Triangle-free graphs with uniquely restricted maximum matchings and their corresponding greedoids

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Abstract

A matching $M$ is uniquely restricted in a graph $G$ if its saturated vertices induce a subgraph which has a unique perfect matching, namely $M$ itself [M.C. Golumbic, T. Hirst, M. Lewenstein, Uniquely restricted matchings, Algorithmica 31 (2001) 139–154]. $G$ is a König–Egerváry graph provided $\chi(G) + \mu(G) = |V(G)|$ [R.W. Deming, Independence numbers of graphs—an extension of the König–Egerváry theorem, Discrete Math. 27 (1979) 23–33; F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number, J. Combin. Theory Ser. B 27 (1979) 228–229], where $\chi(G)$ is the size of a maximum matching and $\mu(G)$ is the cardinality of a maximum stable set. $S$ is a local maximum stable set of $G$, and we write $S \in \Psi(G)$, if $S$ is a maximum stable set of the subgraph spanned by $S \cup N(S)$, where $N(S)$ is the neighborhood of $S$. Nemhauser and Trotter [Vertex packings: structural properties and algorithms, Math. Programming 8 (1975) 232–248], proved that any $S \in \Psi(G)$ is a subset of a maximum stable set of $G$. In [V.E. Levit, E. Mandrescu, Local maximum stable sets in bipartite graphs with uniquely restricted maximum matchings, Discrete Appl. Math. 132 (2003) 163–174] we have proved that for a bipartite graph $G$, $\Psi(G)$ is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted. In this paper we demonstrate that if $G$ is a triangle-free graph, then $\Psi(G)$ is a greedoid if and only if all its maximum matchings are uniquely restricted and for any $S \in \Psi(G)$, the subgraph spanned by $S \cup N(S)$ is a König–Egerváry graph.

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1. Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. We also denote by $G - F$ the partial subgraph of $G$ obtained by deleting the edges of $F$, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$. If $A, B \subset V$ are disjoint and non-empty, then by $(A, B)$ we mean the set $\{ab : ab \in E, a \in A, b \in B\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$. If $|N(v)| = 1$, then $v$ is a pendant vertex of $G$ and by pend($G$) we designate the set of all pendant vertices of $G$. We denote the neighborhood of $A \subset V$ by $N_G(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$ and its closed neighborhood by $N_G[A] = A \cup N(A)$, or shortly, $N(A)$ and $N[A]$, if no ambiguity. $K_n, C_n$ denote,

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A maximal matching

Theorem 1.1

the graph $H$ contain pendant vertices. For instance, such that $G$ is stable set.

Let $S \in \Omega(G)$ be a maximum stable set in the subgraph spanned by $N[A]$, i.e., $A \subseteq V(G)$ is a local maximum stable set of $G$ if $A$ is a maximum stable set in the subgraph spanned by $N[A]$, [17]. Let $\Psi(G)$ stand for the set of all local maximum stable sets of $G$.

Clearly, every set $S \subseteq \text{pend}(G)$ belongs to $\Psi(G)$. Nevertheless, it is not a must for a local maximum stable set to contain pendant vertices. For instance, $\{e, g\} \in \Psi(G)$, where $G$ is the graph from Fig. 1.

There exist graphs (called well-covered, [26,27]), where each stable set is contained in a maximum stable set, e.g., the graph $H$ from Fig. 2. However, most of the graphs are not well-covered. Since there is no maximum stable set $S$ of $G$ such that $\{b, d\} \subseteq S$, the graph $G$ in Fig. 2 is not well-covered.

The following theorem concerning maximum stable sets in general graphs, due to Nemhauser and Trotter [24], shows that some stable sets can be enlarged to maximum stable sets.

**Theorem 1.1 (Nemhauser and Trotter [24]).** Every local maximum stable set of a graph is a subset of a maximum stable set.

Let us notice that the converse of Theorem 1.1 is trivially true, because $\Omega(G) \subseteq \Psi(G)$. The graph $W$ from Fig. 1 has the property that any $S \in \Omega(W)$ contains some local maximum stable set, but these local maximum stable sets are of different cardinalities: $\{a, d, f\} \in \Omega(W)$ and $\{a\}, \{d, f\} \in \Psi(W)$, while for $\{b, e, g\} \in \Omega(W)$ only $\{e, g\} \in \Psi(W)$.

However, there exists a graph $G$ satisfying $\Psi(G) = \Omega(G)$, e.g., $G = C_n$, for $n \geq 4$.

A matching in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of $M$ share a common vertex. A maximal matching is a matching $M$ of $G$ with the property that if any edge not in $M$ is added to $M$, it is no longer a matching. A maximum matching is a matching of maximum size, denoted by $\mu(G)$. Note that every maximum matching is also maximal, but not every maximal matching must be maximum. A matching is perfect if it saturates all the vertices of the graph. Let us recall that $G$ is a König–Egerváry graph provided $\zeta(G) + \mu(G) = |V(G)|$ [4,28]. As a well-known example, any bipartite graph is a König–Egerváry graph [5,12]. Properties of König–Egerváry graphs were discussed in a number of papers, e.g. [2,10,11,15,19–21,25].

Let us notice that if $S$ is a stable set and $M$ is a matching in a graph $G$ such that $|S| + |M| = |V(G)|$, it follows that $S \in \Omega(G)$, $M$ is a maximum matching, and $G$ is a König–Egerváry graph, because $|S| + |M| \leq \zeta(G) + \mu(G) \leq |V(G)|$ is true for any graph.

