(\theta)] dF(\theta) - k \leq 0
\]

subject to incentive compatibility IC and individual rationality IR, with equality if \( C_t \in \{ C_t^* \} \);

(ii) **workers’ optimal job application** at each time \( t \): \( \forall C_t \equiv [e_t(\theta), \omega_t(\theta)]_{\theta \in \Theta} \), for given \( V_{t+1} \) and \( U_{t+1} \)

\[
U_t^* \geq b + \beta \mu(\Gamma_t^*(C_t)) \int_{\Theta} [\omega_t(\theta) - e_t(\theta) (\theta - V_{t+1} + U_{t+1})] dF(\theta) + \beta U_{t+1}
\]

and \( \Gamma_t^*(C_t) \geq 0 \), with complementarity slackness, where

\[
U_t^* = \max_{C_t' \in \mathcal{C}_t^*} b + \beta \mu(\Gamma_t^*(C_t')) \int_{\Theta} [\omega_t(\theta) - e_t(\theta) (\theta - V_{t+1} + U_{t+1})] dF(\theta) + \beta U_{t+1}
\]

or \( U_t^* = b + \beta U_{t+1} \) if \( \{ \mathcal{C}_t^* \} \) is empty, and

\[
V_t^* = \beta s U_{t+1} + \beta (1 - s) V_{t+1}.
\]

The definition of the dynamic equilibrium is a natural generalization of the static version. At each point in time employers maximize profits and workers apply optimally for jobs, both taking as given the future values of being employed and unemployed and aware that a market tightness is associated with each contract, even if not offered in equilibrium, according to the function \( \Gamma_t^*(C_t) \). Moreover, profits are driven to zero at each point in time by free entry.

It follows that the unemployment rate of workers applying to firms posting a contract of type \( C_t \) at time \( t \) is given by

\[
u_{t+1}(C_{t+1}) = u_t(C_t) \left[ 1 - \mu(\Gamma_t^*(C_t)) \int_{\Theta} e_t(\theta) dF(\theta) \right] + (1 - u_t(C_t)) s. \tag{15} \]

Generalizing the static result, the next Proposition states a dynamic characterization of a symmetric competitive search equilibrium in recursive terms.

**Proposition 5** If \( \{ \mathcal{C}_t, \Gamma_t, U_t, V_t \}_{t=0}^{\infty} \) is a **Competitive Search Equilibrium**, then any pair \( (C_t^*, \gamma_t^*) \) with \( C_t^* \in \mathcal{C}_t \) and \( \gamma_t^* = \Gamma_t^*(C_t^*) \) satisfy the following
(i) for given pair $U_{t+1}$ and $V_{t+1}$, for any time $t$, $C_t = [e_t(\theta), \omega_t(\theta)]_{\theta \in \Theta}$ and $\gamma_t$ solve

$$\max_{e_t(\cdot), \omega_t(\cdot), \gamma_t} b + \beta \mu(\gamma_t) \int_\Theta \left[ \omega_t(\theta) - e_t(\theta) (\theta - V_{t+1} + U_{t+1}) \right] dF(\theta) + \beta U_{t+1}$$

(P4)

subject to $e_t(\theta) \in [0, 1]$, non-negative consumption, the incentive compatibility constraint $IC'$ together with the monotonicity assumption on $e_t(\theta)$, the individual rationality constraint $IR'$ and the free-entry condition (14);

(ii) for given $\{C_t, \gamma_t\}_{t=0}^\infty$, then $\{U_t, V_t\}_{t=0}^\infty$ evolve according to (11) and (12).

Conversely, if a sequence $\{C_t^*, \gamma_t^*\}_{t=0}^\infty$ solves the program $P4$, then there exists an equilibrium $\{C_t^*, \Gamma_t^*, U_t^*, V_t^*\}_{t=0}^\infty$ such that $C_t^* \in C_t^*$ and $\gamma_t^* = \Gamma_t^*(C_t^*)$.

In the rest of the analysis I adopt a recursive notation, dropping the $t$ whenever this causes no confusion, and denoting a variable at time $t-1$ with a $-$ sign.

Proposition 5 shows that for given $U$ and $V$, a (symmetric) equilibrium incentive-compatible and individually-rational contract $C$ and tightness $\gamma$ must solve Problem $P4$. The next Proposition shows that the equilibrium can be equivalently described by a hiring function $e(\theta)$ and a tightness $\gamma$ that solve a simplified program $P5$. Given $e(\theta)$ and $\gamma$, an associated wage function $\omega(\theta)$ can be constructed so that the constraints $IC'$ and $IR'$ are satisfied.\(^{21}\)

**Proposition 6** For given $U'$ and $V'$, any function $[e(\theta)]_{\theta \in \Theta}$ and $\gamma$ which solve Problem $P4$, solve also

$$\max_{e(\cdot), \gamma} \beta \mu(\gamma) \int_\Theta e(\theta) \left[ y - \theta + V' - U' \right] dF(\theta) + b - \gamma k + \beta U'$$

(P5)

s.t.

$$\beta \mu(\gamma) \int_\Theta e(\theta) \left[ y - \theta + \frac{F(\theta)}{f(\theta)} + V' - U' \right] dF(\theta) \geq \gamma k. \quad (16)$$

Furthermore, for any function $[e(\theta)]_{\theta \in \Theta}$ and $\gamma$ which solve problem $P5$, there exists a function $[\omega(\theta)]_{\theta \in \Theta}$ such that the contract $C = [e(\theta), \omega(\theta)]_{\theta \in \Theta}$ and $\gamma$ solve problem $P4$.

**Equilibrium Characterization.** The characterization of the equilibrium allocation for a given continuation utility gap between unemployed and employed workers, $U - V$, is very similar to the static one. In fact, in the dynamic setting, $U - V$ represents the effective outside option of the workers. A generalization of Proposition 2 proves that Problem $P5$ has a unique solution and that the first order conditions are necessary and sufficient to characterize it. I proceed by studying the relaxed problem without the monotonicity assumption on $e(\theta)$. Then, using pointwise maximization

\(^{21}\)The proof of Proposition 6 is similar to the one of Proposition 1 and is therefore omitted.
with respect to \( e(\theta) \) I show that the trading area can be fully described by a cut-off value \( \hat{\theta} \) such that

\[
e(\theta) = \begin{cases} 
1 & \text{if } \theta \leq \hat{\theta} \\
0 & \text{if } \theta > \hat{\theta}
\end{cases},
\]

implying that the optimal \( e(\theta) \) is effectively non-increasing. When the constraint (16) is binding\(^{22}\) and \( \mu(\gamma) \) is everywhere differentiable the equilibrium can be characterized, for given \( U' - V' \), by an array \( \hat{\theta}, \gamma \) and \( \lambda \) satisfying the first order conditions

\[
y - \hat{\theta} - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})} - (U' - V') = 0, \tag{17}
\]

\[
\beta \mu'(\gamma) \int_{\theta}^{\hat{\theta}} \left[ y - \theta - \lambda \frac{F(\theta)}{f(\theta)} - (U' - V') \right] dF(\theta) = k \tag{18}
\]

and the binding constraint (16). The variable \( \lambda \) represents a normalized version of the shadow value of the informational rents\(^{23}\). Notice that when \( \lambda = 0 \), the constraint (16) is slack and \( \gamma \) is simply determined by (18). Then, the full information allocation is achieved. Clearly, asymmetric information reduces job creation, as the surplus of the economy must cover not only the cost of vacancy creation but also the rents needed to extract information from the workers. As intuition suggests, \( \hat{\theta} \) decreases with \( \lambda \). When \( \mu(\gamma) \) is not differentiable at some points, equation (18) will be replaced by inequalities involving the left and right derivatives of \( \mu(\gamma) \) at the points of non differentiability.

Similarly to the static setting, whenever the cost of posting a vacancy \( k \) is positive, then \( \lambda > 0 \) and the equilibrium is away from the full information allocation\(^{24}\).

**Lemma 3** If \( k > 0 \), then the solution to problem P5 requires \( \lambda > 0 \), where \( \lambda = \hat{\lambda}/(1 - \hat{\lambda}) \) and \( \hat{\lambda} \) is the Lagrangian multiplier attached to constraint (16).

