Stability of Nonic Functional Equations in Multi-Banach Spaces

R. Murali [1], Sandra Pinelas [2], A. Antony Raj [3]

[1,3] Department of Mathematics, Sacred Heart College,

Tirupattur - 635 601, TamilNadu, India

[2] Academia Militar, Lisboa.

In this paper, we carry out the following Stability of Nonic Functional Equations

$$\xi(s+5t)-9\xi(s+4t)+36\xi(s+3t)-84\xi(s+2t)+126\xi(s+t)-126\xi(s)$$
$$+84\xi(s-t)-36\xi(s-2t)+9\xi(s-3t)-\xi(s-4t)=9!\xi(t)$$

where $9!=362880$ in Multi-Banach Spaces by using fixed point technique.

Introduction

In 1940, Ulam posed a problem concerning the stability of functional equations: Give conditions in order for a linear function near an approximately linear function to exist. An earlier work was done by Hyers [6] in order to answer Ulam's equation [14] on approximately additive mappings.

During last decades various stability problems for large variety of functional equations have been investigated by several mathematicians. A large list of references concerning in the stability of functional equations can be found.

e.g.([1], [2], [6], [7], [9]).

In 2010, Liguang Wang, Bo Liu and ran Bai [10] proved the stability of a mixed type functional equations on Multi - Banach Spaces. In 2010, Tian Zhou Xu, John Michael Rassias, Wan Xin Xu [13] investigated the generalized Ulam-Hyers stability of the general mixed additive-quadratic-cubic-quartic functional equation

$$f(x+ny)+f(x-ny)=n^2 f(x+y)+n^2 f(x-y)+2(1-n^2)f(x)$$
$$+\frac{n^4-n^2}{12}\left[f(2y)+f(-2y)-4f(y)-4f(-y)\right]$$

for fixed integers n with $n\neq 0,\pm 1$ in Multi- Banach Spaces.

In 2011, Zhihua Wang, Xiaopei Li and Th. M. Rassias[16] proved the Hyers - Ulam stability of the additive - cubic - quartic functional equations

$$11\left[f(x+2y)+f(x-2y)\right]=44\left[f(x+y)+f(x-y)\right]+12f(3y)$$
$$-48f(2y)+60f(y)-66f(x)$$

in Multi - Banach Spaces by using fixed point method.

In 2013, Fridoun Moradlou [5] proved the generalized Hyers-Ulam-Rassias stability of the Euler-Lagrange-Jensen Type Additive mapping in Multi-Banach Spaces.

In 2015, Xiuzhong Yang, Lidan Chang, Guofen Liu[15] estabilished the orthogonal stability of mixed additive-quadratic Jensen type functional equation in Multi-Banach Spaces.

In 2016, John M. Rassias, M. Arunkumar, E. Sathya and T. Namachivayam [8] established the (??) general solution and also proved the Felbin's type fuzzy normed space and intuitionistic fuzzy normed space using direct and fixed point method.

In this paper, we carry out the following Stability of Nonic Functional Equations

$$D\xi(s,t)=\xi(s+5t)-9\xi(s+4t)+36\xi(s+3t)-84\xi(s+2t)+126\xi(s+t)-126\xi(s)$$
$$+84\xi(s-t)-36\xi(s-2t)+9\xi(s-3t)-\xi(s-4t)-9!\xi(t) \quad (1)$$

where $9!=362880$ in Multi-Banach Spaces by using fixed point technique.

It is easily verified that that the function $\xi(s)=s^9$ satisfies the above functional equations. In other words, every solution of the nonic functional equation is called a nonic mapping.

Theorem 1.1 *[3], [12] Let* (X,d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;

(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$..

Now, let us recall regarding some concepts in Multi-Banach spaces.

