Int. J. mech. Sci. Pergamon Press. 1972. Vol. 14, pp. 377–393. Printed in Great Britain

DISTORTION OF THE SEMI-INFINITE SOLID DUE TO TRANSIENT SURFACE HEATING

J. R. Barber
University Department of Mechanical Engineering,
Newcastle-upon-Tyne, NE1 7RU

(*Received* 19 *August* 1971)

Summary—A method is described for finding the normal displacements of the surface of a semi-infinite solid, subjected to arbitrary, transient surface heating. Analytical expressions are derived for the displacements due to instantaneous and continuous point and line heat sources and these can be integrated to produce the solutions to particular problems. The method is illustrated by application to uniform heating over a circular area and over an infinite band. Solutions are also obtained for the displacements due to point and line doublets which can be used to solve problems in which the temperature distribution at the boundary plane is defined.

These results permit the reduction of problems in the non-isothermal contact of solids to equivalent isothermal problems.

NOTATION

r, θ, ϕ	spherical polar co-ordinates
r, ϕ, z	cylindrical co-ordinates
a	radius of circular heat source or width of band source
A	$a/\sqrt{4kt}$
c	specific heat
C	Euler's constant
E	Young's modulus
F	concentrated force acting at the boundary
k	thermal diffusivity
$p_{rr}, p_{\theta\phi}$ etc.	stress components in double suffix notation
q	heat input per unit time
q'	strength of a continuous doublet
Q	heat input at an instantaneous source
Q'	strength of an instantaneous doublet
R	$r/\sqrt{4kt}$
s	alternative notation for radial co-ordinate r
S	$s/\sqrt{4kt}$
t	time
T	temperature
u	radial displacement
w	normal displacement of the surface
α	coefficient of linear thermal expansion
ν	Poisson's ratio
ρ	density

Non-dimensional displacements

$B(A, S)$	band source
$C(A, S)$	circular source
$D(S)$	point doublet
$L(S)$	line source
$P(S)$	point source
$R(R, S)$	ring source

1. INTRODUCTION

THIS paper is concerned with finding the normal displacements at the surface of a large solid which is subjected to non-uniform, transient surface heating. This problem arises in the non-isothermal contact of two solids. For example, if heat is conducted across the interface between two contacting solids of different materials, the thermal displacements produced will affect the distribution of contact pressure between the solids and hence the thermal contact resistance. The particular problem of steady heat flow between two contacting solids with spherical surfaces is analysed by Barber[1] and experimental results for this system are described by Clausing.[2] Other applications of this type occur in hot forming processes. Solutions for the pressure distribution under a heated punch were derived by George and Sneddon.[3]

If heat is generated at the interface between two solids, thermal distortion will again affect the contact pressure distribution. This effect can be found in sliding systems, where the heat is generated by friction,[4] and in electrical contacts, where it is generated by resistive losses due to constriction of current flow at the interface.

If the thermal boundary conditions are independent of the contact pressure distribution, any contact problem can be reduced to an equivalent isothermal contact problem by finding the surface displacements due to temperature changes alone and superposing these on the initial profile of the solid.[5] The theory of isothermal contact of elastic solids has been studied extensively. This paper is concerned with the first part of the problem; finding the surface displacements due to thermal expansion alone.

2. METHOD OF SOLUTION

The problem will be solved by the method described by Barber and Wright.[6] We first find the thermal stresses produced in an infinite solid whose temperature field is symmetrical about a plane and then neutralize the stresses at the plane of symmetry by superposing equal and opposite surface tractions, thus producing the solution for a semi-infinite solid with a stress-free surface.

This method could be applied directly to any particular system. However, a greater mathematical simplicity is achieved by finding the displacement due to point and line heat sources at the surface of the semi-infinite solid. The solutions to a wide range of practical problems can then be written down in the form of definite integrals of these results.

3. CONCENTRATED SOURCES OF HEAT

3.1. *The continuous point source*

We shall first find the normal displacement at a point on the surface of a semi-infinite solid, distance s from a continuous heat source of constant strength q, starting at time $t = 0$.

We consider the infinite solid in which a continuous heat source of strength $2q$ originates at time $t = 0$. Let a general point P have spherical polar co-ordinates r, θ, ϕ referred to the source as origin. The temperature at P is then

$$T = (q/2\pi k\rho c r)\,\mathrm{erfc}\,(r/\sqrt{4kt}) \tag{1}$$

where k, ρ, c are respectively the thermal diffusivity, density and specific heat of the material, all assumed constant (see Carslaw and Jaeger,[7] Section 10:4). It should be noted that Carslaw and Jaeger define the "strength" of a source as $q\rho c$. This terminology is not used here.

This temperature field is spherically symmetrical and hence the principal thermal stresses at P are

$$p_{rr} = \frac{-2\alpha E}{(1-\nu)\, r^3}\int_0^r Tr^2\,\mathrm{d}r, \tag{2}$$

$$p_{\theta\theta} = p_{\phi\phi} = \frac{\alpha E}{(1-\nu)}\left(\frac{1}{r^3}\int_0^r Tr^2\,\mathrm{d}r - T\right), \tag{3}$$

where α, E, ν are respectively the coefficient of thermal expansion, Young's modulus and Poisson's ratio for the material (see Timoshenko and Goodier,[8] Art. 152).

Only radial displacements will be produced by this stress system and any plane which includes the origin will remain plane.

Suppose we now divide the infinite solid at the plane $\phi = 0$ and maintain the surfaces plane by applying appropriate surface tractions. These tractions will be equal to the stresses transmitted across the plane in the infinite solid and consist of two shear stresses, $p_{\theta\phi}$, $p_{r\phi}$ (both of which are zero at all points by symmetry), and a tension $p_{\phi\phi}$ which is given by equation (3). Furthermore, there will be no heat flow across the plane of symmetry in the infinite solid and hence the temperature field defined by equation (1) is also that which would be produced in the semi-infinite solid by a continuous heat source, strength q, at the origin and zero heat flow across the rest of the boundary, since half of the heat will flow into each half space.

To find the normal displacements of the surface of the semi-infinite solid due to such a temperature field, we simply relax the tractions $p_{\phi\phi}$ which were required to keep the surface plane. The displacements produced on relaxation will be equal and opposite to those due to a stress distribution $p_{\phi\phi}$ on the surface of an isothermal solid and these can be found by integrating the classical solution for a point load.

The normal displacement (w) of the surface of a semi-infinite solid due to a point tensile load F at a distance r is

$$w = F(1-\nu^2)/\pi E r, \tag{4}$$

where w is defined as positive in the direction away from the solid (Timoshenko and Goodier,[8] Art. 138).