Proposition 6 shows that at each point in time the competitive search equilibrium \( \gamma \) and \( e(\theta) \) are functions only of the expected values of \( V \) and \( U \), which evolve according to the law of motions (11) and (12). Hence, the equilibrium \( U^{CE} \) and \( V^{CE} \) corresponds to a fixed point, given by

\[
U^{CE} = \beta \mu(\gamma^{CE}) \int_{\theta}^{\hat{\theta}} \left[ y - \theta \right] dF(\theta) + b - \gamma^{CE} k \left(1 - \beta \right) \left[ 1 + \frac{\beta}{1 - \beta (1 - \gamma^{CE})} F(\hat{\theta}^{CE}) \right] \tag{19}
\]

\[
V^{CE} = \frac{\beta s}{1 - \beta (1 - s)} U^{CE} \tag{20}
\]

\(^{22}\)When the constraint is not binding, the informational problem is irrelevant and the competitive search equilibrium is constrained efficient as in the standard result.

\(^{23}\)Similarly to the static setting, \( \lambda \equiv \hat{\lambda}/(1 + \hat{\lambda}) \), where \( \hat{\lambda} \) is the Lagrangian multiplier attached to the constraint (16).

\(^{24}\)The proof is analogous to the proof of Lemma 2 and therefore omitted.
where, the equilibrium $\theta^{CE}$ and $\gamma^{CE}$ solve Problem P5, for $U = U^{CE}$ and $V = V^{CE}$, that is, satisfy the binding constraint (16) and equations (17) and (18). The unemployment rate $u$, which is the only state variable of the economy, does not affect this problem. This implies that $\gamma$ and $e(\theta)$ (and $\omega(\theta)$ as well) together with $V$ and $U$, achieve the steady state values directly in the first period and stay constant over time. The transitional dynamics of the competitive search equilibrium will then be characterized uniquely by the transition of the unemployment rate.

**Steady State.** Denote the steady state competitive search equilibrium by

$$SS = \left\{ \left[ e^{CE}(\theta) \right]_{\theta \in \Theta}, \gamma^{CE}, U^{CE}, V^{CE}, u^{SS} \right\}.$$ 

In steady state not only $e(\theta)$, $\gamma$, $V$ and $U$ are constant, but also the unemployment rate $u$ is. The steady state equilibrium is given by the binding constraint (16) and equations (17) and (18) with $U = U^{CE}$ and $V = V^{CE}$, (19), (20) and

$$u^{SS} = s \left[ s + \mu \left( \gamma^{CE} \right) F(\theta^{CE}) \right]^{-1}.$$  

(21)

In the analysis of the static economy, I have shown how decreasing the worker’s outside option $b$ can generate a Pareto improvement. In the dynamic environment, the effective worker’s outside option corresponds to $U^{CE} - V^{CE}$ and is endogenously determined in equilibrium.

4 Dynamic Efficiency

In this section I explore the efficiency properties of the dynamic competitive search equilibrium. The static analysis shows that in the benchmark environment, where $b$ cannot be transferred or destroyed, the competitive search equilibrium is constrained efficient. On the contrary, in the analogous dynamic environment, the social planner internalizes an externality coming from informational imperfections. The planner can improve upon the decentralized equilibrium allocation, because of an extra instrument to provide incentives. He can reward or punish workers reporting a low type, not only through the instantaneous consumption level, but also through continuation utilities.

First, I characterize the social planning problem and show the main result of the paper: the competitive equilibrium is constrained inefficient whenever the unemployment rate is away from the steady state level. Moreover, I show that the steady state competitive search equilibrium, although it satisfies the necessary conditions for constrained efficiency, is not socially optimal according to the utilitarian welfare criterion. Finally I show that a simple time-varying tax on job creation can implements the constrained efficient allocation of resources.

\[25\text{Note that in equilibrium the continuation utility of the employed turns out to be smaller than the continuation utility of the unemployed. This is natural once I define the continuation value of the employed net from wage and disutility.}\]
4.1 Social Planning Problem

As in the static setting, the social planner controls the matching process by deciding how many vacancies to open at the beginning of each period. He does not observe the types of the matched workers and has to induce them to truthfully reveal them. Moreover, there is lack of commitment from the side of the workers, who can always decide not to produce and to consume their private non-transferable utility $b$. I impose on the planner the same anonymity restriction that I impose on the decentralized economy: the pool of unemployed workers is anonymous. Once workers decide to consume $b$ at time $t$, they can always join back the pool of unemployed at time $t + 1$. Given these constraints, together with the resource constraint of the economy, the social planner decides how to allocate intertemporally non-negative consumption among employed and unemployed workers.

An allocation is a sequence of functions $[e_t(\theta)]_{\theta \in \Theta}$ representing the hiring decision for a worker who meets an employer at the end of time $t$ and reports type $\theta$ and a sequence of functions $[c_t(\theta)]_{\theta \in \Theta}$ denoting the expected value of consumption of a worker hired at the end of time $t$ reporting type $\theta$, a sequence of consumption values for unmatched workers $C_t^U$, a sequence of consumption values $C_t^V$ for employed workers matched at time $\tau \leq t$, and a sequence of tightness values $\gamma_t$.$^{26}$ Notice that the consumption profile over time is irrelevant for the analysis, given that agents have linear utility, types are fixed over time within a match and there is no commitment problem after the match is implemented.

Denote by $V$ the continuation value of being employed, net from wages and disutility. It represents just the value of the current transfer plus the discounted expected value of being separated and becoming unemployed in the future, that is,

$$V_t = C_t^V + \beta s U_{t+1} + \beta (1 - s) V_{t+1}. \quad (22)$$

The value of being unemployed at time $t$ is instead

$$U_t = C_t^U + \beta \mu (\gamma_t) \int_{\bar{\theta}}^{\tilde{\theta}} [c_t(\theta) - e_t(\theta)(\theta + U_{t+1} - V_{t+1})] dF(\theta) + \beta U_{t+1}. \quad (23)$$

The *ad interim* utility of a matched worker of type $\theta$ reporting type $\tilde{\theta}$ at time $t$ is given by

$$v_t(\theta, \tilde{\theta}) = c_t(\tilde{\theta}) - e_t(\tilde{\theta})(\theta - V_{t+1}) + [1 - e_t(\tilde{\theta})] U_{t+1} \quad \text{for all } \tilde{\theta}, \theta \in \Theta.$$ 

A straightforward generalization of the static analysis gives that an allocation is incentive-

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$^{26}$Without loss of generality, because of linear preferences, I assume that matched workers who are not hired at time $t$ get the same consumption, $C_t^U$, of the unmatched ones.
compatible when $e(\cdot)$ is non-increasing and
\[
v_t(\theta, \tilde{\theta}) = v_t(\tilde{\theta}, \tilde{\theta}) + \int_{\theta}^{\tilde{\theta}} e_t(y) \, dy \text{ for all } \theta \in \Theta. \tag{24}
\]

Unemployed workers can choose at any point in time to stay in autarky, enjoy an instantaneous utility from leisure of value $b$, and go back to the anonymous pool of unemployed at the beginning of the following period. After some algebra, I can show that an incentive-compatible allocation satisfies also the participation constraints whenever $e(\cdot)$ is non-increasing, $C_t^U \geq b$ and
\[
\int_{\theta}^{\tilde{\theta}} \left[ c_t(\theta) - e_t(\theta) \left( \theta + \frac{F(\theta)}{f(\theta)} - V_{t+1} + U_{t+1} \right) \right] \, dF(\theta) + C_t^U - b \geq 0. \tag{25}
\]

The intertemporal resource constraint ensures that aggregate consumption is covered by aggregate output. Assume that the social planner can transfer resources intertemporally at the fixed interest rate $r = \beta^{-1} - 1$, by borrowing at the beginning of time $t$ at price $\beta$ and by paying back at the beginning of time $t+1$. I assume that the economy does not have external resources so that the intertemporal resource constraint has to hold. Define $P_t$ as the net resources of the planner at time $t$\footnote{That is, $P_t \equiv \sum_{j=t}^{\infty} \beta^j \left\{ u_j \left[ \beta \mu(\gamma_{j+1}) \int_{\theta}^{\tilde{\theta}} \left[ e_{j+1}(\theta) y - c_{j+1}(\theta) \right] dF(\theta) + b - C_{j+1}^V - \gamma_{j+1} \right] - (1 - u_t) C_{j+1}^V \right\}$.}. Then, the resource constraint can be written in recursive terms by using the state variable $P_t$ as
\[
P_t \leq u_t \left[ \beta \mu(\gamma_t) \int_{\theta}^{\tilde{\theta}} \left[ e_t(\theta) y - c_t(\theta) \right] dF(\theta) + b - C_t^U - \gamma_t k \right]
\]
\[
= (1 - u_t) C_t^V + \beta P_{t+1}
\]
for any $t$, where $u_t$ follows the law of motion
\[
u_{t+1} = u_t \left[ 1 - \mu(\gamma_t) \int_{\theta}^{\tilde{\theta}} e_t(\theta) dF(\theta) \right] + (1 - u_t) s. \tag{26}
\]

The social planner, for a given initial rate of unemployment $u_0$, chooses a Pareto optimal pair $U_0$ and $V_0$. The next definition shows how the problem can be stated in a recursive form.