A cycle $C$ is $M$-alternating if for any two incident edges of $C$ exactly one of them belongs to the matching $M$, (see [14]). It is clear that an $M$-alternating cycle should be of even size. The matching $M$ in $G$ is called alternating cycle-free if $G$ has no $M$-alternating cycle. Alternating cycle-free matchings for bipartite graphs were first defined in [14], where these matchings appear in some matroidal problems, and in [9] as a tool for generating all the maximum matchings of a bipartite graph. This kind of matchings was also investigated in connection with the so-called jump-number problem for partially ordered sets (see [3,22,23]).
A matching \( M = \{aibi : ai, bi \in V(G), 1 \leq i \leq k\} \) of a graph \( G \) is called a uniquely restricted matching if \( M \) is the unique perfect matching of \( G[[ai, bi : 1 \leq i \leq k]] \), [7]. For bipartite graphs, this notion was first introduced in [14] under the name clean matching. It appears also in the context of matrix theory, as a constrained matching (see [8]). Recently, a generalization of this concept, namely, a subgraph restricted matching has been studied in [6].

Kroghdal found that a matching \( M \) of a bipartite graph is uniquely restricted if and only if \( M \) is alternating cycle-free (see [14]). This statement was observed for general graphs by Golumbic et al. [7].

A greedoid is a type of set system generalizing the notion of matroid.

**Definition 1.2 (Björner and Ziegler [1], Korte et al. [13]).** A greedoid is a pair \((E, \mathcal{F})\), where \( \mathcal{F} \subseteq 2^E \) is a non-empty set system satisfying the following conditions:

- **(Accessibility)** for every non-empty \( X \in \mathcal{F} \) there is an \( x \in X \) such that \( X - \{x\} \in \mathcal{F} \);
- **(Exchange)** for \( X, Y \in \mathcal{F}, |X| = |Y| + 1 \), there is an \( x \in X - Y \) such that \( Y \cup \{x\} \in \mathcal{F} \).

It is worth observing that if \( \Psi(G) \) is a greedoid and \( S \in \Psi(G), |S| = k \geq 2 \), then by accessibility property, there is a chain

\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, \ldots, x_{k-1}\} \subset \{x_1, \ldots, x_k\} = S,
\]

such that \( \{x_1, x_2, \ldots, x_j\} \in \Psi(G) \), for all \( j \in \{1, \ldots, k - 1\} \). Such a chain we call an accessibility chain of \( S \). For example, \( \{a\} \subset \{a, b\} \subset \{a, b, c\} \) is an accessibility chain of the set \( S = \{a, b, c\} \in \Psi(G_2) \), where \( G_2 \) is presented in Fig. 3.

In [18] we have proved the following result.

**Theorem 1.3.** For a bipartite graph \( G \), \( \Psi(G) \) is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted.

The case of bipartite graphs owning a unique cycle, whose family of local maximum stable sets forms a greedoid is analyzed in [16] (see, for example, the graph \( G_1 \) from Fig. 3). Clearly, for a forest \( T \), the family \( \Psi(T) \) forms a greedoid, because all its maximum matchings are uniquely restricted. This result had been proved directly in [17].

There is no simple way to generalize Theorem 1.3. In other words, there exist non-bipartite graphs with or without uniquely restricted maximum matchings, whose families of local maximum stable sets form greedoids. For instance, the families \( \Psi(G_2), \Psi(G_3), \Psi(G_4) \) of the graphs in Fig. 3 are greedoids. Notice that all maximum matchings of \( G_2 \) are uniquely restricted, \( G_3 \) has both uniquely restricted maximum matchings and non-uniquely restricted maximum matchings, and all maximum matchings of \( G_4 \) are not uniquely restricted. Observe also that \( G_1 \) and \( G_2 \) are König–Egerváry graphs, while only \( G_1 \) is bipartite.

In this paper, we characterize the triangle-free graphs whose family of local maximum stable sets are greedoids. Namely, we demonstrate that for a triangle-free graph \( G \), the family \( \Psi(G) \) is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted and for every \( S \in \Psi(G) \), the subgraph spanned by \( S \cup N(S) \) is a König–Egerváry graph.

### 2. Preliminary results

As we will see in the sequel, triangle-free graphs whose families of local maximum stable sets form a greedoid have to be König–Egerváry graphs. This section discusses various properties of maximum matchings in König–Egerváry graphs, which play a key role in achieving the main goals of the paper.
The following result shows that, under certain conditions, a matching of a König–Egerváry graph can be extended to a maximum matching $M$ in $G$ such that it follows that to some vertex in $S$ between the maximum matching in $G$ includes instance, the graph $G$ is a König–Egerváry graph and Lemma 2.4. If $G$ is a bipartite graph $\cup$ see in the following result.

In a König–Egerváry graph, maximum matchings have a special property, emphasized by the following statement.

**Lemma 2.1** (Levit and Mandrescu [20]). In a König–Egerváry graph, the $\alpha$-critical edges are also $\mu$-critical, and they coincide in a bipartite graph.

In a König–Egerváry graph, maximum matchings have a special property, emphasized by the following statement.

**Lemma 2.2** (Levit and Mandrescu [19]). Every maximum matching $M$ of a König–Egerváry graph $G$ is contained in each $(S, V(G) - S)$ and $|M| = |V(G) - S|$, where $S \in \Omega(G)$.

Clearly, every matching can be enlarged to a maximal matching, which is not necessarily a maximum matching. For instance, the graph $G_1$ in Fig. 4 does not contain any maximum matching including the matching $M = \{e_1, e_2\}$. The following result shows that, under certain conditions, a matching of a König–Egerváry graph can be extended to a maximum matching.

**Lemma 2.3.** If $G$ is a König–Egerváry graph, $\hat{S} \in \Psi(G)$, $H = G[N(\hat{S})]$ is also a König–Egerváry graph, and $\hat{M}$ is a maximum matching in $H$, then there exists a maximum matching $M$ in $G$ such that $\hat{M} \subseteq M$.

