Cost, revenue and profit efficiency measurement in DEA: A directional distance function approach

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Abstract

Estimation of efficiency of firms in a non-competitive market characterized by heterogeneous inputs and outputs along with their varying prices is questionable when factor-based technology sets are used in data envelopment analysis (DEA). In this scenario, a value-based technology becomes an appropriate reference technology against which efficiency can be assessed. In this contribution, the value-based models of Tone (2002) are extended in a directional DEA set up to develop new directional cost- and revenue-based measures of efficiency, which are then decomposed into their respective directional value-based technical and allocative efficiencies. These new directional value-based measures are more general, and include the existing value-based measures as special cases. These measures satisfy several desirable properties of an ideal efficiency measure. These new measures are advantageous over the existing ones in terms of 1) their ability to satisfy the most important property of translation invariance; 2) choices over the use of suitable direction vectors in handling negative data; and 3) flexibility in providing the decision makers with the option of specifying preferable direction vectors to incorporate their preferences. Finally, under the condition of no prior unit price information, a directional value-based measure of profit inefficiency is developed for firms whose underlying objectives are profit maximization. For an illustrative empirical application, our new measures are applied to a real-life data set of 50 US banks to draw inferences about the production correspondence of banking industry.

Keywords: Data envelopment analysis; Cost efficiency; Revenue efficiency; Profit efficiency; Translation invariance; Directional distance function

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1. Introduction

Since its inception by Charnes, Cooper, and Rhodes (1978), data envelopment analysis (DEA) has been gaining increasing popularity in the literature as a convenient tool for estimating the efficiencies of firms characterized by multi-input-multi-output production technologies. The non-parametric methodology of DEA has been used for measuring and analyzing a number of efficiency concepts, including cost efficiency (CE) and revenue efficiency (RE).

One of the most important aspects in applied production analysis of firms is the measurement of their cost and revenue efficiencies (Farrell, 1957), on which we concentrate within the framework of the directional distance function (DDF) by Chambers, Chung, and Färe (1996, 1998). For the first time, Färe, Grosskopf, and Lovell (1985) developed procedures for the empirical implementations of the CE and RE measures in DEA. Since then, the aspect of measuring cost and revenue efficiencies has been explored in many studies. See, e.g., Ray and Kim (1995), Cooper, Thompson, and Thrall (1996), Schaffnit, Rosen, and Paradi (1997), Sueyoshi (1997), Puig-Junoy (2000), Kuosmanen and Post (2001, 2003), Tone (2002), Tone and Sahoo (2005), Tone and Sahoo (2006), Maniadakis and Thanassoulis (2004), Sengupta and Sahoo (2006), Jahanshahloo, Soleimani-Damaneh, and Mostafaee (2008), Mostafaee and Saljooghi (2010), Sahoo, Kerstens, and Tone (2012), among others.

Both the CE- and RE-based DEA models developed by Färe, Grosskopf, and Lovell (1985) require not only input and output quantity data but also their prices at each firm. These models can be of limited use in actual applications when market imperfections exist (Camanho & Dyson, 2008; Park & Cho, 2011; Sahoo & Tone, 2013). This is because these models are based on a number of simplifying assumptions. First, factor inputs are homogeneous across firms; their prices are exogenously given, and are measured and known with full certainty. In real-life applications, however, when production is expanded, firms experience changes in the organization of their processes or in the characteristics of their inputs that are economically more attractive than the replicated alternatives of those already in use. Therefore, the techniques and inputs used at higher scale are very different from those used at lower scale. Hence, factor inputs are thus heterogeneous, and as a result, their prices may vary across firms. Since inputs vary in their quality, the construction of factor-based technology set in DEA becomes problematic.

Further, input prices are not exogenous, but they vary in accordance with the actions by firms (Chamberlin, 1933; Engel & Rogers, 1996; Robinson, 1933). Also, firms often face ex ante price uncertainty while making their production decisions (McCall, 1967; Sandmo, 1971; Camanho &
Dyson, 2005). While costs and revenues are all well measured, physical input and output quantities and their prices are often not. Economic theory suggests that firms enjoying some degree of monopoly power should charge different prices if there is heterogeneity in the productivity of their inputs. This is empirically valid since most firms are observed facing an upward-sloping supply curve in their purchase decisions. This observation also suggests that the assumption of facing common unit prices by firms, i.e., the law of one price, which has long been maintained as a necessary and sufficient condition for Pareto efficiency in competitive markets (Kuosmanen et al., 2006), is not at all justified in revealing the proper CE behavior of firms.

Second, the CE measure by Färe, Grosskopf, and Lovell (1985) can be of limited value in actual applications even when (physical) inputs are homogeneous. This is because, as pointed out by Camanho and Dyson (2008), the CE measure reflects only input inefficiencies (technical inefficiency and/or allocative inefficiency) but not market (price) inefficiencies (deviation from fully competitive setting leading to price differences between firms). Therefore, as a remedy, they suggested a comprehensive CE measure that accounts for both inputs and market inefficiencies.

Third, in many real-life applications the price data on inputs and outputs are synthetically constructed, and hence, represent average, rather than marginal prices. Since managers make decisions at the margin, analysis of efficiency using average prices can distort measures of allocative efficiency (Fukuyama & Weber, 2008).

Therefore, when inputs/outputs are heterogeneous, in order to account for situations where the input/output prices vary between firms as a result of negotiations or to reflect the qualitative differences in the resources/products, the alternative CE/RE model of Tone (2002) should be followed by setting up technology in a cost-output/input-revenue space. Using the directional DEA structure, Fukuyama and Weber (2004) and Färe and Grosskopf (2006) extended this alternative value-based CE model to develop the directional input-cost distance function (DICDF), which, in turn, provides a directional measure of value-based technical inefficiency.

Using the DICDF, we develop two new directional cost- and revenue-based measures of efficiency, i.e., DCE and DRE, which all satisfy the property of translation invariance. This property is considered most important for any efficiency measure (Ali & Seiford, 1990; Cooper et al., 1999; Lovell & Pastor, 1995; Pastor, 1996). Furthermore, we develop two new directional input- and output-oriented value-based measures of technical efficiency (TE). We then decompose our new DCE and DRE measures into their respective directional value-based TE and allocative efficiency (AE) components. These new DCE and DRE measures are more general, and include the Tone (2002)’s CE and RE measures as special cases. Our proposed new measures satisfy several
desirable properties, such as unit invariance (Cooper, et al., 1999; Lovell & Pastor, 1995) and strong monotonicity (Blackorby & Russell, 1999; Cooper, et al., 1999).