**Definition 4** An allocation is constrained efficient if, for any $t$, it maximizes $U_t$, as defined in expression (23), subject to the promise-keeping constraint for $V_t$, given by (22), the law of motion of $u_t$ given by (26) and feasibility, that is, (i) incentive-compatibility (ii) participation constraint, (iii) resource constraint.
4.2 General Characterization: Dual Problem

In order to analyze the constrained efficient allocation, it is convenient to approach the social planner problem from a dual perspective. The planner Bellman equation is a function of three state variables: the promised utility to employed workers, \( V \), the promised utility to unemployed workers, \( U \), and the unemployment rate, \( u \). The planner maximizes the net resources of the economy, subject to two promise-keeping constraints for \( V \) and \( U \), the law of motion of \( u \) and incentive compatibility and individual rationality. I study a relaxed version of the problem, where I do not impose the monotonicity of \( e(\cdot) \). The relaxed planning problem can be written as

\[
P(V_t, U_t, u_t) = \max_{C_t \in \mathcal{C}_t} \left[ u_t \beta \mu (\gamma_t) \int_{\theta}^{\bar{\theta}} \left[ e_t(\theta) y - c_t(\theta) \right] dF(\theta) \right]
\]

subject to the promise-keeping constraints for \( V \) and \( U \), (22) and (23), the combination of incentive-compatibility and participation constraints which requires

\[
U_t \geq C_t^U + \beta \mu (\gamma_t) \int_{\theta}^{\bar{\theta}} \frac{F(\theta)}{f(\theta)} dF(\theta) + \beta \mu (\gamma_t) \left( b - C_t^U \right) + \beta U_{t+1},
\]

together with \( C_t^U \geq b \), and the law of motion for \( u \), (26), with associated Lagrangian multipliers, respectively, \( \nu_t, \eta_t, \xi_t, \chi_t \) and \( \pi_t \) for any \( t \). It is easy to verify that the result of the optimization gives a monotone \( e(\cdot) \), so that it solves also the full planning problem. Indeed, pointwise maximization, as in the competitive equilibrium analysis, implies that there exists a threshold \( \bar{\theta} \) such that \( e(\theta) = 1 \) iff \( \theta \leq \bar{\theta} \). Moreover, I can solve for the optimal allocation substituting for \( c(\theta) \) from the constraint (23).

Next, I show the main result of the paper: away from the steady state, a competitive equilibrium is always constrained inefficient. I assume that the social planner, as the market economy, does not have access to any external resource. Therefore, for a given \( u_0, U_0 \) and \( V_0 \) are on the Pareto frontier iff \( P(V_0, U_0, u_0) = 0 \).

**Proposition 7** If \( u_0 \neq u^{SS} \), then the competitive search equilibrium allocation is constrained inefficient.

The proof in the Appendix proceeds by contradiction. I assume that the competitive allocation solves the social planning problem and, then, I show that the necessary first order conditions are violated. In particular, the optimality condition that is violated is the one determining the level of the utility promised to the unemployed workers, that is, \( U_{t+1} \). The proof highlights the mechanism driving the inefficiency. The direction of the inefficiency depends on the dynamics of
the unemployment rate. Constrained efficiency requires to set \( U_{t+1} \) so that marginal benefits equates marginal costs. In particular, the planner needs to satisfy the following Euler equation

\[
(1 - u_t) s + [1 - \mu (\gamma_t) F(\theta_t)] u_t - \xi_t u_t - u_{t+1} + \xi_{t+1} u_{t+1} = 0. \tag{28}
\]

The first two terms on the left-hand side represent the net benefit at time \( t \) of a one unit increase of \( U_{t+1} \) for the social planner. The planner can pay one unit less in terms of consumption at time \( t \) to agents who will enjoy the extra unit of continuation utility of being unemployed at time \( t + 1 \), that is, employed workers who are separated, \((1 - u_t) s\), and unemployed workers who are not hired, \([1 - \mu (\gamma_t) F(\theta_t)] u_t\). Moreover at time \( t \), there is a cost coming from the informational problem, that is, \( \xi_t u_t \). An increase of \( U_{t+1} \) makes tighter the incentive-compatibility constraint faced by unemployed workers at time \( t \) and \( \xi_t \) represents the shadow cost of information. At time \( t + 1 \) the planner has to pay the extra unit of utility that he promised to unemployed workers \( u_{t+1} \), but there is a benefit in terms of information, \( \xi_{t+1} u_{t+1} \). The fact that the planner is paying more to the unemployed at time \( t + 1 \), reduces the informational rents that they require to be paid once hired.

The Euler condition (28) implies that the evolution of \( u_t \) affects the dynamics of the shadow cost of information. Suppose, with no loss of generality, that the unemployment rate is decreasing, that is, \( u_t > u_{t+1} \). At the optimum it must be that the "informational cost" experienced at time \( t \) is lower relative to the "informational benefit" obtained at time \( t + 1 \), that is, \( \xi_t < \xi_{t+1} \). Recall that the competitive equilibrium allocation is constant over time and that the equilibrium dynamics are characterized only by the evolution of the unemployment rate. This means that if I assume that the competitive allocation solves the social planning problem then it must be that \( \xi_t = \xi_{t+1} \) and then

\[
\frac{\partial \mathcal{L}}{\partial U_{t+1}} \geq 0 \text{ if } u_{t+1} \geq u_t,
\]

where \( \mathcal{L} \) denotes the Lagrangian associated to the planner problem. It follows that the competitive equilibrium cannot meet this optimality condition away from the steady state. When the unemployment rate is above the steady state level, that is, \( u_t > u_{t+1} \), the planner would like to reduce the workers’ outside option, \( U_{t+1} \), given that the there is a bigger mass of unemployed workers at time \( t \) relative to time \( t + 1 \). If the shadow cost of information is constant then the effective informational benefit at time \( t \) of reducing \( U_{t+1} \) is higher than the informational cost experienced at time \( t + 1 \). On the contrary, when the unemployment rate is below the steady state level, then there is a gain from increasing \( U_{t+1} \) because the effective informational cost at time \( t \) is smaller than the benefit experienced at time \( t + 1 \), given that there are more unemployed workers at time \( t + 1 \). The planner can indeed manipulate \( U_{t+1} \) by changing the future choices of vacancy creation and hiring margins.

The steady state competitive search equilibrium satisfies the necessary conditions of the social planning problem. In fact, when the mass of unemployed workers is constant over time the externality is muted. When the unemployment rate is at the steady state level, the mass of unemployed
is constant over time and the effective informational cost at time \( t \) due to an increase of \( U_{t+1} \) is perfectly offset by the effective informational benefit experienced at time \( t + 1 \).

### 4.3 An Example

In this section I propose a simple example of feasible Pareto improvement, that is, the natural dynamic generalization of the *money burning* result described in section 2.2. This example keeps the simplifying assumption that all the workers are *ex-ante* unemployed, that is, \( u_0 = 1 \), and, hence, focuses on the direction of the inefficiency associated with decreasing unemployment rate. The economy can achieve a Pareto improvement by reducing the worker’s effective outside option.