Let $(\wp, \|.\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\wp^k$ the linear space $\wp \oplus \wp \oplus \wp \oplus \ldots \oplus \wp$ consisting of $k-$ tuples $(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k \in \wp$. The linear operations on $\wp^k$ are defined coordinate wise. The zero element of either $\wp$ or $\wp^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1,2,\ldots,k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 1.2 *[4] A Multi- norm on* $\{\wp^k : k \in \mathbb{N}\}$ *is a sequence* $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ *such that* $\|.\|_k$ *is a norm on* $\wp^k$ *for each* $k \in \mathbb{N}, \|x\|_1 = \|x\|$ *for each* $x \in \wp$, *and the following axioms are satisfied for each* $k \in \mathbb{N}$ *with* $k \geq 2$:

1. $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1 \ldots x_k)\|_k$, for $\sigma \in \Psi_k, x_1, \ldots, x_k \in \wp$;
2. $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1 \ldots x_k)\|_k$

for $\alpha_1 \ldots \alpha_k \in \mathbb{C}, x_1, \ldots, x_k \in \wp$;

3. $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$, for $x_1, \ldots, x_{k-1} \in \wp$;
4. $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ for $x_1, \ldots, x_{k-1} \in \wp$.

In this case, we say that $((\wp^k, \|.\|_k) : k \in \mathbb{N})$ is a multi - normed space.

Suppose that $((\wp^k, \|.\|_k) : k \in \mathbb{N})$ is a multi - normed spaces, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [4].

$$(a) \|(x, \ldots x)\|_k = \|x\|, \text{for} x \in \wp,$$

$$(b) \max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\|, \forall x_1, \ldots, x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach space, then $(\wp^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; In this case, $((\wp^k, \|.\|_k) : k \in \mathbb{N})$ is a multi - Banach space.

Lemma 1.3 *[4] Suppose that* $k \in \mathbb{N}$ *and* $(x_1 \ldots x_k) \in \wp^k$. *For each* $j \in \{1 \ldots k\}$, *let* $(x_n^j)_{n=1,2\ldots}$ *be a sequence in* $\wp$ *such that* $\lim_{n \to \infty} x_n^j = x_j$. *Then*

$$\lim_{n\to\infty}\left(x_n^1-y_1,\dots,x_n^k-y_k\right)=\left(x_1-y_1,\dots,x_k-y_k\right) \tag{2}$$

holds for all $\left(y_1,\dots,y_k\right)\in\wp^k$.

Definition 1.4 *[4] Let* $\left((\wp^k,\|.\|_k):k\in\mathrm{N}\right)$ *be a multi - normed space. A sequence* (x_n) *in* $\wp$ *is a multi-null sequence if for each* $\eta>0,$ *there exists* $n_0\in\mathrm{N}$ *such that*

$$\sup_{k\in\mathrm{N}}\left\|\left(x_n,\dots,x_{n+k-1}\right)\right\|_k\le\eta \qquad (n\ge n_0). \tag{3}$$

Let $x\in\wp,$ we say that the sequence (x_n) is multi-convergent to x in $\wp$ and write $\lim_{n\to\infty}x_n=x$ if (x_n-x) is a multi - null sequence.

Definition 1.5 *Let* X *be a set. A function* $d:X\times X\to[0,\infty]$ *is called a generalized metric on* X *if and only if* d *satisfies*

- $d(x,y)=0$ if and only if $x=y;$
- $d(x,y)=d(y,x)$ for all $x,y\in X;$
- $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X.$

Theorem 1.6 *Let* Z *be an linear space and let* $\left((A^k,\|.\|):K\in\mathrm{N}\right)$ *be a multi-Banach space. Suppose that* η *is a nonnegative real number and* $\xi:Z\to A$ *is a mapping satisfying*

$$\sup_{k\in\mathrm{N}}\left\|\left(D\xi(s_1,t_1),\dots,D\xi(s_k,t_k)\right)\right\|_k\le\eta \tag{4}$$