Thus, the normal displacement at a radius s on the surface of the heated semi-infinite solid is

$$w = -\int_0^\infty\int_0^{2\pi} \frac{p_{\phi\phi}(1-\nu^2)\, r\,\mathrm{d}\theta\,\mathrm{d}r}{\pi E\sqrt{(s^2-2sr\cos\theta+r^2)}}, \tag{5}$$

$$= \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c}\int_0^\infty\int_0^{2\pi} \frac{[(2kt/r^2)\,\mathrm{erf}(r/\sqrt{4kt}) - (4kt/\pi)^{\frac{1}{2}}\,(e^{-r^2/4kt}/r) - \mathrm{erfc}(r/\sqrt{4kt})]\,\mathrm{d}\theta\,\mathrm{d}r}{\sqrt{(s^2-2sr\cos\theta+r^2)}} \tag{6}$$

from equations (1), (3) and (5).

It is convenient to make the change of variables $R = r/\sqrt{4kt}$ and $S = s/\sqrt{4kt}$ in which case equation (6) becomes

$$w = \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c}P(S) \tag{7}$$

where

$$P(S) = \int_0^\infty\int_0^{2\pi} \frac{\{[\mathrm{erf}(R)/2R^2] - (\mathrm{e}^{-R^2}/R\sqrt{\pi}) - \mathrm{erfc}\,(R)\}}{\sqrt{(S^2-2SR\cos\theta+R^2)}}\,\mathrm{d}\theta\,\mathrm{d}R. \tag{8}$$

A convergent series representation in powers of S can be obtained for this integral and is

$$P(S) = 2\pi\ln(S/2) + \pi(1+C) + 2\pi\sum_{i=1}^{\infty}\frac{(2i+1)!!\,(-S^2)^i}{(2i+2)!!\,i!\,i}, \tag{9}$$

where $C = 0{\cdot}577216\ldots$ is Euler's constant and the notation $(2i+1)!!$ is used for the product $1.3.5.\ldots.(2i+1)$. At large values of S, $P(S)$ approximates to the asymptotic expression

$$P^*(S) = -2\sqrt{\pi}\sum_{i=0}^{n}\frac{(2i+1)!!\,(2i-1)!!}{(2i+3)\,i!\,4^i\,S^{(2i+3)}}, \tag{10}$$

where $(-1)!!$ takes the value $+1$. This expression diverges at large values of n, but in the range $S > 4$ it converges to five significant figures in the first six terms and agrees with equation (9) to the same accuracy.

It will be noted from equation (7) that the displacement (w) only depends on position and time through the single non-dimensional parameter $S = s/\sqrt{4kt}$. The relationship is shown graphically in Fig. 1. The dotted lines a, b represent equations (9) and (10) respectively with the series truncated to six terms.

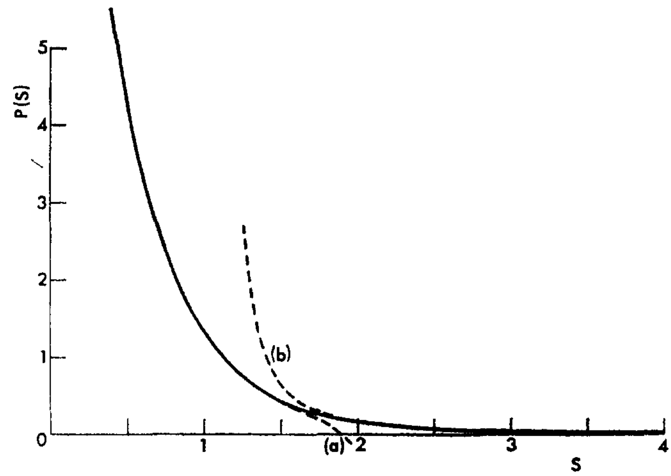

FIG. 1. Normal surface displacement due to the continuous point heat source. The solid line represents the function $P(s)$, defined by equations (7)–(9), whilst the dotted lines (a, b) represent equations (9) and (10) respectively with the series truncated to six terms.

3.1.1. *The steady-state point source.* As time (t) tends to infinity, the temperature distribution due to the continuous point source defined by equation (1) tends to the limiting value

$$T = q/2\pi k\rho c r. \tag{11}$$

However, the thermal displacement (w) due to this temperature field, given by equation (7), is not bounded as t tends to infinity, but tends to

$$w = q\alpha(1+\nu)/[\ln(16kt) - 1 - C - 2\ln(s)]\,4\pi k\rho c. \tag{12}$$

The singular term in equation (12), $\ln(16kt)$, is independent of s and therefore represents a rigid body displacement of the finite domain. Thus, the relative displacement between two finite points at radii s and s_0 tends to a limit at large t given by

$$w(s) - w(s_0) = \frac{q\alpha(1+\nu)}{2\pi k\rho c}\ln\left(\frac{s_0}{s}\right). \tag{13}$$

This equation may be regarded as giving the local effect of a steady-state point source on the boundary of a large finite solid. A similar type of singularity is encountered in corresponding two-dimensional potential flow and isothermal contact problems.

3.1.2. *The instantaneous point source.* The displacement due to an instantaneous point heat source at the origin can readily be found by differentiating equation (7) with respect to time. Thus,

$$\begin{aligned}\frac{dw}{dt} &= -\frac{S}{2t}\frac{dw}{dS}\\ &= \frac{q\alpha(1+\nu)}{2\pi k\rho c t}\sum_{i=0}^{\infty}\frac{(2i+1)!!\,(-S^2)^i}{(2i+2)!!\,i!}.\end{aligned} \tag{14}$$

The instantaneous point source solution is obtained by letting $q\,dt$ tend to a constant quantity of heat Q. Thus,

$$w = \frac{Q\alpha(1+\nu)}{2\pi k\rho ct}\sum_{i=0}^{\infty}\frac{(2i+1)!!\,(-S^2)^i}{(2i+2)!!\,i!}, \tag{15}$$

$$= \frac{Q\alpha(1+\nu)}{4\pi k\rho ct}\,\Phi\left(\tfrac{3}{2}, 2, -S^2\right) \tag{16}$$

where Φ is a degenerate hypergeometric function (see Gradshteyn and Ryzhik,[9] Section 9.21).

For large values of S, we obtain the expression

$$w = \frac{Q\alpha(1+\nu)}{4\pi^{\frac{3}{2}}\,\rho ckt}\sum_{i=0}^{n}\frac{(2i+1)!!\,(2i-1)!!}{4^i\,i!\,S^{(2i+3)}} \tag{17}$$

from expression (10). This expression diverges at large n, but converges for small n.