Consider the competitive search equilibrium allocation. Given that all the workers start out as unemployed, the *ex-ante* welfare coincides with the *ex-ante* value of being unemployed, that is, \( U_0 \). By Proposition (7), it follows that this allocation is constrained inefficient. When \( 1 - \mu(\gamma^{CE})F(\hat{\theta}^{CE}) < \lambda^{CE} \), the equilibrium allocation can be Pareto improved by reducing the workers’ outside option. In Guerrieri (2005) I show that there exists a set of the parameter space \((b,k,y)\) such that \( 1 - \mu(\gamma^{CE})F(\hat{\theta}^{CE}) < \lambda^{CE} \).

I now construct an example of feasible allocation that, in this case, Pareto dominates the competitive search equilibrium. Suppose \( V_t^* = V^{CE} \) for any \( t = 1, 2, ..., \) \( \hat{\theta}_t^* = \hat{\theta}^{CE} \) and \( U_t^* = U^{CE} \) for any \( t = 2, 3, ...; \gamma_t^* = \gamma^{CE} \) for any \( t = 2, 3, ... \) and \( \gamma_1^* = \gamma^{CE} + \varepsilon, U_1^* \) is given by

\[
\beta \mu(\gamma_1^*) \int_{\theta}^{\hat{\theta}^{CE}} [\alpha (y - \theta) + V^{CE} - U^{CE}] dF(\theta) + b - \gamma_1^* k + \beta U^{CE},
\]

\( \hat{\theta}_0^* \) and \( \gamma_0^* \) solve problem P2 at time 0 and the *ex-ante* welfare is given by

\[
\beta \mu(\gamma_0^*) \int_{\theta}^{\hat{\theta}_0^*} [\alpha (y - \theta) + V^{CE} - U_1^*] dF(\theta) + b - \gamma_0^* k + \beta U_1^*.
\]

First, the Envelope Condition shows that *ex-ante* welfare can be improved by reducing \( U_1^* \) whenever \( 1 - \mu(\gamma^{CE})F(\hat{\theta}^{CE}) < \lambda^{CE} \). If at the end of period 1 I perturb the competitive equilibrium level of \( \gamma \), choosing \( \gamma_1^* = \gamma + \varepsilon \), leaving everything else at the level of competitive equilibrium, then, by definition, \( U_1^* \) will be marginally lower than the competitive equilibrium level. Hence, this will increase the *ex-ante* welfare, so that for any \( \hat{\theta}_0 \) and \( \gamma_0 \) the allocation proposed is Pareto improving. In order to make the allocation also feasible I can choose \( \hat{\theta}_0^* \) and \( \gamma_0^* \) solving problem P5 for given \( V^{CE} \) and \( U_1^* \), completing the argument.

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28 Notice that unfortunately problem P6 is not concave so that I cannot state that the first order conditions are also sufficient to characterize the constrained efficient allocation.

29 This condition is a sufficient condition. Even though it is restrictive, it is useful to understand the analogy with the money burning example.
4.4 Implementation

Given that it is generically possible to improve upon the competitive equilibrium allocation of resources, a natural question arises: which policy can correct the market inefficiency? I now show that the government can implement the constrained efficient allocation by combining a time-varying tax/subsidy on vacancy posting, $\tau^V_t$, together with a time-varying tax/subsidy on job hiring, $\tau^H_t$.

Imagine that firms pay $\tau^V_t = \gamma (\text{where } \tau^V_t \text{ can be negative})$ to post a contract associated to the market tightness $\gamma$ at the beginning of time $t$ and a tax $\tau^H_t$ (which can be negative as well) if they hire a worker at the end of time $t$. The competitive equilibrium can be characterized using natural generalizations of Propositions 5 and 6. At each time $t$, agents take as given the taxes $\tau^V_t, \tau^H_t$ fixed by the government together with the continuation utilities $V_{t+1}$ and $U_{t+1}$, and solve the following problem:

$$\max_{\theta, \gamma} \beta \mu(\gamma) \int_{\theta}^{\hat{\theta}} \left[ y - \tau^H_t - \theta + V_{t+1} - U_{t+1} \right] dF(\theta) + b - \gamma k + \tau^V_t + \beta U_{t+1}$$ \hspace{1cm} (P7)$$

s.t.

$$\beta \mu(\gamma) \int_{\theta}^{\hat{\theta}} \left[ y - \tau^H_t - \theta - \frac{F(\theta)}{F(\hat{\theta})} + V_{t+1} - U_{t+1} \right] dF(\theta) \geq \gamma k - \tau^V_t.$$ \hspace{1cm} (P7)

Then, to complete the characterization of the equilibrium one has to find the sequences $\{U_t, V_t\}_{t=0}^{\infty}$ evolving according to (11) and (12), using the optimal sequences $\{C_t, \gamma_t\}_{t=0}^{\infty}$. Moreover, in order for this to be a feasible equilibrium, the sequences of taxes $\{\tau^V_t, \tau^H_t\}$ need to satisfy the intertemporal government budget constraint, that is,

$$\sum_{t=0}^{\infty} \beta^t \left[ \beta \mu(\gamma_t) F(\theta_t) u_t \tau^V_t - u_t \tau^V_t \right] = 0. \hspace{1cm} (29)$$

In order to this be the case, it must be that the $\tau_t$ is positive in some periods of time and negative in others. The government cannot tax lump-sum all the workers because the unemployed do not have any asset to be taxed. This implies that the only way to correct the market inefficiency is to make firms cross-subsidize themselves across different periods of time. By doing so, the tax makes the firms internalize the cost that they impose on other firms in different periods and, so, restores efficiency.

**Proposition 8** The constrained efficient allocation can be implemented with a non-linear tax/subsidy on vacancy posting and a linear tax/subsidy on job creation.

In order to implement the constrained efficient allocation, the government needs to subsidize job creation in periods of high unemployment rate and to tax it in periods of low unemployment rate. This allows to price correctly the externality that firms in period $t + 1$ impose on firms in period $t$, and, hence, induce the optimal behavior of the economy.
4.5 Utilitarian welfare

Finally, I consider a particular welfare criterion commonly used in the literature: the utilitarian criterion. When utility is perfectly transferable the Pareto frontier is linear and this criterion can be used without loss of generality in order to determine the efficient allocation. However, in the context of this environment, where $b$ is not transferable, the Pareto frontier is typically not linear. Nevertheless, the utilitarian welfare can represent the long run expected welfare of a worker \textit{ex-ante}. For this reason, it is interesting to notice that the competitive search equilibrium does not maximize the utilitarian welfare function, neither at the steady state, since unemployed workers are worse off than what the utilitarian planner would prescribe.

\textbf{Proposition 9} If $P$ is differentiable, then the competitive search steady state equilibrium does not maximize the utilitarian welfare function.

The proof of Proposition 7 shows that the competitive search steady state equilibrium satisfies the first order condition of the social planner problem. Then, either it is not an optimum, since the problem is not concave, and then it cannot maximize the utilitarian welfare function, or it is a point on the Pareto frontier. If this is the case, from the Envelope conditions, given that $P$ is assumed differentiable, then

\begin{align*}
P_U &= -u^{CE}(1 - \lambda^{CE}) \\
P_V &= -(1 - u^{CE}).
\end{align*}

It follows directly that the allocation does not maximize the total output of the economy, equal to $uU + (1 - u)V$, as long as $\lambda^{CE} > 0$, given that

\[
\frac{dU}{dV} = -\frac{1 - u^{CE}}{u^{CE}(1 - \lambda^{CE})} < -\frac{1 - u^{CE}}{u^{CE}}
\]

completing the proof.