Note that our value-based DCE measure is developed based on the assumption that physical outputs are homogenous, but not physical inputs. Similarly, the value-based DRE measure is developed based on the assumption that physical inputs are homogenous, but not physical outputs. However, when both physical inputs and outputs are heterogeneous, our DCE and DRE measures cannot be applied to measure the respective cost and revenue efficiencies. To deal with this situation, we develop a directional value-based measure of profit (in)efficiency that is based on a technology set comprising of all feasible input-cost (input-spending) and output-revenue (output-earnings) by observed firms. This measure will be more meaningful for a firm when its underlying behavioral objective is profit maximization.

While none of the existing CE, RE, and AE measures is translation-invariant, our proposed new measures satisfy this property that enables them to effectively deal with negative data. These new measures are flexible in the sense that they provide the decision makers with the option of specifying preferable direction vectors to incorporate their decision-making preferences. Specially, they can deal with value judgments (preference) as to which specific input-cost to reduce or which specific output-earnings to increase by a firm to improve its overall performance. Though the contribution of this paper is mainly theoretical, to demonstrate its ready applicability in empirical work, we conduct an illustrative empirical analysis based on a data set of 50 US banks.

The remainder of the paper unfolds as follows. Section 2 gives a brief review of methods aimed at measuring CE and RE. Section 3 represents the main contribution of the paper, where we present our new directional CE, RE and profit (in)efficiency measures and then discuss their properties. Section 4 demonstrates the ready applicability of our proposed measures on a real-life data set of 50 US banks for the year 1996. Finally, Section 5 concludes with remarks.

2. Preliminaries

Throughout this paper, we assume to deal with \( n \) observed decision making units (DMUs); each uses \( m \) inputs to produce \( s \) outputs. Let \( x_j = (x_{j1}, \ldots, x_{jm})^T \in \mathbb{R}^m_+ \) and \( y_j = (y_{j1}, \ldots, y_{js})^T \in \mathbb{R}^s_+ \) be, respectively, the input and output vectors of DMU\(_j\), \( j \in J = \{1, \ldots, n\} \). Let \( c_j = (c_{j1}, \ldots, c_{jm})^T \in \mathbb{R}^m_0 \) and \( p_j = (p_{j1}, \ldots, p_{js})^T \in \mathbb{R}^s_0 \) be, respectively, the non-negative price vectors of input and output of DMU\(_j\). The superscript \( T \) stands for a vector transpose. Let the input-
spending and output-earnings of DMU, be \( \bar{x} \) and \( \bar{y} \) respectively, where \( \bar{x} = c \ast x \) and \( \bar{y} = p \ast y \). Here, \( \ast \) denotes the component-wise multiplication of vectors. We further assume \( o \) as the index of DMU under evaluation.

We now define four production technologies depending upon data availability. If both physical input and output data are observed, and are homogeneous, we represent technology as

\[
T_{x,y} = \left\{ (x, y) \in \mathbb{R}_{\geq 0}^{m \times r} \mid x \in \mathbb{R}_{\geq 0}^m \text{ can produce } y \in \mathbb{R}_{\geq 0}^r \right\}. \tag{1}
\]

If physical outputs are observed (and are homogeneous) but not physical inputs, then we can represent the technology by considering all feasible input-spending and physical output vectors as

\[
T_{\bar{x},y} = \left\{ (\bar{x}, y) \in \mathbb{R}_{\geq 0}^{m \times r} \mid \bar{x} \in \mathbb{R}_{\geq 0}^m \text{ can produce } y \in \mathbb{R}_{\geq 0}^r \right\}. \tag{2}
\]

If physical inputs are observed (and are homogeneous) but not physical outputs, then we can represent the technology by considering all feasible physical input and output-earnings vectors as

\[
T_{x,\bar{y}} = \left\{ (x, \bar{y}) \in \mathbb{R}_{\geq 0}^{m \times r} \mid x \in \mathbb{R}_{\geq 0}^m \text{ can produce } \bar{y} \in \mathbb{R}_{\geq 0}^r \right\}. \tag{3}
\]

Finally, if both physical inputs and physical outputs are not observed, we can represent the technology by considering all feasible input-spending and output-earnings vectors as

\[
T_{\bar{x},\bar{y}} = \left\{ (\bar{x}, \bar{y}) \in \mathbb{R}_{\geq 0}^{m \times r} \mid \bar{x} \in \mathbb{R}_{\geq 0}^m \text{ can produce } \bar{y} \in \mathbb{R}_{\geq 0}^r \right\}. \tag{4}
\]

If each of these technologies satisfies properties such as no free lunch, free (strong) disposability, closure, convexity and boundedness, following Banker, Charnes, and Cooper (1984), the non-parametric DEA representations of these technologies can, respectively, be represented under variable returns to scale (VRS) as

\[
T_{x,y}^{DEA} = \left\{ (x, y) \in \mathbb{R}_{\geq 0}^{m \times r} \mid x \geq \sum_{j \in J} \lambda_j x_j, y \leq \sum_{j \in J} \lambda_j y_j, \sum_{j \in J} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\}. \tag{5}
\]

\[
T_{\bar{x},y}^{DEA} = \left\{ (\bar{x}, y) \in \mathbb{R}_{\geq 0}^{m \times r} \mid \bar{x} \geq \sum_{j \in J} \lambda_j \bar{x}_j, y \leq \sum_{j \in J} \lambda_j y_j, \sum_{j \in J} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\}. \tag{6}
\]

\[
T_{x,\bar{y}}^{DEA} = \left\{ (x, \bar{y}) \in \mathbb{R}_{\geq 0}^{m \times r} \mid x \geq \sum_{j \in J} \lambda_j x_j, \bar{y} \leq \sum_{j \in J} \lambda_j \bar{y}_j, \sum_{j \in J} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\}. \tag{7}
\]

\[
T_{\bar{x},\bar{y}}^{DEA} = \left\{ (\bar{x}, \bar{y}) \in \mathbb{R}_{\geq 0}^{m \times r} \mid \bar{x} \geq \sum_{j \in J} \lambda_j \bar{x}_j, \bar{y} \leq \sum_{j \in J} \lambda_j \bar{y}_j, \sum_{j \in J} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\}. \tag{8}
\]

The assumption of VRS is maintained in our DEA technology constructs ((5)-(8)) for three reasons: (1) assuming that some data can be negative, one may be not able to define an efficient frontier passing through the origin, as is assumed under constant returns to scale (CRS). Thus, an
assumption of CRS breaks down for negative data (Silva Portela and Thanassoulis, 2010), (2) the assumption of CRS is not consistent with some directional DEA models based on specific direction vectors, which all directly deal with both positive and negative data (Sahoo et al., 2011), and (3) the real-life situations do not always exhibit CRS.