**Proof.** Let $G = (V, E)$. According to Theorem 1.1, there is a stable set $S'$ in $G$ such that $S = \hat{S} \cup S' \in \Omega(G)$. Since $H$ is a König–Egerváry graph and $\hat{M}$ is a maximum matching in $H$, it follows that

$$|\hat{S}| + |\hat{M}| = \alpha(H) + \mu(H) = |V(H)| = |\hat{S}| + |N(\hat{S})|,$$

i.e., $|\hat{M}| = |N(\hat{S})|$. Let $M$ be a maximum matching in $G$. Then, by Lemma 2.2 we get

$$|M| = |V - S| = |N(\hat{S})| + |N(S') - N(\hat{S})| = |\hat{M}| + |V - S - N(\hat{S})|.$$ 

Let $M'$ be the subset of $M$ containing edges having an endpoint in $V - S - N(\hat{S})$. Since no edge joins a vertex of $\hat{S}$ to some vertex in $V - S - N(\hat{S})$, it follows that $M'$ is the restriction of $M$ to $G[V - S - N(\hat{S})]$. Consequently, $\hat{M} \cup M'$ is a matching in $G$ that contains $\hat{M}$, and because

$$|\hat{M} \cup M'| = |\hat{M}| + |V - S - N(\hat{S})| = |M|,$$

it follows that $\hat{M} \cup M'$ is a maximum matching in $G$. $\square$

Since any subgraph of a bipartite graph is also bipartite, we obtain the following result.

**Corollary 2.4.** If $G$ is a bipartite graph, $\hat{S} \in \Psi(G)$ and $\hat{M}$ is a maximum matching in $G[N(\hat{S})]$, then there exists a maximum matching $M$ in $G$ such that $\hat{M} \subseteq M$.

Let us notice that Lemma 2.3 cannot be generalized to any subgraph of a non-bipartite König–Egerváry graph. For instance, the graph $G$ depicted in Fig. 5 is a König–Egerváry graph, $\hat{S} = \{a, c, f\} \in \Psi(G)$, and $\hat{M} = \{ab, cd, fh\}$ is a maximum matching in $G[N(\hat{S})]$, which is not a König–Egerváry graph, but there is no maximum matching in $G$ that includes $\hat{M}$.
The following lemma shows that every König–Egerváry graph with a unique perfect matching has at least one pendant vertex (see, for example, the graph $G_1$, depicted in Fig. 6). Notice that there exist non-König–Egerváry graphs having a unique perfect matching with or without pendant vertices (see, for instance, the graphs $G_2, G_3$ from Fig. 6).

**Lemma 2.5.** If $G = (V, E)$ is a König–Egerváry graph having a unique perfect matching, then $S \cap \text{pend}(G) \neq \emptyset$ holds for every $S \in \Omega(G)$.

**Proof.** Let $M = \{a_i b_i : 1 \leq i \leq \mu(G)\}$ be the unique perfect matching of $G = (V, E)$ and $S \in \Omega(G)$. Since $G$ is a König–Egerváry graph, it follows that $|M| = \mu(G) = \alpha(G) = |S|$. By Lemma 2.2, $M \subseteq (S, V - S)$ and, therefore, we may assume that $S = \{a_i : 1 \leq i \leq \mu(G)\}$.

Suppose that $S \cap \text{pend}(G) = \emptyset$. Hence, $|N(a_i)| \geq 2$ for any $a_i \in S$. Under these conditions, we shall build some cycle $C$ having half of edges contained in $M$. We begin with the edge $a_1 b_1$; since $|N(a_1)| \geq 2$, there is some $b \in (V - S - \{b_1\}) \cap N(a_1)$, say $b_2$. We continue with $a_2 b_2 \in M$. Further, $N(a_2)$ contains some $b \in (V - S - \{b_2\})$. If $b_1 \in N(a_2)$, we are done, because $G[\{a_1, a_2, b_1, b_2\}] = C_4$. Otherwise, we may suppose that $b = b_3$, and we add to the growing cycle the edge $a_3 b_3$. Since $G$ has a finite number of vertices, after a number of edges from $M$, we must find some edge $a_k b_j$ having $1 \leq j < k$. So, the cycle $C$ we found has

$$V(C) = \{a_i, b_i : j \leq i \leq k\},$$

$$E(C) = \{a_i b_i : j \leq i \leq k\} \cup \{a_i b_{i+1} : j \leq i < k\} \cup \{a_k b_j\}.$$

Clearly, half of edges of $C$ are contained in $M$, and this allows us to find a new perfect matching in $G$, which contradicts the hypothesis on the uniqueness of $M$. □

3. Main results

Let us notice that there are graphs, having unique perfect matchings, where maximum stable sets do not admit an accessibility chain (see, for example, the graph $G_1$ from Fig. 4).

**Lemma 3.1.** If $G = (V, E)$ is a König–Egerváry graph having a unique perfect matching, then every $S \in \Omega(G)$ has an accessibility chain.

**Proof.** Since $G$ has a perfect matching and it is a König–Egerváry graph, $\alpha(G) = \mu(G) = n$, where $|V(G)| = 2n$.

We prove by induction on $n$ that every $S \in \Omega(G)$ has an accessibility chain.

For $n = 1$, the assertion is clearly true.

For $n = 2$, let $S = \{x_1, x_2\} \in \Omega(G)$, $N(S) = \{y_1, y_2\}$ and $x_1 y_1, x_2 y_2 \in M$, where $M$ is its unique perfect matching. Then, by Lemma 2.5, at least one of $x_1, x_2$ is pendant, say $x_1$. Hence, the chain is $\{x_1\} \subset \{x_1, x_2\} = S$.