Fig. 2 shows the displacement at a particular value of s as a function of time t due to an instantaneous source at $t = 0$. The maximum displacement occurs when

$$t = s^2/6{\cdot}32k. \tag{18}$$

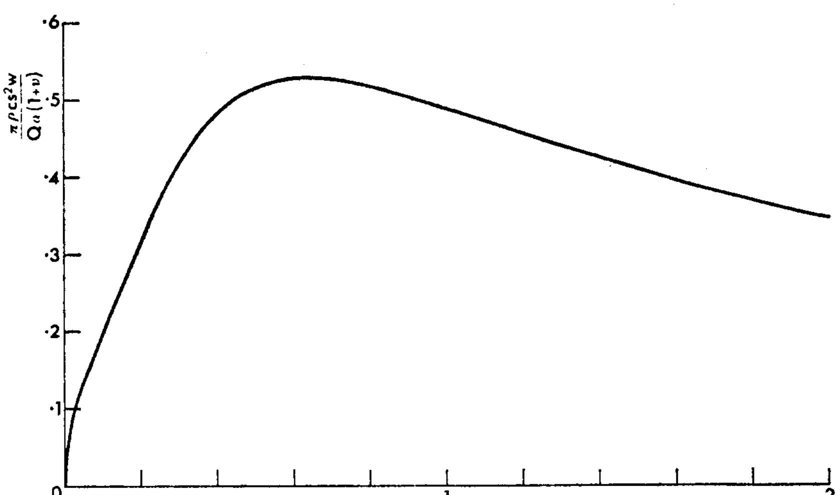

FIG. 2. Normal surface displacement due to the instantaneous point source as a function of time [equation (16)].

3.2. *The continuous line source*

The normal surface displacement due to a continuous line heat source of strength q per unit length on the surface of a semi-infinite solid can be found either by integrating the point source solution or from first principles, using the two-dimensional equivalents of equations (1)–(4). We shall give a shortened form of the derivation from first principles for conditions of plane strain. Corresponding plane stress results can be obtained by a similar method (see below, Section 3.2.1).

The temperature field due to the line source is

$$T = \frac{-q}{2\pi k\rho c}\,Ei\left(\frac{-r^2}{4kt}\right), \tag{19}$$

see Carslaw and Jaeger,[7] Section 10:4.

The corresponding infinite body stresses in the cylindrical co-ordinate system (r, ϕ, z) are

$$p_{rr} = \frac{-\alpha E}{(1-\nu)\,r^2}\int_0^r Tr\,\mathrm{d}r, \tag{20}$$

$$p_{\phi\phi} = \frac{\alpha E}{(1-\nu)}\left(\frac{1}{r^2}\int_0^r Tr\,\mathrm{d}r - T\right), \tag{21}$$

see Timoshenko and Goodier,[8] Art. 151.

A tensile line load, F per unit length on the surface of a semi-infinite solid produces a displacement

$$w = \frac{2F(1-\nu^2)}{\pi E}\ln\left(\frac{r_0}{r}\right) \tag{22}$$

(see Love,[10] Section 150), where r_0 is an arbitrary datum at which w is defined to be zero to remove the infinite rigid body displacement.

The normal surface displacement at a distance s from the source on the heated semi-infinite solid is obtained by neutralizing the normal stresses $p_{\phi\phi}$ at the plane $\phi = 0$ with equal and opposite tractions, giving

$$w = -\int_{-\infty}^{+\infty}\frac{2p_{\phi\phi}(1-\nu^2)}{\pi E}\ln\left(\frac{r_0}{|r-s|}\right)\mathrm{d}r, \tag{23}$$

$$= \frac{-q\alpha(1+\nu)\sqrt{kt}}{\pi^2\,k\rho c}\,L(S) \tag{24}$$

where

$$L(S) = \int_{-\infty}^{+\infty}\left[Ei(-R^2) + \frac{1-\mathrm{e}^{-R}}{R^2}\right]\ln\left(\frac{R_0}{|R-S|}\right)\mathrm{d}R \tag{25}$$

$$= \pi^2|S| + 4\sqrt{\pi}\sum_{i=0}^{\infty}\frac{(-2S^2)^i}{(2i-1)\,(2i+1)!!}. \tag{26}$$

An asymptotic expression for large values of S is

$$L^*(S) = -4\sqrt{\pi}\sum_{i=0}^{n}\frac{(2i-1)!!}{(2i+3)\,(2S^2)^{i+1}}. \tag{27}$$

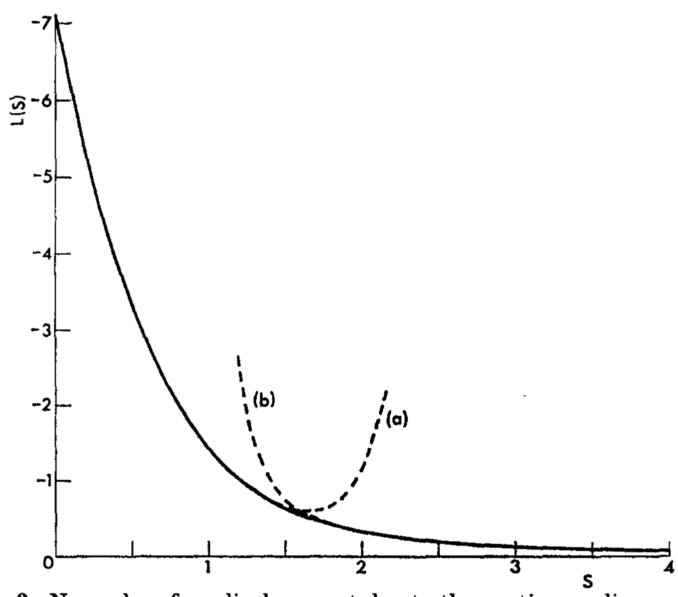

FIG. 3. Normal surface displacement due to the continuous line source. The solid line represents the function $L(S)$ defined by equations (24)–(26), whilst the dotted lines (a, b) represent equations (26) and (27) with the series truncated to six terms.

These results are shown graphically in Fig. 3; the continuous line represents the exact solution, whilst the dotted lines a and b represent equations (26) and (27) respectively with the series truncated to six terms.