2.1 Cost efficiency measurement

If the underlying behavioral objective of a DMU is cost minimization, its CE measure, $\gamma_o$, can be obtained as the optimal objective value of the following linear programming (LP) problem (Färe et al., 1985):

$$\gamma_o = \min_{\lambda, \xi} \frac{1}{C^o} \sum_{i=1}^{m} C_{io}^o x_i$$

s.t. $\sum_{j \in J} \lambda_j x_j^o \leq x_i, \quad i = 1, \ldots, m,$

$$\sum_{j \in J} \lambda_j y_j^o \geq y_{ro}, \quad r = 1, \ldots, s,$$

$$\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J,$$

(9)

where $C^o = \sum_{i=1}^{m} C_{io}^o x_i^o$ is the observed cost, and $C^* = \sum_{i=1}^{m} C_{io}^* x_i^*$ is the minimum cost, of DMU$_o$.

Here, $\gamma_o$ is defined as the ratio of the minimum cost to the observed cost, Hence, $0 < \gamma_o \leq 1$.

The model (9) that is based on $T_{x,y}^{DEA}$ in (5) is constructed on the assumption that input prices are available, and physical inputs and outputs are observed. However, as argued earlier, when (physical) inputs are heterogeneous, in order to account for situations where input prices vary between firms as a result of negotiations or to reflect the qualitative differences in the resources, the alternative value-based CE model of Tone (2002) that is based on $T_{x,y}^{DEA}$ in (6) should be used. This alternative CE measure, $\gamma_{o}^{CE}$, can be represented as

$$\gamma_{o}^{CE} = \min_{\lambda, \xi} \frac{1}{C^o} \sum_{i=1}^{m} \tilde{x}_i$$

s.t. $\sum_{j \in J} \lambda_j \tilde{x}_j \leq \tilde{x}_i, \quad i = 1, \ldots, m,$

$$\sum_{j \in J} \lambda_j y_j^o \geq y_{ro}, \quad r = 1, \ldots, s,$$

$$\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J,$$

(10)

where $C^o = \sum_{i=1}^{m} \tilde{x}_i^o$ is the observed cost of DMU$_o$. 
In addition, the input-oriented value-based TE measure, \( \rho_{o}^{\text{IVTE}} \), can be set up as

\[
\rho_{o}^{\text{IVTE}} = \min_{\lambda, \beta} \bar{\theta} \\
\text{s.t.} \quad \sum_{j \in J} \lambda_{i} \bar{x}_{ij} \leq \bar{x}_{io}, \quad i = 1, \ldots, m, \\
\sum_{j \in J} \lambda_{j} y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s, \\
\sum_{j \in J} \lambda_{j} = 1, \quad \lambda_{j} \geq 0, \quad \forall j \in J.
\]

\( (11) \)

Obviously, one can have the following relationship:

\[
\gamma_{o}^{CE} \leq \rho_{o}^{\text{IVTE}} \leq 1. \tag{12}
\]

Using (12) one can define the input-oriented AE (price efficiency) as

\[
\alpha_{o}^{\text{IAE}} = \frac{\gamma_{o}^{CE}}{\rho_{o}^{\text{IVTE}}} \leq 1. \tag{13}
\]

From (13), the CE measure \( \gamma_{o}^{CE} \) can be expressed as the product of the (input-oriented) value-based AE and TE, i.e.,

\[
\gamma_{o}^{CE} = \alpha_{o}^{\text{IAE}} \times \rho_{o}^{\text{IVTE}}. \tag{14}
\]

Thus, it is clear that a DMU will be cost-efficient (i.e., \( \gamma_{o}^{CE} = 1 \)) if it is both value-based technical- and allocative-efficient. If \( \gamma_{o}^{CE} < 1 \), it incurs higher costs due to not being able to use, the most efficient technology (i.e., technical inefficiency) and/or the cost-minimizing input mix (i.e., allocative inefficiency).

Fukuyama and Weber (2004) presented an alternative approach for estimating the value-based technical inefficiency using the concept of DDF of Chambers, Chung, and Färe (1996, 1998), which generalizes the Shephard’s distance functions (Shephard, 1953, 1970). The DDF seeks to simultaneously minimize inputs and maximize outputs of a given DMU using a pre-specified direction vector, \( g = (-g^{-}, g^{+}) \). Applying the input-oriented DDF \( g = (-g^{-}, 0) \) relative to \( T_{\alpha, \gamma}^{\text{DEA}} \) in (6), the directional input-cost distance function (DICDF) can be set up as

\[
\beta^* = \max_{\lambda, \beta} \beta \\
\text{s.t.} \quad \sum_{j \in J} \lambda_{i} \bar{x}_{ij} \leq \bar{x}_{io} - \beta g^{-}_{i} , \quad i = 1, \ldots, m, \\
\sum_{j \in J} \lambda_{j} y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s, \\
\sum_{j \in J} \lambda_{j} = 1, \quad \lambda_{j} \geq 0, \quad \forall j \in J.
\]

\( (15) \)
\( \beta^* \) can be interpreted as the directional value-based measure of technical inefficiency. Furthermore, the Nerlovian cost inefficiency measure is defined as

\[
\sum_{i=1}^{m} \bar{x}_{io} - \sum_{i=1}^{m} \bar{x}_{i}^* \over \sum_{i=1}^{n} \bar{s}_{i}^*,
\]

(16)

where \( \bar{x}^* \) is the optimal solution vector of the model (10). For further details concerning this cost inefficiency measure, interested readers may refer to Fukuyama and Weber (2004).