Suppose that the assertion is true for $k < n$.

Let $G = (V, E)$ be a König–Egerváry graph of order $2n$, $M = \{a_i b_i : 1 \leq i \leq \mu(G)\}$ be its unique perfect matching, and $S \in \Omega(G)$. 
of local maximum stable sets are greedoids. For example, none of the graphs $G_i$, $i = 1, 2, 3$, are greedoids. For instance, the graphs $G_1$, $G_3$, $G_5$ in Fig. 7 are non-bipartite König–Egerváry graphs with unique perfect matchings, while $G_2$ is a König–Egerváry graph having a unique perfect matching, namely $M_1 = \{e_1, e_4\}$ as a uniquely restricted maximum matching, but is not uniquely restricted. The non-bipartite König–Egerváry graph $G$ depicted in Fig. 8 has $M_1 = \{ab, cd, ux, vy, zt\}$, $M_2 = \{be, cd, ux, vy, zt\}$ as maximum matchings, but only $M_1$ is uniquely restricted.

According to Lemma 2.2, $M \subseteq (S, V - S)$, and by Lemma 2.5, we may assume that $a_1 \in S \cap \text{pend}(G)$. Clearly, $H = G - \{a_1, b_1\}$ is a König–Egerváry graph having a unique perfect matching, namely $M_H = M - \{a_1b_1\}$.

Hence, $S_{n-1} = S - \{a_1\} \in \Omega(H)$, and by induction hypothesis, there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{n-2}\} \subset \{x_1, x_2, \ldots, x_{n-1}\} = S_{n-1}$$

such that $\{x_1, x_2, \ldots, x_k\} \in \Psi(H)$ for any $k \in \{1, \ldots, n - 1\}$. Since $N_G(a_1) = \{b_1\}$, it follows that

$$N_G(\{x_1, x_2, \ldots, x_k\} \cup \{a_1\}) = N_H(\{x_1, x_2, \ldots, x_k\}) \cup \{b_1\},$$

and therefore $\{x_1, x_2, \ldots, x_k\} \cup \{a_1\} \in \Psi(G)$ for any $k \in \{1, \ldots, n - 1\}$. Clearly, $\{a_1\} \in \Psi(G)$, and consequently, we obtain the chain:

$$\{a_1\} \subset \{a_1, x_1\} \subset \{a_1, x_1, x_2\} \subset \cdots \subset \{a_1, x_1, x_2, \ldots, x_{n-2}\}$$

$$\subset \{a_1, x_1, x_2, \ldots, x_{n-1}\} = \{a_1\} \cup S_{n-1} = S,$$

and $\{a_1, x_1, x_2, \ldots, x_k\} \in \Psi(G)$ for any $k \in \{1, \ldots, n - 1\}$. □

As an example, let us consider the König–Egerváry graph $C_5 + e$ in Fig. 7, that evidently has a unique perfect matching. For $\{x, u, v\} \in \Omega(C_5 + e)$ the chain of local maximum stable sets is $\{x\} \subset \{x, u\} \subset \{x, u, v\}$. Notice that such a chain does not exist for every $S \in \Psi(C_5 + e)$; e.g., $\{y, z\} \in \Psi(C_5 + e)$, but $\{y\}, \{z\} \not\in \Psi(C_5 + e)$.

In [18] we have proved that if $G$ is a bipartite graph having a unique perfect matching, then $\Psi(G)$ is a greedoid. However, there are non-bipartite graphs with unique perfect matchings, whose families of local maximum stable sets are not greedoids. For instance, the graphs $C_5 + e, C_5 + 3e$ in Fig. 7 are non-bipartite König–Egerváry graphs with unique perfect matchings, $\Psi(C_5 + 3e)$ is a greedoid, while $\Psi(C_5 + e)$ is not a greedoid, because $\{u, v\} \in \Psi(C_5 + e)$, but $\{u\}, \{v\} \not\in \Psi(C_5 + e)$.

Let us also notice that there exist both bipartite and non-bipartite graphs without perfect matchings, whose families of local maximum stable sets are greedoids. For example, none of the graphs $G_i, 1 \leq i \leq 3$, in Fig. 3 has a perfect matching, $G_1$ is bipartite, and all $\Psi(G_i), 1 \leq i \leq 3$, are greedoids.

If one of the maximum matchings of a graph is uniquely restricted, this is not necessarily true for all its maximum matchings. For instance, the bipartite graph $H$ from Fig. 8 has $M_1 = \{e_1, e_4\}$ as a uniquely restricted maximum matching, while $M_2 = \{e_2, e_3\}$ is a maximum matching, but is not uniquely restricted. The non-bipartite König–Egerváry graph $G$ depicted in Fig. 8 has $M_1 = \{ab, cd, ux, vy, zt\}$, $M_2 = \{be, cd, ux, vy, zt\}$ as maximum matchings, but only $M_1$ is uniquely restricted.

The following lemma shows that in a triangle-free graph the existence of an accessibility chain is equivalent to the fact that one can have a chain of stable sets, where each additional vertex added to the stable set increases the size of the open neighborhood by at most one element.

**Lemma 3.2.** If $A = B \cup \{v\}$ is a stable set in a triangle-free graph $G$, and $B \in \Psi(G)$, then $A \in \Psi(G)$ if and only if $|N(A)| \leq |N(B)| + 1$. 
Proof. Assume that $A = B \cup \{v\} \in \Psi(G)$. Since $G$ is triangle-free, it follows that

$$|N(A)| - |N(B)| = |N(A) - N(B)| = |N(v) - N(B)| \leq 1,$$

because otherwise, if $\{x, y\} \subset N(v) - N(B)$, then $\{x, y\} \cup B$ is a stable set in $N[B \cup \{v\}]$, and larger than $A = B \cup \{v\}$, in contradiction to the assumption that $A \in \Psi(G)$.