$s_1,\dots,s_k,t_1,\dots,t_k\in Z.$ Then there exists a unique nonic mapping $N:Z\to A$ such that

$$\sup_{k\in\mathrm{N}}\left\|\left(\xi(s_1)-N(s_1),\dots,\xi(s_k)-N(s_k)\right)\right\|_k\le\frac{19}{6867840}\eta \tag{5}$$

forall $s_1,\dots,s_k\in Z.$

Proof. Letting $s_1=s_2=,\dots,=s_k=0$ and replacing $t_1,\dots,t_k$ by $2s_1,\dots,2s_k$ in (4), we obtain that

$$\begin{aligned}\sup_{k\in\mathrm{N}}&\left\|\left(\xi(10s_1)-8\xi(8s_1)+27\xi(6s_1)-48\xi(4s_1)-362838\xi(2s_1),\dots,\right.\right.\\&\left.\left.\xi(10s_k)-8\xi(8s_k)+27\xi(6s_k)-48\xi(4s_k)-362838\xi(2s_k)\right)\right\|_k\le\eta\end{aligned} \tag{6}$$

forall $s_1,\dots,s_k\in Z.$

Letting $s_1,\dots,s_k$ by $5s_1,\dots5s_k$ and replacing $t_1,\dots,t_k$ by $s_1,\dots s_k$ in (4), we get

$$\begin{aligned}&\sup_{k\in\mathrm{N}}\left\|\left(\xi(10s_1)-9\xi(9s_1)+36\xi(8s_1)-84\xi(7s_1)+126\xi(6s_1)-126\xi(5s_1)+\right.\right.\\&84\xi(4s_1)-36\xi(3s_1)+9\xi(2s_1)-362881\xi(s_1),\dots,\xi(10s_k)-9\xi(9s_k)+36\xi(8s_k)-84\xi(7s_k)+\\&\left.\left.126\xi(6s_k)-126\xi(5s_k)+84\xi(4s_k)-36\xi(3s_k)+9\xi(2s_k)-362881\xi(s_k)\right)\right\|_k\le\eta\end{aligned} \tag{7}$$

forall $s_1,\dots,s_k\in Z.$ Unifying (6) and (7),

$$\sup_{k\in\mathrm{N}}\left\|\left(9\xi(9s_1)-44\xi(8s_1)+84\xi(7s_1)-99\xi(6s_1)+126\xi(5s_1)-134\xi(4s_1)+36\xi(3s_1)\right.\right.$$

$$-362847\xi(2s_1)+362881\xi(s_1),...,9\xi(9s_k)-44\xi(8s_k)+84\xi(7s_k)$$
$$-99\xi(6s_k)+126\xi(5s_k)-134\xi(4s_k)+36\xi(3s_k)-362847\xi(2s_k)+362881\xi(s_k))\|_k \le 2\eta \quad (8)$$

forall $s_1,...,s_k \in Z$. Letting $s_1,...,s_k$ by $4s_1,...,4s_k$ and replacing $t_1,...,t_k$ by $s_1,...,s_k$ in (4), we arrive

$$\sup_{k\in N}\|(\xi(9s_1)-9\xi(8s_1)+36\xi(7s_1)-84\xi(6s_1)+126\xi(5s_1)-126\xi(4s_1)+84\xi(3s_1)$$
$$-36\xi(2s_1)+9\xi(s_1)-9!\xi(s_1),...,\xi(9s_k)-9\xi(8s_k)+36\xi(7s_k)-84\xi(6s_k)+126\xi(5s_k)$$
$$-126\xi(4s_k)+84\xi(3s_k)-36\xi(2s_k)+9\xi(s_k)-9!\xi(s_k))\|_k \le \eta \quad (9)$$

forall $s_1,...,s_k \in Z$. Multiplying by 9 in (9), we arrive

$$\sup_{k\in N}\|(9\xi(9s_1)-81\xi(8s_1)+324\xi(7s_1)-756\xi(6s_1)+1134\xi(5s_1)-1134\xi(4s_1)+756\xi(3s_1)$$
$$-324\xi(2s_1)-3265839\xi(s_1),...,9\xi(9s_k)-81\xi(8s_k)+324\xi(7s_k)-756\xi(6s_k)+1134\xi(5s_k)$$
$$-1134\xi(4s_k)+756\xi(3s_k)-324\xi(2s_k)-3265839\xi(s_k))\|_k \le 9\eta \quad (10)$$

forall $s_1,...,s_k \in Z$. It follows from (8) and (10), we get

$$\sup_{k\in N}\|(37\xi(8s_1)-240\xi(7s_1)+657\xi(6s_1)-1008\xi(5s_1)+1002\xi(4s_1)-720\xi(3s_1)$$
$$-362523\xi(2s_1)+3628720\xi(s_1),...,37\xi(8s_k)-240\xi(7s_k)+657\xi(6s_k)-1008\xi(5s_k)$$
$$+1002\xi(4s_k)-720\xi(3s_k)-362523\xi(2s_k)+3628720\xi(s_k))\|_k \le 11\eta \quad (11)$$

forall $s_1,...,s_k \in Z$.