2.2 Revenue efficiency measurement

If the underlying behavioral objective of a DMU is revenue maximization, its RE measure, \( \eta_o \), can be obtained as the inverse of the optimal objective value of the following LP problem:

\[
\frac{1}{\eta_o} = \text{Max} \frac{1}{R^o} \sum_{r=1}^{R} p_{ro} y_r \\
\text{s.t.} \sum_{j} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, ..., m, \\
\sum_{j} \lambda_j y_{jr} \geq y_{ir}, \quad r = 1, ..., s, \\
\sum_{j} \lambda_j = 1, \lambda_j \geq 0, \quad \forall j \in J,
\]

(17)

where \( R^o = \sum_{i=1}^{R} p_{ro} y_{ro} \) is the observed revenue, and \( R^* = \sum_{r=1}^{R} p_{ro} y_{ro}^* \) is the maximum revenue of DMU. Here, \( \eta_o \) is defined as the ratio of the observed revenue to the maximum revenue. Hence, \( 0 < \eta_o \leq 1 \).

The model (17) that is based on \( T_{x,y}^{DEA} \) in (5) is constructed on the assumption that output prices are available, and physical inputs and outputs are observed. However, as argued in Sahoo and Tone (2013), when (physical) outputs are heterogeneous, in order to account for situations where output prices vary between firms to reflect the qualitative differences in their products, the alternative value-based RE model of Tone (2002) that is based on \( T_{x,y}^{DEA} \) in (7) should be used. This alternative value-based RE measure, \( \eta_{o}^{RE} \) can be set up as
\[
\frac{1}{\eta_{o}^{RE}} = \max_{\lambda, \tau} \frac{1}{R^{o}} \sum_{r=1}^{s} \tilde{y}_{r},
\]

s.t. \[
\sum_{j=1}^{J} \tilde{\lambda}_{j}x_{ij} \leq x_{i o}, \quad i = 1,...,m,
\]
\[
\sum_{j=1}^{J} \tilde{\lambda}_{j}y_{j} \geq \tilde{y}_{j}, \quad r = 1,...,s,
\]
\[
\sum_{j=1}^{J} \tilde{\lambda}_{j} = 1, \quad \tilde{\lambda}_{j} \geq 0, \quad \forall j \in J,
\]

where \( R^{o} = \sum_{j=1}^{J} \tilde{y}_{j} \) is the observed revenue of DMU\(_{o}\).

Similar to the relationship in (14), \( \eta_{o}^{RE} \) can be decomposed as
\[
\eta_{o}^{RE} = \alpha_{o}^{OAE} \times \rho_{o}^{OVTE},
\]

where \( \alpha_{o}^{OAE} \) and \( \rho_{o}^{OVTE} \) represent, respectively, the output-oriented valued-based AE and TE.

3. Our proposed approach

3.1 Directional input-oriented value-based TE, input-oriented AE, and CE measures

In order to develop a directional value-based TE measure, consider the model (15) whose optimal solution vector is \( (\beta^*, \lambda^*) \). Since \( \sum_{j=1}^{J} \lambda_{j} = 1 \), one can deduce from the \( i \)th input-cost constraint the following relation:
\[
\sum_{j=1}^{J} \lambda_{j}^{*} \tilde{x}_{ij} \leq \tilde{x}_{io} - \sum_{j=1}^{J} \lambda_{j}^{*} \tilde{x}_{ij} \Rightarrow \beta^{*} \leq \frac{\tilde{x}_{io} - \sum_{j=1}^{J} \lambda_{j}^{*} \tilde{x}_{ij}}{g_{i}^{-}}
\]
\[
\leq \frac{\tilde{x}_{io} - \min_{j=1}^{J} \{ \tilde{x}_{ij} \}}{g_{i}^{-}} \leq \begin{cases} \frac{\max_{j=1}^{J} \{ \tilde{x}_{ij} \}}{g_{i}^{-}} & \text{if } \tilde{x}_{io} \leq \max_{j=1}^{J} \{ \tilde{x}_{ij} \} \\ \frac{\max_{j=1}^{J} \{ \tilde{x}_{ij} \} - \min_{j=1}^{J} \{ \tilde{x}_{ij} \}}{g_{i}^{-}} & \text{otherwise} \end{cases}
\]

(19)

To guarantee that \( \beta^{*} \leq 1 \), the direction vector \( g \) satisfies the following primary condition:
\[
\max_{i=1,...,m} \left\{ \frac{\tilde{x}_{io} - \min_{j=1}^{J} \{ \tilde{x}_{ij} \}}{g_{i}^{-}} \right\} \leq 1.
\]

(20)

For example, consider the following direction vectors, which all fulfill the primary condition (20).
\[
g_{i}^{-} = \tilde{x}_{io}, \quad g_{i}^{+} = 0, \quad i = 1,...,m, \quad r = 1,...,s
\]

(21)
However, the fixed direction vector
g_{i}^{\ast} = \max_{j \in J} \{ \bar{\tau}_{ij} \}, \quad g_{i}^{-} = 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s
(22)

g_{i}^{\ast} = \bar{\tau}_{i0} - \min_{j \in J} \{ \bar{\tau}_{ij} \}, \quad g_{i}^{-} = 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s
(23)

g_{i}^{\ast} = \max_{j \in J} \{ \bar{\tau}_{ij} \} - \min_{j \in J} \{ \bar{\tau}_{ij} \}, \quad g_{i}^{-} = 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s
(24)

However, the fixed direction vector
\begin{align*}
g_{i}^{\ast} &= 1, \quad g_{i}^{-} = 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s
\end{align*}
(25)
does not satisfy the primary condition (20).

It is thus clear how the optimal solution to (15) is dependent on the direction vector \( g \). Because this explicit dependence was not considered in the DICDF approach, the solutions do not necessarily provide a direct measure of the value-based TE.

For any pre-specified direction vector \( g \) satisfying the primary condition (20), one can define the directional input-oriented value-based TE (DIVTE) measure as
\[
\rho_{o}^{DIVTE} = 1 - \beta^{\ast}
(26)
\]

\( \rho_{o}^{DIVTE} \) generalizes the input-oriented value-based TE measure. Specifically, if one chooses the direction vector (21), \( \rho_{o}^{DIVTE} \) turns to \( \rho_{o}^{IVTE} \).