3.2.1. *Plane stress.* For a thin semi-infinite plate with no stresses acting on its sides, the derivation is similar except that the stresses given by equations (20) and (21) are multiplied by the constant $(1-\nu)$ (see Timoshenko and Goodier,[8] Art. 150) and the displacement given by equation (22) is divided by $(1-\nu^2)$ (Timoshenko and Goodier,[8] Art. 36). Hence the displacement due to a line continuous heat source at the edge of a thin plate is

$$w = \frac{-q\alpha\sqrt{kt}}{\pi^2 k\rho c} L(S). \tag{28}$$

Similarly, in the following equations, (29)–(32) and (53), the corresponding plane stress results can be obtained by deleting the term $(1+\nu)$.

3.2.2. *The steady-state line source.* As t tends to infinity, equation (26) predicts an unbounded rigid body displacement of the finite domain, as with the point heat source (Section 3.1.1), but the relative normal displacement of two points on the surface distance s and s_0 from the source tends to the limit

$$w(s) - w(s_0) = \frac{q\alpha(1+\nu)(s_0 - s)}{2k\rho c}. \tag{29}$$

Alternatively, this result can be obtained from equation (13) by integration.

3.2.3. *The instantaneous line source.* The displacement due to an instantaneous line heat source at the origin, of strength Q per unit length is obtained by differentiating equation (26) as in Section 3.1.2 giving

$$w = \frac{2Q\alpha(1+\nu)}{\pi^{3/2}\,\rho c\sqrt{kt}} \sum_{i=0}^{\infty} \frac{(-2S^2)^i}{(2i+1)!!} \tag{30}$$

$$= \frac{Q\alpha(1+\nu)\, e^{-S^2}\operatorname{erf}(S)}{\pi\rho c\sqrt{(kt)}\, S}. \tag{31}$$

At large values of S, this can be approximated by the asymptotic expression

$$w^* = \frac{2Q\alpha(1+\nu)}{\pi^{3/2}\rho c\sqrt{kt}} \sum_{i=0}^{n} \frac{(2i-1)!!}{(2S^2)^{i+1}}. \tag{32}$$

4. DISTRIBUTED HEAT SOURCES

The surface displacements of a semi-infinite solid with heat sources distributed over the boundary can be found by a suitable integration of the point or line source solutions derived in the preceding sections. We shall derive a number of simple solutions of this type which will serve as illustrations of the method and which can also be used as elemental solutions for numerical integration.

4.1. *Ring heat source*

Suppose heat is supplied continuously after time $t = 0$ at a rate $q/2\pi r$ per unit length around a ring of radius r with its centre at the origin. The displacement at a radius s can be obtained by integrating equation (7) giving

$$w = \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c} R(R, S), \tag{33}$$

where

$$R(R,S) = \int_0^{2\pi} P[\sqrt{(R^2-2RS\cos\theta+S^2)}]\frac{d\theta}{2\pi} \tag{34}$$

$$= 2\pi\ln(X)+\pi(1+C)+2\pi\sum_{i=0}^{\infty}\frac{(2i+1)!!(-R^2)^i}{(2i+2)!!\,i!\,i}\sum_{j=0}^{i}({}^iC_j)^2(S/R)^{2j} \tag{35}$$

where ${}^iC_j = [j!(i-j)!]/i!$ is the binomial coefficient and $X = R/2$ if $R>S$ and otherwise $= S/2$. We note from equation (34) that

$$R(X,0) = R(0,X) = P(X). \tag{36}$$

4.1.1. *The steady-state ring source.* The relative displacement due to a steady-state ring source of strength $q/2\pi r$ per unit length is most conveniently obtained by integrating equation (13) and taking the reference point (s_0) to be distant from the source (i.e. $s_0 \gg r$). This gives

$$w(s)-w(s_0) = \frac{q\alpha(1+\nu)}{2\pi k\rho c}\ln\left(\frac{s_0}{s}\right) \quad \text{for } s>r \tag{37}$$

and

$$= \frac{q\alpha(1+\nu)}{2\pi k\rho c}\ln\left(\frac{s_0}{r}\right) \quad \text{for } s\leqslant r. \tag{38}$$

Thus, the displacement is constant at all points within the ring. Comparing equations (38) and (13) it is apparent that the displacement outside the ring is equal to that which would be produced by an equal point source at the centre.

4.2. *The uniform circular heat source*

The displacement due to a uniform circular heat source of radius a and strength $q/\pi a^2$ per unit area can be found by integrating equation (33) with respect to r. Thus,

$$w = \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c}C(A,S), \tag{39}$$

where $A = a/\sqrt{4kt}$ and

$$C(A,S) = 2\int_0^A R(R,S)\frac{R\,dR}{A^2} \tag{40}$$

$$= 2\pi\ln(S/2)+\pi(1+C)$$
$$+2\pi\sum_{i=1}^{\infty}\frac{(2i+1)!!(-S^2)^i}{(2i+2)!!\,i!\,i}\sum_{j=0}^{i}({}^iC_j)^2\frac{(A/S)^{2j}}{(j+1)} \quad \text{for } S\geqslant A \tag{41}$$

and

$$= 2\pi\ln(A/2)+\pi(C+S^2/A^2)$$
$$+2\pi\sum_{i=1}^{\infty}\frac{(2i+1)!!(-A^2)^i}{(2i+2)!!\,i!\,i}\sum_{j=0}^{i}({}^iC_j)^2\frac{(S/A)^{2j}}{(i-j+1)} \quad \text{for } S<A. \tag{42}$$

These results are illustrated graphically in Fig. 4.

If A tends to zero, this result reduces to that for the steady-state point source (Section 3.1.1). In other words,

$$C(0,S) = P(S). \tag{43}$$

This can be seen by comparing equations (9) and (41).

If S tends to zero, we obtain the displacement at the centre of the heated circle, which is

$$w = \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c}\left[2\pi \ln(A/2)+\pi C+2\pi \sum_{i=1}^{\infty} \frac{(2i+1)!!\,(-A^2)^i}{(2i+2)!!\,(i+1)!\,i}\right]. \tag{44}$$