Conversely, since $|N(A)| \leq |N(B)| + 1$, we get $\alpha(G[N[v] - N(B)]) = 1$. The fact that $B \in \Psi(G)$ implies $\alpha(G[N[B]]) = |B|$.

It is clear that

$$N[A] = N[B \cup \{v\}] = (N[v] - N(B)) \cup N[B].$$

Consequently, we obtain $|A| = \alpha(G[N[A]])$, because

$$|A| \leq \alpha(G[N[A]]) \leq \alpha(G[N[v] - N(B)]) + \alpha(G[N[B]]) = 1 + |B| = |A|.$$

Thus $A$ is a local maximum stable set of $G$. □

The next finding claims that in a triangle-free graph $G$, the existence of an accessibility chain forces the existence of two extra chains:

- a chain of König–Egerváry subgraphs $H_i$, $1 \leq i \leq \alpha(G)$, of $G$ (including $G$ itself);
- a chain of uniquely restricted maximum matchings, one in each $H_i$.

**Lemma 3.3.** If $G$ is a triangle-free graph and some $S \in \Omega(G)$ has an accessibility chain

$$\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{j-1}\} \subset \{x_1, x_2, \ldots, x_2\} = S,$$

then the following assertions are true:

(i) $G[N[\{x_1, x_2, \ldots, x_i\}]]$ is a König–Egerváry graph for any $i \in \{1, \ldots, \alpha(G)\}$. In particular, $G$ itself is a König–Egerváry graph.

(ii) $G$ has a uniquely restricted maximum matching.

**Proof.** Let us denote $S_i = \{x_1, x_2, \ldots, x_i\}$ and $S_0 = \emptyset$.

Since $S_{i-1} \in \Psi(G)$, $S_i = S_{i-1} \cup \{x_i\} \in \Psi(G)$ and $G$ is triangle-free, Lemma 3.2 implies that $|N(x_i) - N[S_{i-1}]| \leq 1$, because $|N(x_i) - N[S_{i-1}]| = |N(S_i) - N[S_{i-1}]| = |N[S_i]| - |N[S_{i-1}]|$.

Let $I = \{i : 1 \leq i \leq \alpha, |N(x_i) - N[S_{i-1}]| = 1\}$ and $\{y_i\} = N(x_i) - N[S_{i-1}], i \in I$. Hence, $M = \{x_i y_i : i \in I\}$ is a matching in $G$.

**Claim 1:** $M$ is a maximal matching in $G$.

Since $|N(x_i) - N[S_{i-1}]| \leq 1$ holds for all $i \in \{1, \ldots, \alpha\}$, where $S_0 = N[S_0] = \emptyset$, and $\{y_i\} = N(x_i) - N[S_{i-1}]$, for all $i \in I$, it follows that $N(S) = \{y_i : i \in I\}$, and this ensures that $M$ is a maximal matching in $G$.

**Claim 2:** $M$ is a maximum matching in $G$, and consequently, $G$ is a König–Egerváry graph.

Since $|V| \geq \alpha(G) + \mu(G)$ holds for any graph, and in our case

$$|V| = |N[S]| = |S| + |N(S)| = |S| + |\{y_i : i \in I\}| = \alpha(G) + |M|,$$

we infer that $|M| = \mu(G)$. In other words, $M$ is a maximum matching in $G$, and $G$ is a König–Egerváry graph.

**Claim 3:** $M$ is a uniquely restricted maximum matching in $G$ and any $H_k = G[N[S_k]]$ is a König–Egerváry graph.

We use induction on $k = |S_k|$ to show that: $H_k$ is a König–Egerváry graph and the restriction of $M$ to $H_k$, which we denote by $M_k$, is a uniquely restricted maximum matching in $H_k$.

For $k = 1$, $S_1 = \{x_1\} \in \Psi(G)$ and this implies that $N(x_1) = \{y_1\}$, unless $x_1$ is an isolated vertex. In this case, $H_1$ is a König–Egerváry graph and $M_1 = \{x_1 y_1\}$ is a uniquely restricted maximum matching in $H_1$. If $x_1$ is an isolated vertex, though, $H_1$ is a König–Egerváry graph and $M_1 = \emptyset$ is a uniquely restricted maximum matching in $H_1$. 


Suppose that the assertion is true for all \( j \leq k - 1 \). Let us notice that
\[
N[S_k] = N[S_{k-1}] \cup \{x_k\} \cup (N(x_k) - N[S_{k-1}])
\]
and this assures that \( H_k \) is a König–Egerváry graph and \( M_{k-1} \) is a maximum matching in \( H_k \). Hence, denoting \( M_k = M_{k-1} \), we may infer that \( M_k \) is the restriction of \( M \) to \( H_k \) and that \( M_k \) is a uniquely restricted maximum matching in \( H_k \), because \( N(x_k) \subseteq N[S_{k-1}] \).

Case 2: \( N(x_k) - N[S_{k-1}] = \{y_k\} \).