Based on Theorems 2 and 4 in Fukuyama and Weber (2004), \( \rho_{o}^{DIVTE} \) is weakly monotonic with respect to variation in input-costs, and is translation-invariant with respect to outputs. However, unlike what was claimed in Theorem 2 in Fukuyama and Weber (2004), \( \rho_{o}^{DIVTE} \) is not always translation-invariant with respect to input-costs for the non-fixed (dependent on the input-output data) direction vectors, because any change in the input-output data may affect such direction vectors. For instance, it can be seen that the model (15) is not translation-invariant for the direction vectors (21) and (22), but it is for the direction vectors (23) and (24).

Now we have the following theorem on translation invariance.

**Theorem 3.1.1 (Translation invariance).** \( \rho_{o}^{DIVTE} \) is translation-invariant with respect to input-costs if the translation of input-costs data has no effect on the pre-assigned direction vector \( g \). \( \rho_{o}^{DIVTE} \) is also translation-invariant with respect to outputs.

**Proof.** Let \( \tilde{\tau}_{ij} = \bar{\tau}_{ij} + \tau_{i} \) \((i = 1, \ldots, m)\) be the translated \( i \)th input-cost of DMU\( j \), \( j \in J \). Because these translations have no effect on \( g \), the constraints of model (15) based on the translated data can be represented as
\[
\sum_{j \in J} \lambda_j (x_{ij} + \tau_j) \leq x_{io} + \tau_i - \beta g_i^- , \quad i = 1, \ldots, m,
\]
\[
\sum_{j \in J} \lambda_j y_{rj} \geq y_{ro} , \quad r = 1, \ldots, s,
\]

(27)

Since \( \sum_{j \in J} \lambda_j = 1 \), we eliminate \( \tau_j \) on both sides of the input-cost constraints, and obtain the same constraints as in (15). Therefore, the solution set remains unchanged, thus indicating the property of translation invariance. This property with respect to outputs can also be proved in an analogous manner.

We now have the following theorem that provides a condition for the unit independency of the efficiency measure \( \rho_o^{DIVTE} \) in (26).

**Theorem 3.1.2.** If the direction vector \( g \) is chosen in such a way that each component \( g_i^- \) \( (i = 1, \ldots, m) \) has the same units of measurement as the \( i \)th input-cost, then \( \rho_o^{DIVTE} \) will be unit-invariant.

**Proof.** Let us rescale the \( i \)th input-cost and the \( r \)th output by the scalars \( \tau_i > 0 \) and \( w_r > 0 \), respectively. Since both \( g_i^- \) and \( x_{io} \) have the same units of measurement, the corresponding component of the new direction vector is \( \tau_i g_i^- \). Hence, we have

\[
\sum_{j \in J} \lambda_j (x_{ij} + \tau_j) \leq (x_{io} + \tau_i) - \beta (g_i^-), \quad i = 1, \ldots, m,
\]
\[
\sum_{j \in J} \lambda_j (w_r y_{rj}) \geq (w_r y_{ro}), \quad r = 1, \ldots, s.
\]

By eliminating \( \tau_i \) and \( w_r \) on both sides of these expressions, one can obtain the same constraints as in (15). Thus, the solution set remains unchanged, which completes the proof.

For example, the direction vector (21) provides the property of unit invariance.

In order to evaluate the CE score, we develop, based on \( T_{x,y}^{DEA} \) in (6), a new directional CE (DCE) measure as

\[
\chi_o^{DCE} = \min_{\lambda, \delta} \quad 1 - \sum_{i=1}^{m} \frac{g_i^-}{G} \delta_i^- \quad \text{s.t.} \quad \sum_{j \in J} \lambda_{ij} x_{ij} \leq x_{io} - \beta_i^- g_i^- , \quad i = 1, \ldots, m,
\]
\[
\sum_{j \in J} \lambda_{rj} y_{rj} \geq y_{ro} , \quad r = 1, \ldots, s,
\]
\[
\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J, \quad (28)
\]
where the direction vector \( g \) satisfies the primary condition (20), and \( G^- = \sum_{i=1}^{m} g_i^- \). Here, \( \beta_i^- \) represents the rate of improvement in the \( i \)th input-cost of DMU\(_o\).

Because the direction vector \( g \) satisfies (20), it can be easily established that \( \beta_i^* \leq 1, \ i = 1, \ldots, m \), for each optimal solution \( (\lambda^*, \beta^*) \) to (28). This implies that \( \gamma_o^{DCE} \) satisfies the \textit{efficiency requirement} condition, i.e., \( 0 < \gamma_o^{DCE} \leq 1 \).

Note that \( \gamma_o^{DCE} \) is a generalization of \( \gamma_o^{CE} \). Specifically, if \( x_o > 0 \) and the direction vector (21) is considered, then the model (28) can be transformed as

\[
\begin{align*}
\text{Min} & \quad \frac{1}{C'} \sum_{i=1}^{m} \left( 1 - \beta_i^- \right) \bar{x}_o \\
\text{s.t.} & \quad \sum_{j \in J} \lambda_j^* \bar{x}_j \leq \left( 1 - \beta_i^- \right) \bar{x}_o, \quad i = 1, \ldots, m, \\
& \quad \sum_{j \in J} \lambda_j^* y_{oj} \geq y_{ro}, \quad r = 1, \ldots, s, \\
& \quad \sum_{j \in J} \lambda_j^* = 1, \lambda_j^* \geq 0, \ \forall j \in J.
\end{align*}
\]

(29)

By letting \( \bar{x}_i = \left( 1 - \beta_i^- \right) \bar{x}_o \), the model (29) can be easily transformed to the model (10). In addition, if input prices are common across DMUs, then \( \gamma_o \) can be easily derived from \( \gamma_o^{DCE} \).