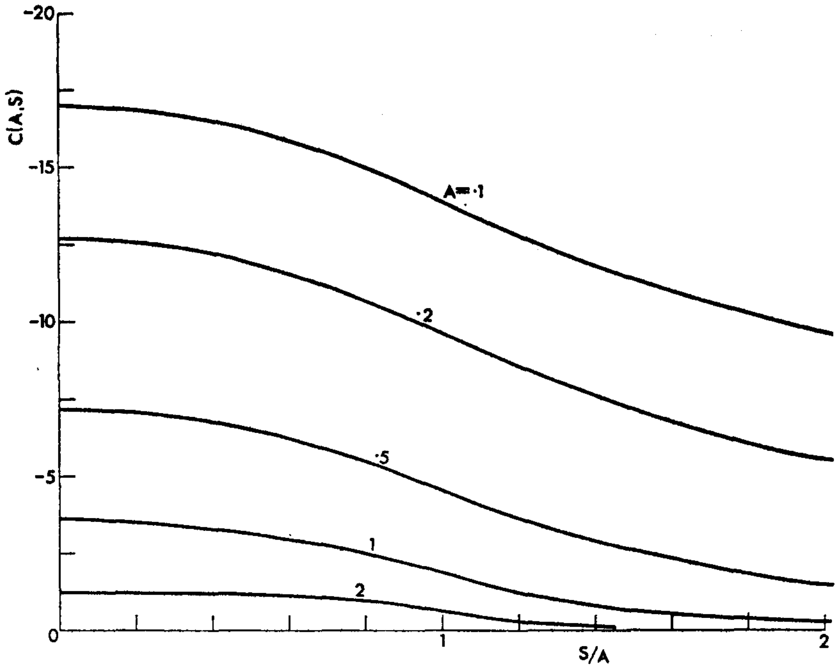

FIG. 4. Normal surface displacement due to the continuous circular source [equations (39)–(42)].

This is an analytic expression for the integral evaluated numerically by Barber and Wright.[6] From equations (36) and (40) we have

$$\tfrac{1}{2}A^2\, C(A,0) = \int_0^A RP(R)\,\mathrm{d}R \tag{45}$$

and it can be shown that this integral tends to the limit $-\pi$ as A tends to infinity (E. N. Fox, private communication). Thus, for large values of A we can use the result

$$\tfrac{1}{2}A^2\, C(A,0) = -\pi - \int_A^{\infty} RP(R)\,\mathrm{d}R \tag{46}$$

and the asymptotic expression (10) for $P^*(R)$ to obtain a corresponding expression for $C(A,0)$, which is

$$C^*(A,0) = \frac{-2\pi}{A^2} + 2\sqrt{\pi}\sum_{i=0}^{n} \frac{[(2i-1)!!]^2}{4^i(2i+3)\,i!\,A^{(2i+3)}}. \tag{47}$$

If we substitute this result into equation (39) and write

$$Q = qt/\pi a^2 \tag{48}$$

for the cumulative quantity of heat supplied per unit area since $t = 0$, we find that

$$\lim_{a\to\infty}(w) = \frac{2Q\alpha(1+\nu)}{\rho c}. \tag{49}$$

4.3. *Sector of a circle*

It follows by considerations of symmetry that if only a sector of the circle, which subtends an angle θ, is heated at a rate $q/\pi a^2$ per unit area, the displacement at the centre will be

$$w = \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c} C(A, 0)\frac{\theta}{2\pi}. \tag{50}$$

This result provides a new elemental solution which can be used for the purposes of integration.

Suppose we wish to find the displacement at a point P inside a uniformly heated area on the surface. We consider the heated area to be made up of a series of infinitesimal sectors centred on P. Thus, if the boundary of the heated area is defined by the equation

$$R = \mathscr{F}(\theta) \tag{51}$$

relative to the point P as origin; it follows that the displacement at P will be

$$w = \frac{-q\alpha(1+\nu)}{4\pi^2 k\rho c}\left(\int_0^{2\pi} C(R, 0)\, R^2\, \mathrm{d}\theta \Big/ \int_0^{2\pi} R^2\, \mathrm{d}\theta\right), \tag{52}$$

where q is the total rate of heat supply to the entire area.

As an example, we can apply this argument to the result expressed by equation (49), from which it follows that, if a large ($a^2 \gg 4kt$) circular area of the surface is heated uniformly, the central region (comprising all points whose distance from the boundary is much greater than $\sqrt{4kt}$) will be flat, the displacement being given by equation (49), whilst the displacement at the boundary of the heated region will be half that at the centre.

This method of integration can be extended to points outside the heated area and to cases of non-uniform heating. However, it is particularly valuable for the numerical computation of displacement at points within the heated area, since the function $C(R, 0)$ has no singular points. A corresponding evaluation using the point source solution $P(R)$ requires special treatment in the range where R is small.

4.4. *The infinite band source*

Two-dimensional problems can be solved by integrating the solution for an infinite line source (Section 3.2). For example, if heat is supplied at a rate $q/2a$ per unit area over the band $-a < r < a$, the displacement at a distance s from the centre of the band is

$$w = \frac{-q\alpha(1+\nu)\sqrt{kt}}{\pi^2 k\rho c} B(A, S), \tag{53}$$

where

$$B(A, S) = \int_{-A}^{+A} L(|R+S|)\frac{\mathrm{d}R}{2A} \tag{54}$$

$$= \pi^2 |S| + 4\sqrt{\pi}\sum_{i=0}^{\infty}\frac{(-2)^i\{[(S+A)^{2i+1} - (S-A)^{2i+1}]/2A\}}{(4i^2-1)(2i+1)!!} \tag{55}$$

$$= \pi^2 |S| + 4\sqrt{\pi}\sum_{i=0}^{\infty}\frac{(-2S^2)^i}{(4i^2-1)(2i+1)!!}\sum_{j=0}^{i} {}^{2i+1}C_{2j+1}\left(\frac{A}{S}\right)^{2j} \quad \text{for } S^2 > A^2 \tag{56}$$

and

$$= \frac{\pi^2(S^2+A^2)}{2A} + 4\sqrt{\pi}\sum_{i=0}^{\infty}\frac{(-2)^i}{(4i^2-1)(2i+1)!!}\left[\frac{(S+A)^{2i+1} - (S-A)^{2i+1}}{2A}\right]$$

$$\text{for } S^2 < A^2. \tag{57}$$

We note from equations (26) and (56) that

$$B(0, S) = L(S) \tag{58}$$

[cf. equation (43)].

5. SURFACE TEMPERATURE SPECIFIED

The solutions derived in Sections 3 and 4 can be used to write down surface displacements as a definite integral of the heat input to the solid. However, the boundary conditions may be defined in terms of the surface temperature, in which case the point source solution can only be used if we first solve the heat conduction problem.

A more convenient method is to make use of the solution for a "point doublet". A doublet can be regarded as two sources of strength q and $-q$ separated by a small distance δz, the limit being taken as δz tends to zero whilst $q\delta z$ is maintained constant and finite. Alternatively, the solution (which is actually the differential of Green's function for the semi-infinite solid) can be derived by differentiating the point source solution with respect to a space parameter.