5 Transferability restores Full Information.

In this section I consider the alternative environment where $b$ is freely transferable. In this case, I interpret $b$ as home production. If $b$ is high enough, the social planner can achieve the full information allocation, by transferring utility between unemployed and employed workers. In particular, given risk neutrality, the social planner could tax, at no social cost, the unemployed workers in order to finance the informational rents of matched workers, which is isomorphic to subsidize job creation and tax lump sum both unemployed and employed workers. Similarly, the full information allocation could be achieved if the planner had access to enough external resources.
Consider the full information allocation which is characterized by the hiring cut-off value $\hat{\theta}^{FI}$ such that

$$\hat{\theta}^{FI} = y - (U^{FI} - V^{FI}),$$

and $\gamma^{FI}$ implicitly defined by

$$\beta \mu' (\gamma^{FI}) \int_{\theta}^{\hat{\theta}} y - \theta - (U^{FI} - V^{FI}) dF(\theta) = k,$$

where

$$U^{FI} - V^{FI} = \frac{\beta \mu (\gamma^{FI}) \int_{\hat{\theta}}^{\theta} y - \theta dF(\theta) + b - \gamma^{FI} k}{1 - \beta (1 - s) + \beta \mu (\gamma^{FI}) F(\hat{\theta})^{FI}}.$$

When $b$ is transferable and high enough, a subsidy to job creation $\tau$ subsidized by taxing lump-sum both employed and unemployed workers, makes the competitive equilibrium reach the full information allocation. Budget balance requires that the subsidy $\tau$ is financed by a tax $T_t$ equal to $u_t (\gamma) F(\hat{\theta}) \tau$ at time $t$. Notice that this policy could also be rewritten in terms of a direct lump-sum transfer from unemployed to employed workers.

**Proposition 10** Suppose that $b$ is transferable and high enough. Then the full information allocation can be decentralized by subsidizing job creation with a lump-sum tax on workers, both employed and unemployed.

This result begs the question whether the competitive search equilibrium can achieve the full information allocation when $b$ is transferable. Indeed, in the next proposition I show that this is the case when $b$ is high enough.

**Proposition 11** Suppose that $b$ is transferable and the following inequality holds

$$b \geq \frac{k \gamma^{FI}}{\beta \mu (\gamma^{FI})},$$

then the competitive search equilibrium achieves the full information allocation.

When $b$ is transferable the optimal wage schedule takes the form

$$\omega(\theta) = \begin{cases} 
\omega(\hat{\theta}) + \hat{\theta} + U - V & \text{if } \theta \leq \hat{\theta} \\
\omega(\hat{\theta}) & \text{if } \theta > \hat{\theta}
\end{cases},$$

where firms can set a negative value for $\omega(\hat{\theta})$ as long as

$$\omega(\hat{\theta}) \geq -b. \quad (30)$$
This contract has a natural interpretation as bond posting. The firms ask workers to sign a contingent promise, after the match and before they observe the realization of the shock. Matched workers sign a promise that they will pay an application fee of value $-\omega(\tilde{\theta})$. On top of that, if they are hired, they will receive a wage of value $\tilde{\theta} + U - V$. The constraint (30) means that workers can credibly promise to pay ex-post a value not greater than the value of the home production they obtain when unemployed.\(^{30}\)

This implies that the social planner can restore the full information allocation for a larger set of parameters than competitive search can. The difference comes from the fact that firms cannot extract resources from workers who are unmatched, while the social planner can impose an ex-ante entry cost for the search market.\(^{31}\)

6 Conclusions

In this paper, I have modeled the interaction between informational imperfections and matching frictions in labor markets. My focus has been to analyze the ability of labor markets with these features to decentralize the efficient allocation of resources.

The two crucial ingredients of my model are: first, bilateral trade between employers and workers, delivered by a standard matching environment, and, second, the presence of private information on the side of the workers on the potential surplus of a match. Firms have to design contracts not only in order to attract workers, but also in order to give the correct incentives to make them reveal their information without walking away. In this setup a new type of externality arises, which can lead to inefficient equilibrium unemployment. In order to highlight the role of this externality, I have used the equilibrium notion of competitive search, which correctly internalizes the standard search externality generated by matching frictions. On the other hand, I have used general contracts under private information, that are efficient in standard competitive markets. I have shown that the combination of asymmetric information and the competition among exclusive contracts, which emerges naturally in the matching environment, drives an externality which is not internalized by competitive search. All along, I have framed the efficiency analysis in terms of constrained efficiency, by defining a social planning problem subject to the same constraints faced by the decentralized economy.

My model shows that the competitive search equilibrium is constrained inefficient whenever the unemployment rate is away from the steady state level. I also show that a simple time-varying tax/subsidy to job creation restores efficiency. A natural business cycle interpretation would suggest

\(^{30}\)I do not allow for non-negative consumption.

\(^{31}\)Though, allowing for a broader interpretation of competitive search, I could think of market makers who impose an application fee to all the workers who search for a match. This delivers a problem that is isomorphic to the one of the social planner I have described above. In this case, the competitive search equilibrium will be able to restore the full information allocation exactly for the same set of parameters, that is for $\beta b \geq k^{FL}$. 
that decentralized economies may react inefficiently to booms and to recessions. In particular, there is insufficient creation in recessions and excessive creation in booms. An interesting area for future research is to introduce aggregate shocks explicitly and to study the business cycle implications of the model. The implementation policy I propose, suggests that countercyclical subsidies may be an optimal response to business cycle shocks.
7 Appendix

Proof of Proposition 1. The proof follows closely Acemoglu and Shimer (1999a) and proceeds in two steps: step 1 shows that any equilibrium solves problem P1 and step 2 shows that any solution to P1 is part of an equilibrium.

Step 1. Let \( \{C, \Gamma, U\} \) be an equilibrium with \( C^* \in C \) and \( \gamma^* = \Gamma(C^*) \). I show that \( \{C^*, \gamma^*\} \), where \( C^* = [e^*(\theta), \omega^*(\theta)]_{\theta \in \Theta} \), solves P1. First, profit maximization ensures that \( \{C^*, \gamma^*\} \) solves constraint (2) together with IC and IR.

Suppose now that another pair \( \{C, \gamma\} \) satisfies IC, IR and achieves an higher value of the objective, that is,

\[
\mu(\gamma) \int_{\theta} \left[ \omega(\theta) - e(\theta)(\theta + b) \right] dF(\theta) + b > U,
\]

I show that it must violates constraint (2). Since \( \{C, \Gamma, U\} \) is an equilibrium, optimal job application implies

\[
\mu(\Gamma(C)) \int_{\theta} \left[ \omega(\theta) - e(\theta)(\theta + b) \right] dF(\theta) + b \leq U
\]

and given IR this implies that \( \mu(\Gamma(C)) < \mu(\gamma) \) and so \( \Gamma(C) > \gamma \). Then, combining this with profit maximization, it follows that

\[
\frac{\mu(\Gamma(C))}{\Gamma(C)} \int_{\theta} \left[ e(\theta)y - \omega(\theta) \right] dF(\theta) - k < \frac{\mu(\gamma)}{\gamma} \int_{\theta} \left[ e(\theta)y - \omega(\theta) \right] dF(\theta) - k \leq 0.
\]

This implies that \( \{C, \gamma\} \) violates (2), completing the proof of the first step.

Step 2. This step shows that for any solution \( \{C^*, \gamma^*\} \) to problem P1, there is an equilibrium \( \{C, \Gamma, U\} \) with \( C = \{C^*\} \) and \( \Gamma(C^*) = \gamma^* \). Set

\[
U = \mu(\gamma^*) \int_{\theta} \left[ \omega^*(\theta) - e^*(\theta)(\theta + b) \right] dF(\theta) + b
\]

and let \( \Gamma(C) \) satisfy

\[
U = \mu(\Gamma(C)) \int_{\theta} \left[ \omega(\theta) - e(\theta)(\theta + b) \right] dF(\theta) + b,
\]

or \( \Gamma(C) = 0 \) if either IC or IR are not satisfied. It follows that \( \{C, \Gamma, U\} \) satisfies the optimal application for jobs.