Then we have
\[
|V(H_k)| = |S_{k-1} \cup \{x_k\}| + |M_{k-1} \cup \{x_ky_k\}| = |S_{k-1}| + |M_{k-1}| \leq \alpha(H_k) + \mu(H_k) \leq |V(H_k)|,
\]
and this assures that \( H_k \) is a König–Egerváry graph and \( M_k \) is a maximum matching in \( H_k \). The edge \( x_ky_k \) is \( \alpha \)-critical in \( H_k \), because \( \{y_k\} = N(x_k) - N[S_{k-1}] \), and hence, \( x_ky_k \) is also \( \mu \)-critical in \( H_k \), according to Lemma 2.1. Therefore, any maximum matching of \( H_k \) contains the edge \( x_ky_k \). Since \( M_k = M_{k-1} \cup \{x_ky_k\} \) and \( M_{k-1} \) is a uniquely restricted maximum matching in \( H_{k-1} = H_k - \{x_k, y_k\} \), it follows that \( M_k \) is a uniquely restricted maximum matching in \( H_k \).

The graph \( G \) in Fig. 9 shows that even if some \( S \in \Omega(G) \) has an accessibility chain, this is not necessarily true for all maximum stable sets.

The following result demonstrates that the case of triangle-free graphs is different.

**Proposition 3.4.** If \( G \) is a triangle-free graph, then the following assertions are equivalent:

(i) there exists some \( S \in \Omega(G) \) having an accessibility chain;
(ii) \( G \) is a König–Egerváry graph and there exists a uniquely restricted maximum matching in \( G \);
(iii) each \( S \in \Omega(G) \) has an accessibility chain.

**Proof.** The implication “(i) ⇒ (ii)” is true by Lemma 3.3.

(ii) ⇒ (iii) Let \( M \) be a uniquely restricted maximum matching in \( G \) and \( S \in \Omega(G) \). According to Lemma 2.2, \( M \subseteq (S, V(G) - S) \) and \( |M| = |V(G) - S| = \mu(G) \). Therefore, \( M \) is the unique perfect matching of \( H = G[N[S_\mu]] \), where
\[
S_\mu = \{x : x \in S, x \text{ is an endpoint of an edge in } M\}.
\]

Since \( N(S_\mu) = V(G) - S \) and \( S_\mu \) is stable, we infer that \( S_\mu \) is a maximum stable set in \( H \), i.e., \( S_\mu \in \Psi(G) \). In addition, \( H \) is a König–Egerváry graph, because
\[
|V(H)| = |N[S_\mu]| = |S_\mu| + |M| = \alpha(H) + \mu(H).
\]

By Lemma 3.1, there exists a chain
\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{\mu-1}\} \subset \{x_1, x_2, \ldots, x_{\mu-1}, x_\mu\} = S_\mu,
\]
Theorem 3.5. (ii) generates the uniquely restricted maximum matching such that all \(V\). \(\{x_1, x_2, \ldots, x_k\}, 1 \leq k \leq \mu\) are local maximum stable sets in \(H\). The equality \(N_H[S_k] = N_G[S_k]\) implies that \(S_k \in \Psi(G)\) for all \(k \in \{1, \ldots, \mu(G)\}\).

Since \(N(S_{\mu}) = V(G) - S\), one can just add the remaining vertices in \(S - S_{\mu}\) to \(S_{\mu}\) one at a time and eventually obtain an accessibility chain for \(S\). In some more detail it reads as follows.

Let now \(x \in S - S_{\mu}\). Then \(N(x) \subseteq V(G) - S\), and, therefore, \(N(S_{\mu} \cup \{x\}) = V(G) - S\). Since \(S_{\mu}\) is a maximum stable set in \(H\) and \(S_{\mu} \cup \{x\}\) is stable in \(H \cup \{x\} = G[N(S_{\mu} \cup \{x\})]\), we get that \(S_{\mu} \cup \{x\}\) is a maximum stable set in \(H \cup \{x\}\), i.e., \(S_{\mu} \cup \{x\} \in \Psi(G)\). If there still exists some \(y \in S - S_{\mu} - \{x\}\), we infer, in the same manner, that \(S_{\mu} \cup \{x, y\} \in \Psi(G)\). In this way we build the following chain:

\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{\mu-1}, x_{\mu}\} = S_{\mu} \subset S_{\mu} \cup \{x_{\mu+1} = x\}
\]

\[
\subset S_{\mu} \cup \{x_{\mu+1}\} \cup \{x_{\mu+2} = y\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{2-1}\} \subset \{x_1, x_2, \ldots, x_{2-1}, x_2\} = S,
\]

where for all \(1 \leq j \leq x, \{x_1, x_2, \ldots, x_j\} \in \Psi(G)\).

Clearly, (iii) implies (i), and this completes the proof. \(\square\)

For instance, the accessibility chain \(\{a\} \subset \{a, c\} \subset \{a, c, d\} \subset \{a, c, d, g\} \in \Omega(G)\) of the graph \(G\) from Fig. 10 generates the uniquely restricted maximum matching \(M = \{ab, ce, df\}\).

Let us notice that the graph \(G\) in Fig. 11 is a König–Egerváry graph whose maximum matchings are uniquely restricted. According to Proposition 3.4, each \(S \in \Omega(G)\) has an accessibility chain. However, \(\Psi(G)\) is not a greedoid, because, for example, \(\{b, c\} \in \Psi(G)\), while \(\{b, c\} \not\in \Psi(G)\). The following theorem will show us another reason, why the family \(\Psi(G)\) of the graph \(G\) from Fig. 11 is not a greedoid, namely the subgraph spanned by \(N[\{b, c\}]\) is not a König–Egerváry graph.

Moreover, the proof of Theorem 3.5 allows us to see that in a triangle-free graph \(G\), whose \(\Psi(G)\) forms a greedoid, the vertices of every maximum stable set can be ordered in such a way that deleting one at a time, we can get a chain of König–Egerváry subgraphs containing uniquely restricted maximum matchings.