If the temperature at the boundary ($z = 0$) of the semi-infinite solid is a known function of position and time, the temperature field in the solid can be found by placing a corresponding distribution of point doublets at the plane $z = 0$ in the infinite solid, with their axes in the z direction. The proof of this result and a full discussion of the use of doublets in the solution of heat conduction problems can be found in Carslaw and Jaeger,[7] Sections 10:8 and 14:9. In the following section we shall find the displacement due to point and line continuous doublets from first principles and derive the steady state and instantaneous doublet solutions as limiting cases.

5.1. *Displacement due to a point continuous doublet*

Consider an infinite solid in which are located two equal and opposite point continuous sources, q, at the point $(0, 0, \delta z/2)$ and $-q$, at $(0, 0, -\delta z/2)$, where δz is a small quantity. Equations (2) and (3) give the stresses produced by each source *referred to itself as origin*. We therefore use the stress transformation relations to refer these stresses to the true origin. In particular, the stresses due to the positive source on the plane $z = 0$ are

$$p'_{\phi\phi} = p_{\phi\phi} + O(\delta z^2), \tag{59}$$

$$p'_{r\phi} = (\delta z/2r)(p_{\phi\phi} - p_{rr}) + O(\delta z^3), \tag{60}$$

$$p'_{\theta\phi} = 0, \tag{61}$$

where $p_{\phi\phi}$ and p_{rr} are defined by equations (2) and (3).

Superposing the corresponding stresses due to the equal and opposite source $-q$ at $(0, 0, -\delta z/2)$, we get

$$p'_{\phi\phi} = 0, \tag{62}$$

$$p'_{r\phi} = (\delta z/r)(p_{\phi\phi} - p_{rr}) + O(\delta z^3), \tag{63}$$

$$p'_{\theta\phi} = 0. \tag{64}$$

The shear stress $p'_{r\phi}$ on the half-plane containing the positive source is directed towards the origin.

Substituting for $p_{\phi\phi}$ and p_{rr} from equations (2) and (3) and dropping the prime on $p'_{r\phi}$ we get

$$p_{r\phi} = \frac{\delta z \alpha E}{1-\nu}\left(\frac{3}{r^4}\int_0^r Tr^2\,\mathrm{d}r - \frac{T}{r}\right). \tag{65}$$

The temperature field due to the point doublet is not an even function of z and there will therefore generally be a component of normal displacement w at the plane $z = 0$ in the infinite solid. The displacement at a radius s due to the positive source acting alone is purely radial relative to the source itself and is given by

$$u = \frac{1+\nu}{1-\nu}\frac{\alpha}{s^2}\int_0^s Tr^2\,\mathrm{d}r \tag{66}$$

(see Timoshenko and Goodier,[8] Art. 152).

Resolving this in the z direction, we get at the plane $z = 0$

$$w_1 = \frac{1+\nu}{1-\nu}\frac{\delta z\alpha}{2s^3}\int_0^s Tr^2\,\mathrm{d}r. \tag{67}$$

For the equal and opposite source, both T and δz change sign so that the normal displacements are of the same sign and the sum is equal to $2w_1$.

To find the displacement of the semi-infinite solid due to the point doublet, we must superpose surface forces at the boundary equal and opposite to $p_{r\phi}$.

A ring of shear force, F per unit length and radius r, directed towards the origin, produces a displacement at a radius s given by

$$w = \frac{-F(1+\nu)(1-2\nu)}{E} \quad \text{for } s<r \tag{68}$$

and

$$= 0 \quad \text{for } s>r \tag{69}$$

(see Appendix).

Thus, the normal surface displacement at a radius s due to the surface stress distribution $-p_{r\phi}$ is

$$w_2 = \int_s^\infty \frac{p_{r\phi}(1+\nu)(1-2\nu)}{E}\,\mathrm{d}r \tag{70}$$

$$= \frac{(1+\nu)(1-2\nu)}{1-\nu}\delta z\alpha\int_s^\infty\left(\frac{3}{r^4}\int_0^r Tr^2\,\mathrm{d}r - \frac{T}{r}\right)\mathrm{d}r \quad \text{[from equation (65)]} \tag{71}$$

$$= \frac{(1+\nu)(1-2\nu)}{1-\nu}\delta z\alpha\left(\frac{1}{s^3}\int_0^s Tr^2\,\mathrm{d}r\right). \tag{72}$$

To find the displacement of the surface of a semi-infinite solid with the temperature field due to a point doublet and a stress free surface, we superpose this value on the displacement at $z = 0$ in the infinite solid. Thus,

$$w = 2w_1 + w_2 = 2(1+\nu)\,\delta z\alpha\left(\frac{1}{s^3}\int_0^s Tr^2\,\mathrm{d}r\right) \tag{73}$$

from equations (67) and (72).

The value of T for a source of strength q acting alone in the infinite solid will be half that given by equation (1). Using this value, we get

$$w = \frac{(1+\nu)\,q\delta z\alpha}{4\pi k\rho c\sqrt{4kt}}D(S), \tag{74}$$

where

$$D(S) = \frac{2}{S^3}\int_0^S R\,\mathrm{erfc}\,(R)\,\mathrm{d}R \tag{75}$$

$$= \frac{\mathrm{erfc}\,(S)}{S} - \frac{\mathrm{e}^{-S}}{\sqrt{(\pi)}\,S^2} + \frac{\mathrm{erf}\,(S)}{2S^3}. \tag{76}$$

The function $D(S)$ also permits the convergent series representation

$$D(S) = \frac{1}{S} - \frac{4}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-S^2)^i}{(2i+3)(2i+1)\,i!}. \tag{77}$$

To complete the doublet solution, we allow δz to tend to zero, whilst the product $q\delta z$ tends to a finite limit q', which is known as the "strength" of the doublet. Thus, the displacement due to a doublet of strength q' is

$$w = \frac{(1+\nu)\,q'\alpha}{4\pi k\rho c\sqrt{4kt}}D(S). \tag{78}$$

5.1.1. *Steady-state point doublet.* As t tends to infinity, S tends to zero and

$$D(S) \to \frac{1}{S} - \frac{4}{3\sqrt{\pi}}. \tag{79}$$

Thus, the steady-state displacement is

$$w = \frac{q'\alpha(1+\nu)}{4\pi k\rho c s}. \tag{80}$$

Comparing equations (4) and (80) we note that a point doublet of strength q' produces the same normal surface displacements as a point tensile load

$$F = \frac{q'\alpha E}{4k\rho c(1-\nu)}. \tag{81}$$

A distribution of doublets of strength

$$q' = 2k\rho c T \tag{82}$$

per unit area produces a temperature distribution T at the boundary $z = 0$ (see Carslaw and Jaeger,[7] Section 10:8). Hence, if the surface of the semi-infinite solid is maintained at a steady temperature T, where T is any function of position, the normal displacement produced will be the same as that due to a distribution of tensile stress at the boundary given by