To complete the proof I now show that it also satisfies firms’ profit maximization. Suppose by contradiction that it is violated by a pair \( \{C', \gamma'\} \) which satisfies IC and IR, but such that

\[
\frac{\mu(\Gamma(C'))}{\Gamma(C')} \int_{\theta} \left[ e'(\theta)y - \omega'(\theta) \right] dF(\theta) - k > 0.
\]

Then, I can choose \( \gamma' > \Gamma(C') \) such that

\[
\frac{\mu(\gamma')}{\gamma'} \int_{\theta} \left[ e'(\theta)y - \omega'(\theta) \right] dF(\theta) - k = 0.
\]
Then, by the construction of \( \Gamma, \gamma' > \Gamma (C') \) and IR imply

\[
U < \mu (\gamma') \int \phi \left[ \omega' (\theta) - e' (\theta) (\theta + b) \right] dF (\theta) + b,
\]

so that the pair \( \{C', \gamma'\} \) satisfies all the constraints, but generates an higher value for the objective function, yielding to a contradiction.

**Proof of Proposition 1.** The proof proceeds in two steps. First I show that any pair \( \{e (\theta), \gamma\} \) solving problem P1 is also a solution to problem P2 and then I show that for any such a pair, I can construct a wage function \( \omega (\theta) \) which solves problem P1.

**Step 1.** First, from constraints IC' and IR'

\[
\int \phi v (\theta, \theta) dF (\theta) = v (\vartheta, \vartheta) + \int \phi \left[ \int \phi e (y) dy \right] dF (\theta),
\]

\[
v (\vartheta, \vartheta) \geq b.
\]

From integration by parts it follows

\[
\int \phi \left[ \int \phi e (y) dy \right] dF (\theta) = \int \phi e (\theta) F (\theta) d\theta,
\]

which combined with (31) gives

\[
\int \phi [\omega (\theta) - e (\theta) (\theta + b)] dF (\theta) + b = \int \phi e (\theta) \frac{F (\theta)}{f (\theta)} dF (\theta) + v (\vartheta, \vartheta).
\]

Using (32) I get

\[
\int \phi \left[ \omega (\theta) - e (\theta) \left( \theta + b + \frac{F (\theta)}{f (\theta)} \right) \right] dF (\theta) \geq 0.
\]

Then, a relaxed version of problem P1, where I leave to check the monotonicity of \( e (\theta) \) for the end, can be rewritten as

\[
U = \max_{\omega (\theta), \gamma, e (\theta)} \mu (\gamma) \int \phi \left[ \omega (\theta) - e (\theta) (\theta + b) \right] dF (\theta) + b
\]

s.t.

\[
\mu (\gamma) \int \phi [e (\theta) y - \omega (\theta)] dF (\theta) = \gamma k
\]

\[
\int \phi \left[ \omega (\theta) - e (\theta) \left( \theta + b + \frac{F (\theta)}{f (\theta)} \right) \right] dF (\theta) \geq 0
\]

where I can eliminate the wage from the program by using the free-entry condition, ending up with problem P2 exactly.

**Step 2.** Now I show that for any \( \{e (\theta), \gamma\} \) solving problem P2, I can construct a wage which, together with the same \( \{e (\theta), \gamma\} \), satisfies problem P1. In particular pointwise maximization gives that at the
optimum

\[ e(\theta) = \begin{cases} 
1 & \text{if } \theta \leq \hat{\theta} \\
0 & \text{if } \theta > \hat{\theta}
\end{cases} \]

where \( \hat{\theta} \) is implicitly defined by

\[ \hat{\theta} = y - b - \frac{F(\hat{\theta})}{f(\theta)}. \]

I can construct the wage schedule

\[ \omega(\theta) = \begin{cases} 
\hat{\theta} + b & \text{if } \theta \leq \hat{\theta} \\
0 & \text{if } \theta > \hat{\theta}
\end{cases} \]

which satisfies the IC constraints, completing the proof.

**Proof of Proposition 2.**

**Step 1.** Existence.
First, notice that Proposition 1 shows that for any solution \( e^*(\theta) \) and \( \gamma^* \) of problem P2 there exists a function \( \omega^*(\theta) \) such that \( \omega^*(\theta) , e^*(\theta) \) and \( \gamma^* \) are a solution of problem P1. Then the existence of a solution of problem P2 is sufficient for the existence of a solution to problem P1.

Next, to show existence of a solution to problem P2, I first show that there exists a solution to the relaxed version of P2, without assuming the monotonicity condition on \( e(\theta) \), and then I show that \( e(\theta) \) is in fact monotone, implying that there exists a solution to the original problem P2.

Pointwise maximization, together with the monotone hazard rate assumption, implies that there exists a threshold \( \hat{\theta} \) such that \( e(\theta) = 1 \) if \( \theta < \hat{\theta} \) and \( e(\theta) = 0 \) otherwise. This shows directly that \( e(\theta) \) is in fact non-increasing and that a solution to the relaxed version of problem P2 is also a solution of the original problem. This allows me to reduce the control variables to \( \hat{\theta} \) and \( \gamma \). The relaxed problem can be written as

\[
U = \max_{\hat{\theta}, \gamma} \mu(\gamma) \int_{\hat{\theta}} y - \theta - b \ dF(\theta) + b - \gamma k
\]

s.t.

\[
F(\hat{\theta}, \gamma) = \mu(\gamma) \int_{\hat{\theta}} \left[ y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) - \gamma k \geq 0.
\]

It is straightforward to see that the objective function is continuous in \( \hat{\theta} \) and \( \gamma \) and that the constraint set is compact, since \( F(\hat{\theta}, \gamma) \) is continuous in both its arguments and is not empty, since, for example, \( \gamma = 0 \) and any \( \hat{\theta} \) satisfies it. Existence follows directly.

**Step 2.** Uniqueness.

Proposition 1 shows that an equilibrium can be characterized by and array \( \hat{\theta}, \gamma \) and \( \lambda \) that must satisfy equations (4), (5) and (??). Notice that equation (4) defines implicitly \( \hat{\theta} \) as a function of \( \lambda \) with \( \partial \hat{\theta} / \partial \lambda < 0 \) which can be substituted for into equations (5) and (??). Now there are two equations in two unknown, \( \hat{\theta} \) and \( \lambda \):

\[
f_1(\gamma, \lambda) = \mu'(\gamma) \int_{\hat{\theta}} y - \theta - b - \frac{F(\theta)}{f(\theta)} dF(\theta) - k
\]

\[
f_2(\gamma, \lambda) = \frac{\mu(\gamma)}{\gamma} \int_{\hat{\theta}} \left[ y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) - k
\]
Notice that $f_1(\gamma, \lambda)$ and $f_2(\gamma, \lambda)$ define implicitly two functions, which I name $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$. Then, the implicit function theorem implies that
\[
\frac{d\gamma_1(\lambda)}{d\lambda} = -\frac{\partial f_1(\gamma, \lambda)}{\partial \lambda} \frac{\partial \gamma}{\partial \lambda} > 0 \quad \text{and} \quad \frac{d\gamma_2(\lambda)}{d\lambda} = -\frac{\partial f_2(\gamma, \lambda)}{\partial \lambda} \frac{\partial \gamma}{\partial \lambda} > 0,
\]
since
\[
y - \hat{\theta} - b - \frac{F(\hat{\theta})}{f(\hat{\theta})} < y - \hat{\theta} - b - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})} = 0.
\]
It follows that the two curves must intersect at most once. Moreover, given that I have proved existence in the previous step, they must intersect exactly at one point, completing the proof.

**Proof of Lemma 2.** The proof proceeds by contradiction. Let assume that the solution to problem P2 is an array $\hat{\theta}, \gamma$ and $\lambda$ with $\lambda = 0$. Then, Proposition 2 implies that $\hat{\theta}$ and $\gamma$ have to satisfy equations (4), (5) and (3) with $\lambda = 0$. Using (4), I obtain
\[
\frac{\mu(\gamma)}{\gamma} \int_{\hat{\theta}}^{\hat{\theta}} \left[ \hat{\theta} - \theta - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k.
\]
Integration by parts implies that
\[
\int_{\hat{\theta}}^{\hat{\theta}} \left[ \hat{\theta} - \theta \right] dF(\theta) = \int_{\hat{\theta}}^{\hat{\theta}} F(\theta) d\theta.
\]
It follows that it must be $k = 0$ yielding a contradiction and completing the proof.