The optimal solution to (28) for DMU\(_o\) yields the following expression:

\[
\begin{align*}
\sum_{j \in J} \lambda_j^* \bar{x}_j &= \bar{x}_o - \beta_i^- g_i, \quad i = 1, \ldots, m, \\
\sum_{j \in J} \lambda_j^* y_{oj} &\geq y_{ro}, \quad r = 1, \ldots, s.
\end{align*}
\]

(30)

Note that all the input-cost constraints in (28) are active (binding) at the optimum; otherwise the optimality would be violated. Based on (30), we define the DCE-based projection of DMU\(_o\) by

\[
\begin{align*}
\bar{x}_o \rightarrow \hat{\bar{x}}_o &= \sum_{j \in J} \lambda_j^* \bar{x}_j, \quad i = 1, \ldots, m, \\
y_{ro} \rightarrow \hat{y}_{ro} &= y_{ro}, \quad r = 1, \ldots, s.
\end{align*}
\]

(31)

The projection point \( (\hat{\bar{x}}_o, \hat{y}_o) \) is obtained by bringing in the maximum possible improvement in the total cost. Thus, \( (\hat{\bar{x}}_o, \hat{y}_o) \) represents the levels of operation of input-costs and outputs that would make DMU\(_o\) cost-efficient, i.e., DCE-efficient.
Similar to Theorem 6 in Tone (2002), it can be demonstrated that the DCE-efficient projection point \( \left( \hat{x}_o, \hat{y}_o \right) \) is also DIVTE-efficient. Despite this result, \( \left( \hat{x}_o, \hat{y}_o \right) \) may not be strongly technical-efficient in a Pareto-Koopmans sense due to the existence of possible output slacks. However, it is strongly technical-efficient with respect to inputs as in Tone (2002).

**Theorem 3.1.3.** Let \( (x_o, y_o) \in T^{DEA}_{x,y} \). Then, given the direction vector \( g \), we have

\[
\sum_{i=1}^{m} (\bar{x}_{io} - \beta_i \gamma g_i) = \sum_{i=1}^{m} \bar{x}_i^* \tag{32}
\]

where \( \bar{x}^* \) is the optimal solution to (10).

**Proof.** Let \( (x^*, \lambda^*) \) and \( (\hat{\beta}^*, \hat{\lambda}^*) \) be, respectively, the optimal solution vectors of the models (10) and (28). Then, \( (x^*, \lambda^*) = (\bar{x}_o - \beta^* \gamma g_i, \hat{\lambda}^*) \) is a feasible solution to (10). From the optimal solution vector \( (x^*, \lambda^*) \) to (10), we have

\[
\sum_{i=1}^{m} \bar{x}_i^* \leq \sum_{i=1}^{m} (\bar{x}_{io} - \beta_i \gamma g_i). \tag{32}
\]

At the same time,

\[
\left( \beta_i^* = \frac{\bar{x}_{io} - \bar{x}_i^*}{g_i}, \forall \lambda, \lambda_j^* = \lambda_j^*, \forall j \right) \text{ is a feasible solution to (28). It then follows that }
\]

\[
\sum_{i=1}^{m} \bar{x}_i^* \geq \sum_{i=1}^{m} (\bar{x}_{io} - \beta_i \gamma g_i), \text{ which completes the proof.} \tag{33}
\]

As an immediate consequence of Theorem 3.1.3, we obtain the following result:

\[
\gamma^DCE \leq \gamma^DIVTE \leq 1. \tag{33}
\]

where \( \bar{x}^* \) is the optimal solution vector to (10). This proves that our definition of the directional CE measure is consistent with that of the Nerlovian cost inefficiency measure (16).

It can now be easily shown that the value of the directional CE measure is no more than that of the directional input-oriented value-based TE, i.e.,

\[
\gamma^DCE \leq \rho^DIVTE \leq 1. \tag{34}
\]

Hence, we define a new directional input-oriented AE by the ratio of \( \gamma^DCE \) to \( \rho^DIVTE \), i.e.,

\[
\alpha^DIAE = \frac{\gamma^DCE}{\rho^DIVTE}. \tag{35}
\]

Based on (35), we express the DCE-based measure as the product of the directional (input-oriented) AE (\( \alpha^DIAE \)) and value-based TE (\( \rho^DIVTE \)) components, i.e.,
With regard to the property of unit invariance for $\gamma_{o}^{DCE}$, we have a result similar to that as stated in Theorem 3.1.2. Specifically, if the direction vector $g^{-}$ and the input-costs for all the DMUs have the same units of measurement (e.g., dollar, pound), then $\gamma_{o}^{DCE}$ will be unit-invariant.

**Theorem 3.1.4 (Monotonicity).** $\gamma_{o}^{DCE}$ is strongly monotonic in input-costs.

**Proof.** Consider two different units, namely DMU$_{o}$ and DMU$_{p}$, who have all the same input-cost/output data excepting their $h$th input-cost; i.e., assume $\bar{x}_{hp} = \bar{x}_{ho} + \delta g_{h}^{-}$ where $\delta > 0$. Let $(\beta^{-}, \lambda^{-})$ be an optimal solution vector for DMU$_{o}$ in (28) and $(\bar{x}^{-}, y^{-})$ be the corresponding projection point. It is easy to show that $(\hat{\beta}^{-}, \hat{\lambda}^{-})$, where $\hat{\beta}_{i}^{-} = \beta_{i}^{-}, i \neq h$, $\hat{\beta}_{h}^{-} = \beta_{h}^{-} + \delta$, $\hat{\lambda} = \lambda^{-}$, is a feasible solution for DMU$_{p}$ in (28) with the following objective function value:

$$1 - \left(1/G^{-}\right) \left[ \sum_{i=1,i\neq h}^{m} \beta_{i}^{-} + \left(\beta_{h}^{-} + \delta\right) \right].$$

Since the model (28) is a minimization problem, and

$$1 - \left(1/G^{-}\right) \left[ \sum_{i=1,i\neq h}^{m} \beta_{i}^{-} + \left(\beta_{h}^{-} + \delta\right) \right] < 1 - \left(1/G^{-}\right) \sum_{i=1}^{m} \beta_{i}^{-},$$

one can conclude that $\gamma_{p}^{DCE} < \gamma_{o}^{DCE}$, which completes the proof. ■

Theorem 3.1.4 shows that an increase in an input-cost with all the other input-costs and outputs being constant will reduce the DCE score. However, it is not always so in case of increment in outputs, for which we have the following remark.