Putting $s_1,...,s_k$ by $3s_1,...3s_k$ and replacing $t_1,...,t_k$ by $s_1,...,s_k$ in (4), we get

$$\sup_{k\in N}\|(\xi(8s_1)-9\xi(7s_1)+36\xi(6s_1)-84\xi(5s_1)+126\xi(4s_1)-126\xi(3s_1)+84\xi(2s_1)$$
$$-36\xi(s_1)-\xi(-s_1)-9!\xi(s_1),...,\xi(8s_k)-9\xi(7s_k)+36\xi(6s_k)-84\xi(5s_k)$$
$$+126\xi(4s_k)-126\xi(3s_k)+84\xi(2s_k)-36\xi(s_k)-\xi(-s_k)-9!\xi(s_k))\|_k \le \eta \quad (12)$$

forall $s_1,...,s_k \in Z$. Multiplying by 37 in (12), we get

$$\sup_{k\in N}\|(37\xi(8s_1)-333\xi(7s_1)+1332\xi(6s_1)-3108\xi(5s_1)+46621\xi(4s_1)-46621\xi(3s_1)$$
$$+3108\xi(2s_1)-13427855\xi(s_1),...,37\xi(8s_k)-333\xi(7s_k)+1332\xi(6s_k)$$
$$-3108\xi(5s_k)+46621\xi(4s_k)-46621\xi(3s_k)+3108\xi(2s_k)-13427855\xi(s_k))\|_k \le 37\eta \quad (13)$$

forall $s_1,...,s_k \in Z$. By (11) and (13), we obtain

$$\sup_{k\in N}\|(93\xi(7s_1)-675\xi(6s_1)-2100\xi(5s_1)-3660\xi(4s_1)+3942\xi(3s_1)$$
$$-365631\xi(2s_1)+17056575\xi(s_1),...,93\xi(7s_k)-675\xi(6s_k)-2100\xi(5s_k)$$
$$-3660\xi(4s_k)+3942\xi(3s_k)-365631\xi(2s_k)+17056575\xi(s_k))\|_k \le 48\eta \quad (14)$$

forall $s_1,...,s_k \in Z$.

Replacing $s_1,...,s_k$ by $2s_1,...,2s_k$ and $t_1,...,t_k$ by $s_1,...,s_k$ in (4), we get

$$\sup_{k\in N}\|(\xi(7s_1)-9\xi(6s_1)+36\xi(5s_1)-84\xi(4s_1)+126\xi(3s_1)-126\xi(2s_1)+84\xi(s_1)$$

$$+9\xi(-s_1)-\xi(-2s_1)-9!\xi(s_1),...,\xi(7s_k)-9\xi(6s_k)+36\xi(5s_k)-84\xi(4s_k)$$

$$+126\xi(3s_k)-126\xi(2s_k)+84\xi(s_k)+9\xi(-s_k)-\xi(-2s_k)-9!\xi(s_k))\|_k \le \eta \quad (15)$$

forall $s_1,...,s_k \in Z$. Multiplying by 93 on both sides in (15), we can get

$$\sup_{k\in N}\|(93\xi(7s_1)-837\xi(6s_1)+3348\xi(5s_1)-7812\xi(4s_1)+11718\xi(3s_1)$$
$$-11625\xi(2s_1)-33740865\xi(s_1),...,93\xi(7s_k)-837\xi(6s_k)+3348\xi(5s_k)$$
$$-7812\xi(4s_k)+11718\xi(3s_k)-11625\xi(2s_k)-33740865\xi(s_k))\|_k \le 93\eta \quad (16)$$