**Theorem 3.5.** If \(G\) is a triangle-free graph, then the following assertions are equivalent:

(i) \(\Psi(G)\) is a greedoid;
(ii) all its maximum matchings are uniquely restricted and the closed neighborhood of every local maximum stable set of \(G\) induces a König–Egerváry graph.

**Proof.** (i) \(\Rightarrow\) (ii) Let \(X \in \Psi(G)\) and \(Y \in \Omega(G)\). By accessibility property, there is a chain

\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, \ldots, x_{k-1}\} \subset \{x_1, \ldots, x_{k-1}, x_k\} = X,
\]

such that \(\{x_1, x_2, \ldots, x_j\} \in \Psi(G), \text{ for all } j \in \{1, \ldots, k - 1\}\). Clearly, also for \(Y \in \Omega(G)\) there is a chain

\[
\{y_1\} \subset \{y_1, y_2\} \subset \cdots \subset \{y_1, y_2, \ldots, y_{x-1}\} \subset \{y_1, y_2, \ldots, y_x\} = Y,
\]

such that \(\{y_1, y_2, \ldots, y_i\} \in \Psi(G), \text{ for all } i \in \{1, \ldots, x - 1\}\).
If $k < |Y|$, using the exchange property, we find some $x_{k+1} \in \{y_1, y_2, \ldots, y_{k+1}\} - X$, such that $X \cup \{x_{k+1}\} \in \Psi(G)$. If still $k + 1 < |Y|$, we use again the exchange property and find some $x_{k+2} \in \{y_1, y_2, \ldots, y_{k+1}, y_{k+2}\} - X \cup \{x_{k+1}\}$, such that $X \cup \{x_{k+1}\} \cup \{x_{k+2}\} \in \Psi(G)$, and so on.

Finally we build a chain
\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, \ldots, x_{k-1}\} \subset X \subset X \cup \{x_{k+1}\} \\
\subset X \cup \{x_{k+1}, x_{k+2}\} \subset \cdots \subset \{x_1, x_2, \ldots, x_\ell\} \subset \Omega(G),
\]
such that $S_i = \{x_1, x_2, \ldots, x_i\} \in \Psi(G)$, for each $i \in \{1, 2, \ldots, \ell\}$. According to Lemma 3.3(i), $G[N[S_i]]$ is a König–Egerváry graph for every $1 \leq i \leq \ell$. Hence, $G[N[X]]$ must be a König–Egerváry graph.

Let $M$ be a maximum matching in $G$ and $S \in \Omega(G)$. Lemma 2.2 implies that both $M \subseteq (S, V(G) - S)$ and $|M| = |V(G) - S|$. Let $S_\mu$ contain the vertices of $S$ matched by $M$ with the vertices of $V(G) - S$. Since $M$ is a perfect matching in $G[N[S_\mu]]$ and $|S_\mu| = |M|$, it follows that $S_\mu$ is a maximum stable set in $G[N[S_\mu]]$, i.e., $S_\mu \in \Psi(G)$. Hence, there is a chain:
\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{\mu-1}\} \\
\subset \{x_1, x_2, \ldots, x_{\mu}\} = S_\mu \subset S_\mu \cup \{x_{\mu+1}\} \subset \cdots \subset S,
\]
such that all $S_i = \{x_1, x_2, \ldots, x_i\} \in \Psi(G)$. By Lemma 3.3, $G[N[S_\mu]]$ has a uniquely restricted maximum matching. Since $M$ is a perfect matching in $G[N[S_\mu]]$, it follows that $M$ is unique in $G[N[S_\mu]]$. Hence, $M$ is a uniquely restricted maximum matching of $G$.

(ii) $\Rightarrow$ (i) Let $S_0 \in \Psi(G)$, i.e., $S_0$ is a maximum stable set, of size say $q$, in $H = G[N[S_0]]$, which is a König–Egerváry graph, according to the hypothesis. Let $M_0$ be a maximum matching in $H$. By Lemma 2.3, there exists a maximum matching in $G$, say $M$, such that $M_0 \subseteq M$. Since $M$ is uniquely restricted in $G$, it follows that $M_0$ is uniquely restricted in $H$. According to Proposition 3.4, there exists a chain
\[
\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_{q-1}\} \subset \{x_1, x_2, \ldots, x_{q-1}, x_q\} = S_0,
\]
such that all $S_k = \{x_1, x_2, \ldots, x_k\}$, $1 \leq k \leq q$, are local maximum stable sets in $H$. Since $N_H[S_k] = N_G[S_k]$, it results that $S_k \in \Psi(G)$, for any $k \in \{1, \ldots, q\}$. In other words, $\Psi(G)$ satisfies the accessibility property.

We have to show now that $\Psi(G)$ satisfies also the exchange property. Let us consider $X, Y \in \Psi(G)$, and $|Y| = |X| + 1 = m + 1$. Hence, there is a chain
\[
\{y_1\} \subset \{y_1, y_2\} \subset \cdots \subset \{y_1, y_2, \ldots, y_m\} \subset Y,
\]
such that $\{y_1\}, \{y_1, y_2\}, \ldots, \{y_1, \ldots, y_m\} \in \Psi(G)$, because $\Psi(G)$ satisfies the accessibility property. Since $Y$ is stable, $X \in \Psi(G)$, and $|X| < |Y|$, it follows that there exists some $y \in Y - X$, such that $y \notin N[X]$. Let $M_X$ be a maximum matching in $H = G[N[X]]$. Since $H$ is a König–Egerváry graph, $X$ is a maximum stable set in $H$, and $M_X$ is a maximum matching in $H$, it follows that
\[
|X| + |M_X| = |N[X]| = |X| + |N(X)|,
\]
i.e., $|M_X| = |N(X)|$. Let $y_{k+1} \in Y$ be the first vertex in $Y$ satisfying the conditions: $y_1, \ldots, y_k \in N[X]$ and $y_{k+1} \notin N[X]$. Since $\{y_1, \ldots, y_k\}$ is stable in $N[X]$, there exists some set $\{x_1, \ldots, x_k\} \subseteq X$ such that for any $i \in \{1, \ldots, k\}$ either $x_i = y_i$ or $x_i y_i \in M_X$.