**Proof of Proposition 7.** The Proof proceeds by contradiction: assume that the competitive equilibrium is constrained efficient and then show that it is impossible. The social planner problem can be rewritten, in recursive notation, as
\[
P(V, U; u) = \max_{C^U, C^V} u \beta \mu(\gamma) \int_{\hat{\theta}}^{\hat{\theta}} \left[ y - \theta - \frac{F(\theta)}{f(\theta)} + V' - U' \right] dF(\theta)
\]
\[
\text{s.t.}
\]
\[
[(1 - u) V] \quad V = C^V + \beta s U' + \beta (1 - s) V'
\]
\[
[\mu] \quad U = C^U + \beta \mu(\gamma) \int_{\hat{\theta}}^{\hat{\theta}} \frac{F(\theta)}{f(\theta)} dF(\theta) + \beta \mu(\gamma) (b - C^U) + \beta U'
\]
\[
[\beta \pi] \quad u' = u \left[ 1 - \mu(\gamma) F(\hat{\theta}) \right] + (1 - u) s
\]
\[
[x] \quad C^u \geq b
\]
Note that if the competitive equilibrium allocation $\hat{\theta}^{CE}, \gamma^{CE}, V^{CE}, U^{CE}$ is constrained efficient for a given initial value of unemployment rate $u_0$, then it must solve the social planner problem, that is, it must satisfy
the first order conditions:

\[ y - \hat{\theta} + V' - U' - (1 - \eta) \frac{F(\hat{\theta})}{f(\theta)} - \pi = 0, \]  

\[ \beta \mu' (\gamma) \int_{\hat{\theta}}^{\theta} \left[ y - \theta + V' - U' - (1 - \eta) \frac{F(\theta)}{f(\theta)} - \pi \right] dF(\theta) \]

\[ = k + \beta \mu' (\gamma) (1 - \eta) (b - C^U), \]

\[ \chi = u (1 - \eta) (1 - \mu (\gamma)), \]

\[ \nu \leq 1 \text{ when } C^V \geq 0, \]

\[ \beta P'_V + u \beta \mu (\gamma) F(\hat{\theta}) + \nu (1 - u) \beta (1 - s) = 0, \]

\[ \beta P'_U - u \beta \mu (\gamma) F(\hat{\theta}) + \nu (1 - u) \beta s + \eta u \beta = 0, \]

\[ P_u = \pi, \]

where the multipliers \( \eta, \pi, \chi, \nu \) must be such that

\[ V = C^V + \beta s U' + \beta (1 - s) V', \]

\[ U = C^U + \beta \mu (\gamma) \int_{\hat{\theta}}^{\theta} \frac{F(\theta)}{f(\theta)} dF(\theta) + \beta U', \]

\[ u' = u \left[ 1 - \mu (\gamma) F(\hat{\theta}) \right] + (1 - u) s, \]

\[ C^U \geq b. \]

Moreover, recall that when \( k > 0 \) the informational constraint is binding, that is, it must be true that

\[ \int_{\hat{\theta}}^{\theta} e(\theta) \left[ c(\theta) - \theta - \frac{F(\theta)}{f(\theta)} + V' - U' \right] dF(\theta) + C^U - b = 0 \]

and at the optimum

\[ P = u \left[ \beta \mu (\gamma) \int_{\hat{\theta}}^{\theta} \alpha (y - c(\theta)) dF(\theta) + b - \gamma k - C^U \right] - (1 - u) C^V + \beta P'. \]

Finally, the Envelope conditions are

\[ P_U = -u \eta, \]

\[ P_V = -(1 - u) \nu, \]

\[ P_u = \beta \mu (\gamma) \int_{\hat{\theta}}^{\theta} \left[ y - \theta + V' - U' - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) \]

\[ + [(1 - \beta \mu (\gamma)) (b - C^U) - \gamma k] + C^V + \beta \pi \left[ 1 - \mu (\gamma) F(\hat{\theta}) - s \right]. \]
Now guess that the steady state competitive search equilibrium \( \theta^{CE}, \gamma^{CE}, u^{CE}, V^{CE}, U^{CE}, C^U = b, C^V = 0 \) and \( B = 0 \) is a solution to this problem.

Moreover, recall that \( \dot{\theta}^{CE}, \gamma^{CE}, U^{CE}, V^{CE}, u_t^{CE} \) and the normalized multiplier \( \lambda^{CE} \) must satisfy \( C^U = b, C^V = P = 0 \) and solve the system of equations

\[
y - \dot{\theta}^{CE} + V^{CE} - U^{CE} - \lambda^{CE} \frac{F(\hat{\theta}^{CE})}{f(\hat{\theta}^{CE})} = 0, \quad (49)
\]

\[
\beta \mu' (\gamma^{CE}) \int_{\theta}^{\hat{\theta}} \left[ y - \theta + V^{CE} - U^{CE} - \lambda^{CE} \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k, \quad (50)
\]

\[
\beta \mu (\gamma^{CE}) \int_{\theta}^{\hat{\theta}} \left[ y - \theta + V^{CE} - U^{CE} - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = \gamma^{CE} k, \quad (51)
\]

\[
U^{CE} = \frac{\beta \mu (\gamma^{CE})}{1 - \beta} \left[ 1 + \frac{\beta}{1 - \beta (1 - s)} \mu (\gamma^{CE}) F(\hat{\theta}^{CE}) \right], \quad (52)
\]

\[
V^{CE} = \frac{\beta s}{1 - \beta (1 - s)} U^{CE}, \quad (53)
\]

and

\[
u_t + 1 = u_t \left[ 1 - \mu (\gamma^{CE}) F(\hat{\theta}^{CE}) \right] + (1 - u_t) s.
\]

First, combining equations (33) and (34) with equations (49) and (50), it follows

\[
\left\{ \begin{array}{l}
(1 - \eta - \lambda^{CE}) \frac{F(\hat{\theta}^{CE})}{f(\hat{\theta}^{CE})} + \pi = 0 \\
(1 - \eta - \lambda^{CE}) \int_{\theta}^{\hat{\theta}} \frac{F(\theta)}{f(\theta)} dF(\theta) + \pi \int_{\theta}^{\hat{\theta}} dF(\theta) = 0
\end{array} \right.
\]

Given that

\[
\frac{F(\hat{\theta}^{CE})}{f(\hat{\theta}^{CE})} \left[ \int_{\theta}^{\hat{\theta}} \frac{F(\theta)}{f(\theta)} dF(\theta) \right]^{-1} > 1,
\]

it must be that

\[
\eta = 1 - \lambda^{CE} \text{ and } \pi = 0.
\]

It follows that \( \chi > 0 \) and \( C^U = b \) as we assumed.

Notice that, using the Envelope condition (46), I can rewrite equation (37) as

\[
u_{t+1} (1 - u_{t+1}) = u_t \mu (\gamma) F(\hat{\theta}) + \nu_t (1 - u_t) (1 - s),
\]

which for \( \nu_{t+1} = \nu_t = \nu^{CE} \) and \( \gamma = \gamma^{CE}, \hat{\theta} = \hat{\theta}^{CE} \), using the law of motion for \( u \), yields

\[
(1 - \nu^{CE}) [u_{t+1} - u_t - (1 - u_t) s] = 0,
\]

implying \( \nu^{CE} = 1 \), as long as job creation is different from zero. Moreover, using the Envelope condition
(47), I can rewrite equation (38) as
\[ (1 - \eta_{t+1}) u_{t+1} = (1 - \eta_t) u_t + (1 - \nu_t) (1 - u_t) s, \]
which, when \( \nu^{CE} = 1, \gamma = \gamma^{CE}, \hat{\theta} = \hat{\theta}^{CE}, 1 - \eta_{t+1} = 1 - \eta_t = \lambda^{CE}, \) using the law of motion of \( u, \) yields
\[ (u^{CE}_t - u^{CE}_{t+1}) \lambda^{CE} = 0. \]

A contradiction follows immediately as long as the unemployment rate is not at the steady state value and \( \lambda^{CE} \) is different from zero, which is the case whenever \( k > 0, \) as we assumed. This implies that, away from the steady state, the competitive search equilibrium is constrained inefficient completing the proof.