forall $s_1,...,s_k \in Z$. By (14) and (16), we get

$$\sup_{k\in N}\|(162\xi(6s_1)-1248\xi(5s_1)+4152\xi(4s_1)-7776\xi(3s_1)-354006\xi(2s_1)+50797440\xi(s_1),...,$$

$$162\xi(6s_k)-1248\xi(5s_k)+4152\xi(4s_k)-7776\xi(3s_k)-354006\xi(2s_k)+50797440\xi(s_k))\|_k \le 141\eta \quad (17)$$

forall $s_1,...,s_k \in Z$. Dividing on both sides by 2 in (17), we get

$$\sup_{k\in N}\|(81\xi(6s_1)-624\xi(5s_1)+2076\xi(4s_1)-3888\xi(3s_1)-177003\xi(2s_1)+25398720\xi(s_1),...,$$

$$81\xi(6s_k)-624\xi(5s_k)+2076\xi(4s_k)-3888\xi(3s_k)-177003\xi(2s_k)+25398720\xi(s_k))\|_k \le \frac{141}{2}\eta$$

(18)

forall $s_1,...,s_k \in Z$. Replacing $t_1,...t_k$ by $s_1,...,s_k$ in (4), we get

$$\sup_{k\in N}\|(\xi(6s_1)-9\xi(5s_1)+36\xi(4s_1)-84\xi(3s_1)+126\xi(2s_1)-126\xi(s_1)-36\xi(-s_1)$$
$$+9\xi(-2s_1)-\xi(-3s_1)-9!\xi(s_1),...,\xi(6s_k)-9\xi(5s_k)+36\xi(4s_k)-84\xi(3s_k)+126\xi(2s_k)$$
$$-126\xi(s_k)-36\xi(-s_k)+9\xi(-2s_k)-\xi(-3s_k)-9!\xi(s_k))\|_k \le \eta \quad (19)$$

forall $s_1,...,s_k \in Z$. Multiplying both sides 81 by (19), we can get

$$\sup_{k\in N}\|(81\xi(6s_1)-729\xi(5s_1)+2916\xi(4s_1)-6723\xi(3s_1)+9477\xi(2s_1)-29400570\xi(s_1),...,$$

$$81\xi(6s_k)-729\xi(5s_k)+2916\xi(4s_k)-6723\xi(3s_k)+9477\xi(2s_k)-29400570\xi(s_k))\|_k \le 81\eta \quad (20)$$

forall $s_1,...,s_k \in Z$. From (18) and (20)

$$\sup_{k\in N}\|(105\xi(5s_1)-840\xi(4s_1)+2835\xi(3s_1)-186480\xi(2s_1)+54799290\xi(s_1)$$

$$105\xi(5s_k)-840\xi(4s_k)+2835\xi(3s_k)-186480\xi(2s_k)+54799290\xi(s_k))\|_k \le \frac{303}{2}\eta \quad (21)$$

forall $s_1,...,s_k \in Z$.

Replacing $s_1,...,s_k = 0$ and $t_1,...,t_k$ by $s_1,...,s_k$ in (4), we obtain

$$\sup_{k\in N}\|(\xi(5s_1)-9\xi(4s_1)+36\xi(3s_1)-84\xi(2s_1)+126\xi(s_1)+84\xi(-s_1)$$
$$-36\xi(-2s_1)+9\xi(-3s_1)-\xi(-4s_1)-9!\xi(s_1),...,\xi(5s_k)-9\xi(4s_k)+36\xi(3s_k)-84\xi(2s_k)$$
$$+126\xi(s_k)+84\xi(-s_k)-36\xi(-2s_k)+9\xi(-3s_k)-\xi(-4s_k)-9!\xi(s_k))\|_k \le \eta \quad (22)$$

forall $s_1,\ldots,s_k \in Z.$

Multiplying on bothsides by 105 in (22), we obtain that

$$\sup_{k\in N}\big\|(105\xi(5s_1)-840\xi(4s_1)+2835\xi(3s_1)-5040\xi(2s_1)-38097990\xi(s_1)$$