$$p_{zz} = \frac{\alpha E T}{2(1-\nu)}. \tag{83}$$

The classical solutions for surface loading of the semi-infinite solid can therefore be used to generate a corresponding series of solutions to the thermal displacement problem. For example, if a circular region of the surface, radius a, is maintained at a constant, uniform temperature T, whilst the rest of the surface is maintained at zero temperature, the displacement is equal to that produced by a uniform tensile stress p_{zz} acting over the heated circle, the value of which is given by equation (83). The displacement due to this stress distribution is given by Timoshenko and Goodier,[8] Art. 138, and is

$$w = \frac{2\alpha T(1+\nu)\,r}{\pi}\left[E\left(\frac{a}{r}\right) - \left(1-\frac{a^2}{r^2}\right)K\left(\frac{a}{r}\right)\right] \tag{84}$$

for $r \geqslant a$ and

$$w = \frac{2\alpha T(1+\nu)\,a}{\pi}E\left(\frac{r}{a}\right) \tag{85}$$

for $r \leqslant a$, where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively.

5.1.2. *Heated-punch problems.* Equation (83) also enables us to write down the contact stresses and hence the total load for a particular type of heated-punch problem (referred to by George and Sneddon,[3] as "type a").

Suppose a rigid punch of known profile is pressed into the surface of a semi-infinite elastic solid and suppose the steady temperature in the contact region is a known function (T) of position, whilst the surface of the semi-infinite solid outside the contact region is maintained at zero temperature.

Let p be the contact pressure distribution under the heated punch, whilst p_0 is the pressure distribution necessary to produce the same indentation under isothermal conditions. The displacement in the contact region is the same in both cases and in the heated case can be regarded as due to the combined action of the actual pressure distribution (p) and an equivalent tensile stress given by equation (83).

It follows by the uniqueness theorem that

$$p - p_0 = \frac{\alpha E T}{2(1-\nu)} \tag{86}$$

throughout the contact area. The total load (F) is obtained by integration over the contact area giving

$$F - F_0 = \iint_A \frac{\alpha E T}{2(1-\nu)} \, dA. \tag{87}$$

These equations are not restricted to any particular shape of punch and the temperature distribution can be completely general in the contact region. The only restrictions are that the temperatures should not be time dependent and the surface temperature outside the contact area should be zero.

If we restrict our attention to the particular case of indentation by a flat-ended cylindrical punch, equation (87) reduces to equation (2.5) of George and Sneddon.[3]

5.1.3. *Instantaneous point doublet.* The displacement due to an instantaneous point doublet is obtained by differentiating equation (78) with respect to time (cf. Section 3.1.2), giving

$$w = \frac{Q'\alpha(1+\nu)}{\pi \rho c s^3} \left(\frac{\operatorname{erf}(S)}{2} - \frac{S e^{-S^2}}{\sqrt{\pi}} \right) \tag{88}$$

or in series form

$$w = \frac{2Q'\alpha(1+\nu)}{\pi^{\frac{3}{2}} \rho c s^3} \sum_{i=0}^{\infty} \frac{(-1)^i S^{2i+3}}{(2i+3)\, i!}, \tag{89}$$

where Q' is the limiting value of $q'\delta t$, as δt tends to zero. Alternatively, it may be regarded as the limiting value of $Q\delta z$, as δz tends to zero, where Q is the strength of two equal and opposite instantaneous point sources separated by a distance δz.

5.2. *Continuous line doublet*

The solution for a line doublet follows the same pattern as that for the point doublet (Section 5.1), except that equations (19)–(21) are used in place of equations (1)–(3).

Thus, the stresses on the plane $z = 0$ due to a continuous line source at $(0, \frac{1}{2}\delta z)$ are

$$p'_{\phi\phi} = p_{\phi\phi} + O(\delta z^2), \tag{90}$$

$$p'_{r\phi} = (\delta z/2r)(p_{\phi\phi} - p_{rr}) + O(\delta z^3), \tag{91}$$

where $p_{\phi\phi}$ and p_{rr} are given by equations (20) and (21) and $p'_{r\phi}$ is directed towards the origin on the half space containing the positive heat source.

Superposing an equal and opposite source at $(0, -\frac{1}{2}\delta z)$, substituting for $p_{\phi\phi}$ and p_{rr} from equations (20) and (21) and dropping the prime on $p'_{r\phi}$ we obtain

$$p_{r\phi} = \frac{\delta z \alpha E}{1-\nu} \left(\frac{2}{r^3} \int_0^r T r \, dr - \frac{T}{r} \right). \tag{92}$$

The radial displacement of the infinite solid at a radius s due to a single source is

$$u = \frac{(1+\nu)\alpha}{(1-\nu)s} \int_0^s T r \, dr. \tag{93}$$

Resolving in the z direction we obtain the normal displacement at the plane $z = 0$

$$w_1 = \frac{1+\nu}{1-\nu} \frac{\delta z}{2s^2} \int_0^s T r \, dr. \tag{94}$$

The displacement due to the two sources is $2w_1$ as in Section 5.1.

Two equal, parallel and opposite line shear forces on the surface of a semi-infinite solid, F per unit length, can be regarded as forming a curve closed at infinity and the displacement at any point between the lines will therefore be

$$w = \frac{-F(1+\nu)(1-2\nu)}{E}, \tag{95}$$

whilst at all other points it will be zero (see Appendix). [Alternatively, this result can be obtained from Love,[10] Art. 150, by the reciprocal theorem or by integrating equation (106).] The shear forces are interpreted as positive if they are directed towards each other.