**Remark 1.** Like $\gamma_{o}^{CE}$, $\gamma_{o}^{DCE}$ does not take into account the non-zero output slacks of non-binding output constraints, which may result in assigning identical CE scores to the DMUs with different outputs (Fukuyama & Weber, 2009). To get rid of this problem in a way different from the one proposed in Fukuyama and Weber (2009), we solve the following problem:

$$\text{Max}_{\lambda, \beta^{+}} \quad 1 + \frac{1}{s} \sum_{i=1}^{s} \beta_{i}^{+}$$

s.t. $\sum_{j \in J} \lambda_{j} x_{ij} = \bar{x}_{io} - \beta_{i}^{-} g_{i}^{-}, \quad i = 1, ..., m,$

$$\sum_{j \in J} \lambda_{j} y_{rj} \geq y_{ro} + \beta_{r}^{+} g_{r}^{+}, \quad r = 1, ..., s,$$

$$\sum_{j \in J} \lambda_{j} = 1, \quad \lambda_{j} \geq 0, \quad \forall j \in J,$$

where $(\beta^{-}, \lambda^{-})$ is the optimal solution to (28). Then, we define the directional CE bias arising from the output slacks as
where $\beta^{*\ast}$ is the optimal solution to (37). We can now define the directional output-slack-adjusted CE ($\gamma^\text{DOSACE}_o$) as

$$\gamma^\text{DOSACE}_o = \gamma^\text{DCE}_o \times B^\text{DOS}_o. \quad (39)$$

With regard to the translation invariance property of $\gamma^\text{DCE}_o$, we present the following theorem, whose proof is similar to that of Theorem 3.1.1.

**Theorem 3.1.5 (Translation invariance).** $\gamma^\text{DCE}_o$ is translation-invariant with respect to input-costs if the translation of input-cost data does not affect the direction vector $g$. It is also translation-invariant with respect to outputs.

If the direction vector $g$ in (28) is not affected by the translation of the input-cost data, then $\gamma^\text{DCE}_o$ will be translation-invariant with respect to input-costs. A simple instance of such direction vector is the fixed direction vector considered in (25). The other instances are the non-fixed direction vectors in (23) and (24). For exposition, let us consider (23), and let $\bar{x}_{ij} = x_{ij} + \tau_i$ ($i = 1,\ldots,m$) be the $i$th translated input-cost of DMU$_j$. We show how the direction vector (23) for the translated data $(\bar{g})$ is the same direction vector for the original data:

$$\bar{g}_i = \bar{x}_{i0} - \text{Min}_{j \in J} \{ \bar{x}_{ij} \} = (x_{i0} + \tau_i) - \text{Min}_{j \in J} \{ x_{ij} + \tau_i \} = x_{i0} + \tau_i - \text{Min}_{j \in J} \{ x_{ij} \} - \tau_i = g_i^-, \quad i = 1,\ldots,m.$$  

### 3.2. Directional output-oriented value-based TE, output-oriented AE, and RE measures

We now present below the output-oriented DDF-based DEA model relative to $T^\text{DEA}_x$ as

$$\beta' = \max_{\lambda, \beta} \beta$$

s.t. $\sum_{j \in J} \lambda_j x_{ij} \leq x_{i0}$, \quad $i = 1,\ldots,m$, 

$$\sum_{j \in J} \lambda_j \bar{y}_{rj} \geq \bar{y}_{r0} + \beta g_r^+$, \quad r = 1,\ldots,s, \quad (40)$$

$$\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J.$$

Based on (40), we now define the new **directional output-oriented value-based TE** (DOVTE) measure as

$$\rho^\text{DOVTE}_o = \frac{1}{1 + \beta'} . \quad (41)$$
Similar to Theorems 2 and 4 in Fukuyama and Weber (2004) and Theorems 3.1.1 and 3.1.2, we present several results that are linked to properties such as monotonicity, unit invariance and translation invariance. For example, \( \rho^{DOTE}_o \) is unit-invariant for the following direction vectors:

\[
g^-_i = 0, \quad g^+_r = \bar{y}_{yo}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \tag{42}
\]

\[
g^-_i = 0, \quad g^+_r = \max_{j \in J} \{ \bar{y}_{yo} \}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \tag{43}
\]

Furthermore, \( \rho^{DOTE}_o \) is both unit-invariant and translation-invariant for each of the following direction vectors:

\[
g^-_i = 0, \quad g^+_r = \max_{j \in J} \{ \bar{y}_{yo} \} - \bar{y}_{yo}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \tag{44}
\]

\[
g^-_i = 0, \quad g^+_r = \max_{j \in J} \{ \bar{y}_{yo} \} - \min_{j \in J} \{ \bar{y}_{yo} \}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s. \tag{45}
\]

We now introduce, based on \( T^{DEA}_{\lambda, \beta} \), a new directional RE (DRE) measure as

\[
\eta^{DRE}_o = \frac{1}{1 + \sum_{i=1}^{n} g^+_r G^+} \left[ \sum \lambda_j x_j - x_{yo} \right], \quad i = 1, \ldots, m,
\]

\[
\text{s.t.} \quad \sum_{j \in J} \bar{y}_{yo} \geq \bar{y}_{yo} + \beta^+ G^+, \quad r = 1, \ldots, s,
\]

\[
\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J,
\]

where \( G^+ = \sum_{i=1}^{n} g^+_r \), and \( \beta^+ \) represents the rate of improvement in the \( r \)th output-earnings of DMU\(_o\).

\( \eta^{DRE}_o \) is a generalization of \( \eta^{RE}_o \). Specifically, if \( \bar{y}_{yo} > 0 \) and the direction vector (42) is considered, then the model (46) is transformed into the following:

\[
\text{Max} \quad \frac{1}{R_o} \sum_{i=1}^{n} (1 + \beta^+_r) \bar{y}_{yo}
\]

\[
\text{s.t.} \quad \sum_{j \in J} \bar{y}_{yo} \geq \bar{y}_{yo} + \beta^+ g^+_r, \quad r = 1, \ldots, s,
\]

\[
\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J.
\]

Letting \( \bar{y}_i = (1 + \beta^+ r) \bar{y}_{yo} \) in (47) yields exactly the model (18).

Solving the model (46) yields the following expression for DMU\(_o\):
\[
\sum_{j \in J} \lambda_j^i x_{ij} \leq x_{io}, \quad i = 1, \ldots, m,
\]
\[
\sum_{j \in J} \lambda_j^r \bar{y}_j = \bar{y}_{ro} + \beta_r^* g_r^+, \quad r = 1, \ldots, s.
\] (48)

Based on (48), we define the DRE-based projection for DMU \(_o\) by
\[
x_{io} \rightarrow \hat{x}_{io} = x_{io}, \quad i = 1, \ldots, m,
\]
\[
\bar{y}_{ro} \rightarrow \hat{\bar{y}}_{ro} = \sum_{j \in J} \lambda_j^r \bar{y}_j, \quad r = 1, \ldots, s.
\] (49)

The projection point \((\hat{x}_o, \hat{\bar{y}}_o)\) represents the maximum possible improvements in the total earnings for DMU \(_o\) to be DRE-efficient.