**Proof of Proposition 8.** In order to prove that the optimal allocation can be implemented using a non-linear tax on vacancy posting and a linear tax on job creation, I need to show that it is a competitive equilibrium when agents take as given a pair of sequences of taxes \( \{\tau^V_t, \tau^H_t\} \) fixed by the government and the government budget constraint is satisfied.

First, consider the optimal allocation \( \hat{\theta}(U,V,u), \gamma(U,V,u), C^U(U,V,u), C^V(U,V,u), V'(U,V,u), U'(U,V,u), u'(U,V,u), \) where from now on I drop the argument, wherever this does not cause any confusion. It must satisfy the first-order conditions of the social planner problem P6, that is, equations (33)-(39) and the Envelope conditions (46)-(48), where the multipliers \( \eta, \pi, \chi, \nu \) must be such that (40)-(43) hold. Moreover, recall that the economy start with zero external resources, that is, \( P_0 = 0. \) Next, I show two features of the optimal allocation that are relevant for this proof: \( C^U = b \) and \( C^V = 0. \) First, notice that condition (36) implies that \( \nu \leq 1. \) Moreover by combining condition (38) with the Envelope condition (46) and using equation (42), I obtain
\[ (1 - \nu) (1 - u) s = (1 - \eta) u'. \]

This immediately implies that \( \eta \leq 1. \) If \( \eta \) is equal to 1 the equilibrium is constrained efficient and zero taxes on both vacancy posting and job creation are optimal. Let me focus on an interesting problem where \( \eta < 1. \) Then equation (54) immediately implies that \( \nu < 1. \) This implies by equation (35), that \( \chi > 0 \) and, by complementary slackness, that \( C^U = b. \) Moreover, \( \nu < 1 \) implies that \( C^V = 0. \)

Now, I need to show that there exist a pair of sequences of taxes \( \{\tau^V_t, \tau^H_t\} \) such that the intertemporal government budget constraint (29) is satisfied and such that the optimal allocation is a competitive equilibrium. Following Proposition 5, first I show that choosing the sequences of continuation utilities for unemployed and employed workers \( \{U_t, V_t\} \) equal to the optimal ones, the equilibrium \( \hat{\theta}_t, \gamma_t \) solve problem P7 for each time \( t. \) The first-order conditions coming from P7 are
\[ y - \tau^H_t - \hat{\theta}_t - \lambda_t \frac{F(\hat{\theta}_t)}{f(\hat{\theta}_t)} - U^{SP}_{t+1} + V^{SP}_{t+1} = 0 \]
\[ \beta \mu' (\gamma_t) \int_0^\theta \left[ y - \tau^H_t - \theta - \lambda_t \frac{F(\theta)}{f(\theta)} - U^{SP}_{t+1} + V^{SP}_{t+1} \right] dF(\theta) = k \]
and the binding constraint
\[
\beta \mu (\gamma_t) \int_0^{\hat{\theta}_t} \left[ y - \tau_t^H - \theta - \frac{F(\theta)}{f(\theta)} - U_{i+1}^{SP} + V_{i+1}^{SP} \right] dF(\theta) = \gamma_t k - \tau_t^Y. \tag{57}
\]

Given that \( C^U = b \) and \( C^V = 0 \), the optimal allocation satisfies equations (55) and (56) for
\[
\lambda_t = 1 - \eta_t \quad \text{and} \quad \tau_t^H = \pi_t. \tag{58}
\]
Moreover it satisfies equation (57) for
\[
\tau_t^Y = \gamma_t k - \beta \mu (\gamma_t) \int_0^{\hat{\theta}_t} \left[ y - \tau_t^H - \theta - \frac{F(\theta)}{f(\theta)} - U_{i+1}^{SP} + V_{i+1}^{SP} \right] dF(\theta). \tag{59}
\]

Then, given the optimal \( \gamma \) and \( \hat{\theta} \), the sequences \( \{U_t, V_t\} \) solve the law of motions (11) and (12) by construction.

Finally I need to check only that the intertemporal government budget constraint (29) is satisfied. Given that the taxes have to satisfy (58) and (59), it follows that
\[
\sum_{t=0}^{\infty} \beta^t \left[ \beta \mu (\gamma_t) F(\hat{\theta}_t) u_t \tau_t - u_t \tau_t^Y \right] = \sum_{t=0}^{\infty} \beta^t u_t \left[ \beta \mu (\gamma_t) \int_0^{\hat{\theta}_t} \left[ y - \theta - \frac{F(\theta)}{f(\theta)} + V_{i+1} - U_{i+1} \right] dF(\theta) - \gamma_t k \right],
\]
and the budget constraint is immediately satisfied given that
\[
F_0 = \sum_{t=0}^{\infty} \beta^t u_t \left[ \beta \mu (\gamma_t) \int_0^{\hat{\theta}_t} \left[ y - \theta - \frac{F(\theta)}{f(\theta)} + V_{i+1} - U_{i+1} \right] dF(\theta) - \gamma_t k \right] = 0,
\]
completing the proof.

**Proof of Proposition 10.** Assume that the government gives a subsidy \( \tau_t \) to firms hiring at time \( t \), by taxing lump-sum employed and unemployed workers. Budget balance imposes that the subsidy \( \tau_t \) is covered by a tax \( T_t \) equal to \( \tau_t \mu (\gamma) \) \( F(\hat{\theta}) \). Then, the equilibrium has to satisfy
\[
U_t = \beta \mu (\gamma) \int_0^{\hat{\theta}_t} [y + \tau_t - \theta + V_{i+1} - U_{i+1}] dF(\theta) + b - \tau_t u_t \mu (\gamma) F(\hat{\theta}) - \gamma k + \beta U_{i+1}
\]
and
\[
V_t = -\tau_t u_t \mu (\gamma) F(\hat{\theta}) + \beta s U_{i+1} + \beta (1 - s) V_{i+1},
\]
where
\[
u_{i+1} = u_t \left[ 1 - \mu (\gamma) F(\hat{\theta}) \right] + (1 - u_t) s.
\]
Consider the full information allocation. From the informational constraint, there must exist an \( H \) such
that
\[ \beta \mu(\gamma^{FI}) \int_{\theta}^{\bar{\theta}} \left[ y - \theta + H - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = \gamma^{FI} k \]
so that a competitive equilibrium allocation reaching the full information outcome requires \( V_t - U_t \geq H \) \( \forall t \).

Then, imposing that \( V_t - U_t = H \) for any \( t \) and combining the equations above it follows that the full information allocation is achievable in competitive equilibrium if the government can sustain a constant value of job creation subsidy \( \tau \) such that
\[ H = -\beta \mu(\gamma^{FI}) \int_{\theta}^{\bar{\theta}} \left[ y + \tau - \theta + H \right] dF(\theta) - b + \gamma^{FI} k + \beta (1 - s) H, \]
by imposing the following time-varying lump-sum tax on workers:
\[ T_t = u_t \left[ \beta \mu(\gamma^{FI}) \int_{\theta}^{\bar{\theta}} (y - \theta) dF(\theta) + b - \gamma^{FI} k \right] 
+ u_t \left[ 1 - \beta (1 - s) + \beta \mu(\gamma^{FI}) F(\theta^{FI}) \right] H. \]

When \( b \geq T_t \) for any \( t \), the policy is feasible completing the proof.

**Proof of Proposition 11.** When \( b \) is transferable, the competitive search equilibrium can be defined exactly as in Proposition 5, except for the participation constraints IR which now become
\[ \nu_t(\bar{\theta}, \bar{\theta}) \geq U_{t-1} - b \text{ for all } \theta \in \Theta. \]

Hence, the constraint in problem P5 can be replaced by
\[ \beta \mu(\gamma) \int_{\theta}^{\bar{\theta}} e(\theta) \left[ y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) + \beta \mu(\gamma) b \geq \gamma k. \]
Substituting the full information values for \( \gamma \) and \( e(\theta) \) I obtain
\[ \beta \mu(\gamma) \int_{\theta}^{\bar{\theta}} e(\theta) \left[ \theta - \theta - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) + \beta \mu(\gamma) b \geq \gamma k. \]
Integrating by parts, it follows that the full information allocation can be implemented whenever
\[ \beta \mu(\gamma^{FI}) b \geq k \gamma^{FI}, \]
completing the proof.
References