$$105\xi(5s_k)-840\xi(4s_k)+2835\xi(3s_k)-5040\xi(2s_k)-38097990\xi(s_k))\big\|_k \le 105\eta \tag{23}$$

forall $s_1,\ldots,s_k \in Z.$ From (21) and (23)

$$\sup_{k\in N}\big\|(-181440\xi(2s_1)+92897280\xi(s_1),\ldots,-181440\xi(2s_k)+92897280\xi(s_k))\big\|_k \le \frac{513}{2}\eta \tag{24}$$

forall $s_1,\ldots,s_k \in Z.$ It follows from (24)that

$$\sup_{k\in N}\big\|(\xi(2s_1)-512\xi(s_1),\ldots,\xi(2s_k)-512\xi(s_k))\big\|_k \le \frac{171}{120960}\eta \tag{25}$$

$$\sup_{k\in N}\left\|\left(\xi(s_1)-\frac{\xi(2s_1)}{2^9},\ldots,\xi(s_k)-\frac{\xi(2s_k)}{2^9}\right)\right\|_k \le \frac{57}{20643840}\eta \tag{26}$$

forall $s_1,\ldots,s_k \in Z.$

Let $\Lambda=\{l:Z\to A\,|\,l(0)=0\}$ and introduce the generalized metric d defined on λ by

$$d(l,m)=\inf\left\{\lambda\in[0,\infty]\,|\sup_{k\in N}\|l(s_1)-m(s_1),\ldots,l(s_k)-m(s_k)\|_k \le \lambda \quad \forall \quad s_1,\ldots,s_k\in Z\right\}$$

Then it is easy to show that Λ, d is a generalized complete metric space, See [11].

We define an operator $\mathrm{J}:\Lambda\to\Lambda$ by

$$\mathrm{J}l(s)=\frac{1}{2^9}l(2s) \qquad s\in Z$$

We assert that J is a strictly contractive operator. Given $l,m\in\Lambda,$ let $\lambda\in[0,\infty]$ be an arbitary constant with $d(l,m)\le\lambda.$ From the definition if follows that

$$\sup_{k\in N}\|l(s_1)-m(s_1),\ldots,l(s_k)-m(s_k)\|_k \le \lambda \quad s_1,\ldots,s_k\in Z.$$

Therefore,

$$\sup_{k\in N}\big\|(\mathrm{J}l(s_1)-\mathrm{J}m(s_1),\ldots,\mathrm{J}l(s_k)-\mathrm{J}m(s_k))\big\|_k$$

$$\le \sup_{k\in N}\left\|\left(\frac{1}{2^9}l(2s_1)-\frac{1}{2^9}m(2s_1),\ldots,\frac{1}{2^9}l(2s_k)-\frac{1}{2^9}m(2s_k)\right)\right\|_k$$

$$\le \frac{1}{2^9}\lambda$$

$s_1,\ldots,s_k\in Z.$

Hence,it holds that

$$d(\mathrm{J}l,\mathrm{J}m) \le \frac{1}{2^9}\lambda d(\mathrm{J}l,\mathrm{J}m) \le \frac{1}{2^9} d(l,m)$$

$\forall l, m \in \Lambda.$

This Means that J is strictly contractive operator on Λ with the Lipschitz constant $L = \frac{1}{2^9}.$

By (26), we have $d(\mathrm{J}\xi,\xi) \le \frac{57}{20643840}\eta.$ According to Theorem 1.1, we deduce the existence of a fixed point of J that is the existence of mapping $N : Z \to A$ such that

$$N(2s) = 2^9 N(s) \qquad \forall s \in Z.$$

Moreover, we have $d\left(\mathrm{J}^n \xi, N\right) \to 0,$ which implies

$$N(s) = \lim_{n\to\infty} \mathrm{J}^n \xi(s) = \lim_{n\to\infty} \frac{\xi(2^n s)}{2^{9n}}$$

