Stability of a Functional Equation for Square Root Spirals

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Abstract—By using an idea of Heuvers, Moak and Boursaw [1], we will prove a Hyers-Ulam-Rassias stability (or a general Hyers-Ulam stability) of the functional equation (1), which is closely related to the square root spiral. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [2] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let $G_1$ be a group and let $G_2$ be a metric group with a metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [3] under the assumption that $G_1$ and $G_2$ are Banach spaces. Later, the result of Hyers was significantly generalized by Rassias [4] (see also [5]). It should be remarked that we can find in [6,7] a lot of references concerning the stability of functional equations (see also [8–10]).

Recently, Heuvers, Moak and Boursaw investigated the general solution of the functional equation

$$f \left( \sqrt{r^2 + 1} \right) = f(r) + \arctan \frac{1}{r},$$

(1)
which is closely related to the square root spiral, for the case that \( f(1) = 0 \) and \( f(r) \) is monotone increasing for \( r > 0 \) (see [1]).

In this note, we follow the method of Heuvers, Moak and Boursaw [3] to prove a Hyers-Ulam-Rassias stability (or a general type of Hyers-Ulam stability) of the functional equation (1).

The main result of this note is presented in the following theorem.

**Theorem 1.** If a mapping \( f : [1, \infty) \rightarrow [0, \infty) \) satisfies the inequality

\[
|f(\sqrt{r^2 + 1}) - f(r) - \arctan \frac{1}{r}| \leq \varphi(r),
\]

for all \( r \in [1, \infty) \), where \( \varphi : [1, \infty) \rightarrow [0, \infty) \) is a mapping which satisfies the condition

\[
\Phi(r) = \sum_{k=0}^{\infty} \varphi(\sqrt{r^2 + k}) < \infty,
\]

for all \( r \in [1, \infty) \), then there exists a unique solution \( F : [1, \infty) \rightarrow [0, \infty) \) of the functional equation (1), which satisfies

\[
|F(r) - f(r)| \leq \Phi(r),
\]

for any \( r \in [1, \infty) \).

**2. PROOF OF THEOREM**

**Proof.** If we replace \( r \) by \( \sqrt{r^2 + k} \) \((k = 1, 2, \ldots, n - 1)\) in (2), then we have

\[
|f(\sqrt{r^2 + 2}) - f(\sqrt{r^2 + 1}) - \arctan \frac{1}{\sqrt{r^2 + 1}}| \leq \varphi(\sqrt{r^2 + 1}),
\]

\[
|f(\sqrt{r^2 + 3}) - f(\sqrt{r^2 + 2}) - \arctan \frac{1}{\sqrt{r^2 + 2}}| \leq \varphi(\sqrt{r^2 + 2}),
\]

\[
\vdots
\]

\[
|f(\sqrt{r^2 + n}) - f(\sqrt{r^2 + n - 1}) - \arctan \frac{1}{\sqrt{r^2 + n - 1}}| \leq \varphi(\sqrt{r^2 + n - 1}),
\]

for any \( r \geq 1 \) and \( n \in \mathbb{N} \). In view of triangle inequality, the inequalities in (2) and in (5) yield

\[
|f(\sqrt{r^2 + n}) - f(r) - \sum_{k=0}^{n-1} \arctan \frac{1}{\sqrt{r^2 + k}}| \leq \sum_{k=0}^{n-1} \varphi(\sqrt{r^2 + k}),
\]

for all \( r \geq 1 \) and each \( n \in \mathbb{N} \). Substitute \( \sqrt{r^2 + m} \) for \( r \) in (6) to obtain

\[
|f(\sqrt{r^2 + m + n}) - \sum_{k=0}^{m+n-1} \arctan \frac{1}{\sqrt{r^2 + k}} - f(\sqrt{r^2 + m}) + \sum_{k=0}^{m-1} \arctan \frac{1}{\sqrt{r^2 + k}}| \leq \sum_{k=m}^{m+n-1} \varphi(\sqrt{r^2 + k}),
\]

for every \( r \geq 1 \) and \( m, n \in \mathbb{N} \). Thus, considering (3), we see that the sequence

\[
\left\{ f(\sqrt{r^2 + n}) - \sum_{k=0}^{n-1} \arctan \frac{1}{\sqrt{r^2 + k}} \right\}_{n=1}^{\infty}
\]
is a Cauchy sequence for any \( r \geq 1 \). Hence, we can define a mapping \( F : [1, \infty) \to [0, \infty) \) by

\[
F(r) = \lim_{n \to \infty} \left\{ f\left(\sqrt{r^2 + n}\right) - \sum_{k=0}^{n-1} \arctan \frac{1}{\sqrt{r^2 + k}} \right\}.
\]

(7)

By (2), (3), and (7), we get

\[
\left| F\left(\sqrt{r^2 + 1}\right) - F(r) - \arctan \frac{1}{r} \right| = \lim_{n \to \infty} \left| f\left(\sqrt{r^2 + n + 1}\right) - f\left(\sqrt{r^2 + n}\right) - \arctan \frac{1}{\sqrt{r^2 + n}} \right|
\]

\[
\leq \lim_{n \to \infty} \varphi\left(\sqrt{r^2 + n}\right) = 0,
\]

for all \( r \geq 1 \), which verifies that \( F \) is a solution of the functional equation (1).

In view of (3), (6), and (7), the validity of the inequality (4) is obvious. It only remains to prove the uniqueness of the mapping \( F \). If a mapping \( H : [1, \infty) \to [0, \infty) \) is a solution of the functional equation (1), then we can apply an induction argument to prove

\[
H\left(\sqrt{r^2 + n}\right) - H(r) = \sum_{k=0}^{n-1} \arctan \frac{1}{\sqrt{r^2 + k}},
\]

for all \( r \geq 1 \) and all \( n \in \mathbb{N} \). Now, let \( G : [1, \infty) \to [0, \infty) \) be a solution of equation (1) and satisfy inequality (4) in place of \( F \). It then follows from (3), (4), and (8) that

\[
|F(r) - G(r)| = \left| F\left(\sqrt{r^2 + n}\right) - G\left(\sqrt{r^2 + n}\right) \right|
\]

\[
\leq 2\Phi\left(\sqrt{r^2 + n}\right)
\]

\[
\to 0, \quad \text{as } n \to \infty,
\]

for any \( r \geq 1 \), which completes our proof. \( \blacksquare \)

If we define \( \varphi : [1, \infty) \to [0, \infty) \) by

\[
\varphi(r) = \frac{1}{r^{2+\alpha}},
\]

for some \( \alpha > 0 \) and for any \( r \geq 1 \), then we have

\[
\Phi(r) = \sum_{k=0}^{\infty} \varphi\left(\sqrt{r^2 + k}\right) \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha/2}} = \zeta\left(1 + \frac{\alpha}{2}\right),
\]

for \( r \geq 1 \), where \( \zeta \) is the Riemann zeta function. This fact, together with Theorem 1, implies the following corollary.

**Corollary 1.** Assume that a mapping \( f : [1, \infty) \to [0, \infty) \) satisfies the inequality

\[
\left| f\left(\sqrt{r^2 + 1}\right) - f(r) - \arctan \frac{1}{r} \right| \leq \frac{1}{r^{2+\alpha}},
\]

for some \( \alpha > 0 \) and for all \( r \in [1, \infty) \). Then, there exists a unique solution \( F : [1, \infty) \to [0, \infty) \) of the functional equation (1) such that

\[
|F(r) - f(r)| \leq \zeta\left(1 + \frac{\alpha}{2}\right),
\]

for any \( r \in [1, \infty) \).
REFERENCES