Thus, the shear stress distribution $-p_{r\phi}$ will produce a displacement

$$w_2 = \int_s^\infty \frac{p_{r\phi}(1+\nu)(1-2\nu)\,dr}{E} \quad \text{[cf. equation (71)]} \tag{96}$$

$$= \frac{(1+\nu)(1-2\nu)}{1-\nu}\,\delta z\alpha \int_s^\infty \left(\frac{2}{r^3}\int_0^r Tr\,dr - \frac{T}{r}\right) dr \quad \text{[from equation (92)]} \tag{97}$$

$$= \frac{(1+\nu)(1-2\nu)}{1-\nu}\,\delta z\alpha \left(\frac{1}{s^2}\int_0^s Tr\,dr\right). \tag{98}$$

Thus, the displacement of the semi-infinite solid due to a line doublet is

$$w = 2w_1 + w_2 = 2(1+\nu)\,\delta z\alpha \left(\frac{1}{s^3}\int_0^s Tr\,dr\right), \tag{99}$$

$$= \frac{\alpha q'(1+\nu)}{4\pi kpc}\left[\frac{1-e^{-S^2}}{S^2} - Ei(-S^2)\right] \tag{100}$$

from equation (19), where

$$q' = \lim_{\delta z \to 0} (q\delta z). \tag{101}$$

A convergent series representation is

$$w = \frac{\alpha q'(1+\nu)}{4\pi kpc}\left(1 - C - 2\ln(S) - \sum_{i=1}^{\infty}\frac{(-S^2)^i}{i(i+1)!}\right). \tag{102}$$

5.2.1. *Steady-state line doublet.* Equation (102) is not bounded as S tends to zero ($t \to \infty$), but we can obtain a bounded solution for relative displacement as in Section 3.1.1. We choose a finite point (s_0) as datum, in which case the steady-state displacement is

$$w(s) - w(s_0) = \frac{\alpha q'(1+\nu)}{2\pi kpc}\ln\left(\frac{s_0}{s}\right). \tag{103}$$

We have already noted that the point doublet produces the same normal surface displacements as an equivalent point load F defined by equation (76). It follows that the line doublet produces the same normal surface displacements as an equivalent line load. Thus, equation (103) can be derived directly from equations (4), (22) and (75).

5.2.2. *Instantaneous line doublet.* The displacement due to an instantaneous line doublet is obtained by differentiating equation (96) with respect to t and is

$$w = \frac{\alpha Q'(1+\nu)}{\pi pcs^2}(1 - e^{-S^2}), \tag{104}$$

$$= -\frac{\alpha Q'(1+\nu)}{\pi pcs^2}\sum_{i=1}^{\infty}\frac{(-S^2)^i}{i!}. \tag{105}$$

6. CONCLUSIONS

The surface displacements of a semi-infinite solid with any known transient or steady-state heat input at the surface can be written down as a definite integral of one of the singular solutions derived in Sections 3 and 4. Alternatively, if the temperature of the surface is specified instead of the heat input, the surface displacements may be expressed as definite integrals of one of the doublet solutions of Section 5.

If the thermal boundary conditions are mixed, this method can only be used if the heat conduction problem is solved first. In such a case, it may be more advantageous to obtain a direct solution for the thermal displacements—for example, by using transform methods on the equations of equilibrium and compatibility. However, there are advantages in the use of point source methods even in mixed boundary value problems, particularly when a numerical solution is required, since numerical integration is a stable and reliable method in comparison with relaxation processes or the numerical solution of differential equations in several variables. Furthermore, a judicious choice of singular solution will contribute to the economy and stability of the process. For example, in finding the displacement within a heated area, it is better to use the sector solution (Section 4.3) than the point source solution, since the latter is unbounded at the origin. On the other hand, the point source solution is more suitable for finding the displacement some distance from the heated area.

The solutions derived in this paper give only the normal surface displacements of the semi-infinite solid. However, the same approach can be used to find the displacements inside the solid and hence the internal stresses. For example, the internal displacements due to the point heat source can be found by adding two components: a purely radial displacement, due to the stress system defined by equations (2) and (3), and a displacement analogous to that given by equation (5), but in which the general equations for the internal displacements due to a point load are used in place of equation (4).

The primary application of these results is in the non-isothermal contact of solids and the distribution of contact pressure in such cases depends only on the normal surface displacements. For particular examples of the use of this method in the solution of non-isothermal contact problems, the reader is referred to refs. 1 and 4. The same method could be applied to find the thermal stresses around a discontinuity in a large solid.

REFERENCES

1. J. R. BARBER, *Int. J. Heat Mass Transfer* **14**, 751 (1971).
2. A. M. CLAUSING, *Int. J. Heat Mass Transfer* **9**, 791 (1966).
3. D. L. GEORGE and I. N. SNEDDON, *J. Math. Mech.* **11**, 665 (1962).
4. J. R. BARBER, *Proc. R. Soc.* **A312**, 381 (1969).
5. J. R. BARBER, *Int. J. Engng Sci.* **9**, 1165 (1971).
6. J. R. BARBER and K. WRIGHT, *Int. J. mech. Sci.* **9**, 811 (1967).
7. H. CARSLAW and J. C. JAEGER, *The Conduction of Heat in Solids* (2nd edn). Clarendon Press, Oxford (1959).
8. S. P. TIMOSHENKO and J. N. GOODIER, *Theory of Elasticity* (3rd edn). McGraw-Hill, New York (1970).
9. I. S. GRADSHTEYN and I. W. RYZHIK, *Tables of Integrals, Series and Products* (4th edn). Academic Press, New York (1965).
10. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (4th edn). Cambridge University Press (1927).

APPENDIX

The normal surface displacement due to a shear force F acting at the surface of a semi-infinite solid is

$$w = \frac{-F(1+\nu)(1-2\nu)x}{2\pi E r^2} \tag{106}$$

(Love,[10] Art. 166), where r is the distance from the point of application of the force and x is the projection of r on its line of action.

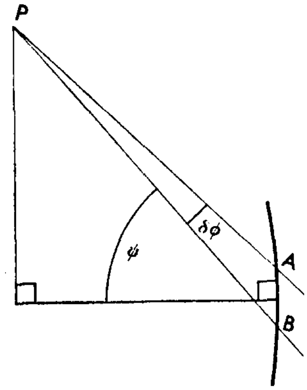

FIG. 5.

If a shear force F per unit length acts along the normal to the line AB in Fig. 5, the displacement produced at a general point P will be

$$w = \frac{-F \times AB(1+\nu)(1-2\nu)\, AP \cos\psi}{2\pi E\, AP^2}, \tag{107}$$

$$= \frac{-F(1+\nu)(1-2\nu)\,\delta\phi}{2\pi E}. \tag{108}$$

Hence, if a constant shear force, F per unit length, acts along the inward normal at all points on a closed curve on the surface, the surface displacement w at any point inside the curve will be

$$w = \int_0^{2\pi} \frac{-F(1+\nu)(1-2\nu)\,d\phi}{2\pi E}, \tag{109}$$

$$= \frac{-F(1+\nu)(1-2\nu)}{E}. \tag{110}$$

If the point P lies outside the closed curve, any small sector $\delta\phi$ must include an even number of line segments, at half of which the shear force will be directed away from P. Thus, at all exterior points

$$w = 0. \tag{111}$$

The argument is not restricted to convex curves. Equations (69), (70) and (95) follow as special cases of this result.