for all $s \in Z.$

Also, $d(\xi,N) \le \frac{1}{1-L} d(\mathrm{J}\xi,\xi)$ implies the inequality

$$d(\xi,N) \le \frac{1}{1-\frac{1}{2^9}} d(\mathrm{J}\xi,\xi)$$

$$\le \frac{19}{6867840}\eta.$$

Setting $s_1 =,\ldots,= s_k = 2^n s, t_1 =,\ldots,= t_k = 2^n t$ in (4) and divide both sides by $2^{9n}.$ Then, using property (a) of multi-norms, we obtain

$$\|DN(s,t)\| = \lim_{n\to\infty} \frac{1}{2^{9n}} \left\|D\xi\left(2^n s, 2^n t\right)\right\|$$

$$\le \lim_{n\to\infty} \frac{1}{2^{9n}} = 0$$

for all $s, t \in Z.$ Hence N is Nonic.

The uniqueness of N follows from the fact that N is the unique fixed point of J with the property that there exists $\ell \in (0,\infty)$ such that

$$\sup_{k\in \mathbb{N}} \left\|\left(\xi(s_1) - N(s_1),\ldots,\xi(s_k) - N(s_k)\right)\right\|_k \le \ell$$

for all $s_1,\ldots,s_k \in Z.$

This completes the proof of the Theorem.

References

[1] **Aoki.T**, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Jpn. 2 (1950), 64-66.

[2] **Czerwik.S**, *Functional Equations and Inequalities in Several Variables,* World Scientific Publishing Co., Singapore, New Jersey, London, (2002).

[3] **Diaz.J.B and Margolis.B**, *A fixed point theorem of the alternative, for contraction on a generalized complete metric space,* Bulletin of the American Mathematical Society, vol. 74 (1968), 305-309.

[4] **Dales, H.G and Moslehian**, *Stability of mappings on multi-normed spaces,* Glasgow Mathematical Journal, 49 (2007), 321-332.

[5] **Fridoun Moradlou**, *Approximate Euler-Lagrange-Jensen type Additive mapping in Multi-Banach Spaces: A Fixed point Approach,* Commun. Korean Math. Soc. 28 (2013), 319-333.

[6] **Hyers.D.H**, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[7] **Hyers.D.H, Isac.G, Rassias.T.M**, *Stability of Functional Equations in Several Variables,* BirkhÃ¤user, Basel, (1998).

[8] **John M. Rassias, Arunkumar.M, Sathya.E and Namachivayam.T**, *Various generalized Ulam-Hyers Stabilities of a nonic functional equations,* Tbilisi Mathematical Journal 9(1) (2016), 159-196.

[9] **Jun.K, Kim.H**, *The Generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. 274 (2002), 867-878.

[10] **Liguang Wang, Bo Liu and Ran Bai**, *Stability of a Mixed Type Functional Equation on Multi-Banach Spaces: A Fixed Point Approach,* Fixed Point Theory and Applications (2010), 9 pages.

[11] **Mihet.D and Radu.V**, *On the stability of the additive Cauchy functional equation in random normed spaces,* Journal of mathematical Analysis and Applications, 343 (2008), 567-572.

[12] **Radu.V**, *The fixed point alternative and the stability of functional equations,* Fixed Point Theory 4 (2003), 91-96.

[13] **Tian Zhou Xu, John Michael Rassias and Wan Xin Xu**, *Generalized Ulam - Hyers Stability of a General Mixed AQCQ functional equation in Multi-Banach Spaces: A Fixed point Approach,* European Journal of Pure and Applied Mathematics 3 (2010), 1032-1047.

[14] **Ulam.S.M**, *A Collection of the Mathematical Problems,* Interscience, New York, (1960).

[15] **Xiuzhong Wang, Lidan Chang, Guofen Liu,***Orthogonal Stability of Mixed Additive-Quadratic Jenson Type Functional Equation in Multi-Banach Spaces,* Advances in Pure Mathematics 5 (2015), 325-332.

[16] **Zhihua Wang, Xiaopei Li and Themistocles M. Rassias,***Stability of an Additive-Cubic-Quartic Functional Equation in Multi-Banach Spaces,* Abstract and Applied Analysis (2011), 11 pages.