We show that $X \cup \{y_{k+1}\} \in \Psi(G)$.

Case 1: $N[X \cup \{y_{k+1}\}] = N[X] \cup \{y_{k+1}\}$.

Clearly, $X \cup \{y_{k+1}\}$ is stable in $G(N[X \cup \{y_{k+1}\})$ and further, the equality
\[
|X \cup \{y_{k+1}\}| = |X| + 1
\]
ensures that $X \cup \{y_{k+1}\} \in \Psi(G)$, because $X \in \Psi(G)$, as well.

Case 2: $N[X \cup \{y_{k+1}\}] \neq N[X] \cup \{y_{k+1}\}$.

Suppose that there are $a, b \in N(y_{k+1}) - N[X]$. Hence, it follows that $\{a, b, x_1, \ldots, x_k\}$ is a stable set (because $G$ is triangle-free) included in $N[y_1, \ldots, y_{k+1}]$ and larger than $\{y_1, \ldots, y_{k+1}\}$, in contradiction with the fact that $\{y_1, \ldots, y_{k+1}\} \in \Psi(G)$. Therefore, there exists a unique $a \in N(y_{k+1}) - N[X]$. Consequently,
\[
N[X \cup \{y_{k+1}\}] = N[X] \cup N[y_{k+1}] = N[X] \cup \{a, y_{k+1}\}
\]
and since $ay_{k+1} \in E(G)$, we obtain that $X \cup \{y_{k+1}\}$ is a maximum stable set in the subgraph $G[N[X \cup \{y_{k+1}\}]]$, i.e., $X \cup \{y_{k+1}\} \in \Psi(G)$. Therefore, $\Psi(G)$ satisfies also the exchange property.

In conclusion, $\Psi(G)$ is a greedoid on the vertex set of $G$. \qed

Since a bipartite graph is also triangle-free and every of its subgraphs is also bipartite, we obtain:

**Corollary 3.6 (Levit and Mandrescu [18]).** For a bipartite graph $G$, the family $\Psi(G)$ is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted.

The graph $G_1$ from Fig. 12 validates the existence of non-bipartite triangle-free König–Egerváry graphs, whose families of local maximum stable sets form greedoids. On the other hand, the graph $G_2$ from the same figure shows the limits of applicability of Theorem 3.5 even for König–Egerváry graphs. Namely, $G_2$ is not triangle-free, its maximum matchings are not uniquely restricted, but $\Psi(G_2)$ is a greedoid.

### 4. Conclusions

We have characterized triangle-free graphs whose family of local maximum stable sets form a greedoid on their vertex sets. Our description is based on the property that some subgraphs of our graph are König–Egerváry graphs. The following lemma gives hope to find another characterization of the above mentioned type of graphs with the help of interconnections between chordless cycles and local maximum stable sets of these graphs.

**Lemma 4.1.** If $\Psi(G)$ is a greedoid, then $\Omega(C_n) \cap \Psi(G) = \emptyset$ for its every cycle $C_n$ of size $n \geq 4$.

**Proof.** Suppose that there exists $S \in \Omega(C_n) \cap \Psi(G)$ in $G$ for some $n \geq 4$. Since $\Psi(G)$ is a greedoid, there is a chain of local maximum stable sets

$$\{x_1\} \subset \{x_1, x_2\} \subset \cdots \subset \{x_1, x_2, \ldots, x_k\} \subset \{x_1, x_2, \ldots, x_{k-1}\} \subset \{x_1, x_2, \ldots, x_k\} = S,$$

where $k = |S| \geq 2$. Hence, $x_1$ must be a pendant vertex in $G$, contradicting the fact that $x_1$ belongs to $V(C_n)$. \qed

The graph $G$ from Fig. 13 satisfies the condition that $\Omega(C_n) \cap \Psi(G) = \emptyset$ for its every cycle $C_n$ of size $n \geq 4$, but $\Psi(G)$ is not a greedoid, because $G$ is a triangle-free graph having a maximum matching, namely $M = \{ab, cf, de\}$, which is not uniquely restricted. In other words, the inverse assertion to Lemma 4.1 is not true. Therefore, we think that it would be interesting to complete this lemma with its corresponding if-and-only-if strengthening.

A linear time algorithm deciding whether a matching in a bipartite graph is uniquely restricted is presented in [7]. It is also shown there that the problem of finding a uniquely restricted maximum matching is \textbf{NP}-complete even for bipartite graphs. In [16] we showed that unicycle bipartite graphs having only uniquely restricted maximum matchings can be recognized in polynomial time. These facts allow us to propose the following open problem: how to recognize a triangle-free graph whose family of local maximum stable sets is a greedoid.
Theorem 1.3 and its generalization, Theorem 3.5, exhibit a close relationship between maximum uniquely restricted matchings and local maximum stable sets of a graph, that could give birth, sometimes, to a greedoid structure on the vertex set of the graph.

Finally, let us notice that Theorem 3.5 implies that if $G$ is triangle-free and $\Psi(G)$ is a greedoid, then $\mu_r(G) = \mu(G)$, where $\mu_r(G)$ is the maximum size of a uniquely restricted matching in $G$. Recall that Golumbic, Hirst and Lewenstein have shown in [7] that $\mu_r(G) = \mu(G)$ holds when $G$ is a tree or has only odd cycles.

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