It can be shown that the projection point \((\hat{x}_o, \hat{\bar{y}}_o)\) is also DOVTE-efficient, but not strongly technical-efficient in a Pareto-Koopmans sense due to the existence of possible input slacks. It can be also shown that \((\hat{x}_o, \hat{\bar{y}}_o)\) is strongly technical-efficient with respect to outputs.

Similar to Theorem 3.1.3, we have that
\[
\eta_o^{DRE} = \frac{1}{1 + \frac{1}{G} \left( \sum_{r=1}^{s} \bar{y}_r^* - \sum_{r=1}^{s} \bar{y}_{ro} \right)}, \quad \eta_o^{DRE} \leq \rho_o^{DOVTE} \leq 1.
\] (50)

where \(\bar{y}^*\) is the optimal solution vector to (18). Further, \(\eta_o^{DRE}\) is strongly monotonic in output-earnings and is translation-invariant with respect to output-earnings if the translation of the output-earnings data has no effect on the assigned direction vector. For instance, the direction vectors - (44) and (45), provide this property. We now have the following relationship:
\[
\eta_o^{DRE} \leq \rho_o^{DOVTE} \leq 1.
\] (51)

One can thus define the directional output-oriented AE (\(\alpha_o^{DOAE}\)) as
\[
\alpha_o^{DOAE} = \eta_o^{DRE} / \rho_o^{DOVTE}.
\] (52)

Rearranging (52) yields the following decomposition of \(\eta_o^{DRE}\) into directional (output-oriented) AE (\(\alpha_o^{DOAE}\)) and value-based TE (\(\rho_o^{DOVTE}\)):
\[
\eta_o^{DRE} = \alpha_o^{DOAE} \times \rho_o^{DOVTE}.
\] (53)

3.3 Directional profit inefficiency measure
In many real-life applications both physical inputs and physical outputs are heterogeneous across firms, in which case our DCE and DRE measures cannot be applied to evaluate the respective cost and revenue efficiencies. In this scenario, a directional measure of profit (in)efficiency based on \( T^{DEA}_{\tau, \gamma} \) needs to be developed. This measure will be more meaningful for firms when their underlying behavioral objectives are profit maximization rather than cost minimization or revenue maximization.

Consider the following directional profit-maximization problem based on \( T^{DEA}_{\tau, \gamma} \) in (8):

\[
\kappa^*_o = \max_{\lambda, \beta^+, \beta^-} \sum_{r=1}^s \frac{g_r^+}{G^+} \beta^+_r + \sum_{i=1}^m \frac{g_i^-}{G^-} \beta^-_i
\]

\[
s.t. \quad \sum_{j \in J} \lambda_j x_{ij} \leq \overline{x}_{io} - \beta^+_i g_i^-, \quad i = 1, ..., m,
\]

\[
\sum_{j \in J} \lambda_j \overline{y}_{ij} \geq \overline{y}_{io} + \beta^+_r g_r^+, \quad r = 1, ..., s,
\]

\[
\sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J.
\]

DMU\(_o\) is directional profit-efficient if \( \kappa^*_o = 0 \). However, if \( \kappa^*_o > 0 \), then DMU\(_o\) is directional profit-inefficient in which case the degree of profit inefficiency is defined as:

\[
\pi_{o}^{D-buff} = \left( \sum_{r=1}^s (\overline{y}_{io} + \beta^+_r g_r^{**}) - \sum_{i=1}^m (\overline{x}_{io} - \beta^-_i g_i^{**}) \right) - \left( \sum_{r=1}^s \overline{y}_{io} - \sum_{i=1}^m \overline{x}_{io} \right)
\]

\[
\frac{\sum_{r=1}^s \beta^+_r g_r^{**} + \sum_{i=1}^m \beta^-_i g_i^{**}}{G^+ + G^-}
\]

\( \pi_{o}^{D-buff} \) can be estimated for each of the following direction vectors:

\[
g_i^-=1, \quad g_r^+=1, \quad i = 1, ..., m, \quad r = 1, ..., s.
\]

\[
g_i^- = \overline{x}_{io}, \quad g_r^+ = \overline{y}_{io}, \quad i = 1, ..., m, \quad r = 1, ..., s.
\]

\[
g_i^- = \max_{j \in J} \{ \overline{x}_{ij} \}, \quad g_r^+ = \max_{j \in J} \{ \overline{y}_{ij} \}, \quad i = 1, ..., m, \quad r = 1, ..., s.
\]

\[
g_i^- = \overline{x}_{io} - \min_{j \in J} \{ \overline{x}_{ij} \}, \quad g_r^+ = \max_{j \in J} \{ \overline{y}_{ij} \} - \overline{y}_{io}, \quad i = 1, ..., m, \quad r = 1, ..., s.
\]

\[
g_i^- = \max_{j \in J} \{ \overline{x}_{ij} \} - \min_{j \in J} \{ \overline{x}_{ij} \}, \quad g_r^+ = \max_{j \in J} \{ \overline{y}_{ij} \} - \min_{j \in J} \{ \overline{y}_{ij} \}, \quad i = 1, ..., m, \quad r = 1, ..., s.
\]

Note that \( \beta^+_r (\forall r) \) and \( \beta^-_i (\forall i) \) in (54) are not all necessarily non-negative. It implies that the model (54) may permit a DMU to become profit-efficient by reducing its spending on some inputs while increasing on the other input costs; and by increasing its earnings on some outputs while
decreasing on the others. However, if one expects a DMU to become profit-efficient by reducing its spending on all its inputs, and by increasing its earnings on all outputs, then one need to impose the non-negativity restrictions on the variables on output-earnings and input-spending, i.e., \( \beta_r^+ \geq 0 \ (\forall r) \) and \( \beta_i^- \geq 0 \ (\forall i) \) in (54).

Furthermore, it might also be possible to discuss the value judgments (preferences) in our proposed measures as to which costs to reduce or which revenues to increase. The firm manager may be willing to make the firm cost-, revenue- or profit-efficient without reducing its spending on some specific inputs and/or increasing its earnings on some specific outputs. In such situations, (s)he may choose a direction vector whose components associated with these inputs/outputs are zero. (S)he may also like to set the upper or lower bounds for the required input-spending or output-earnings changes by adding constraints \( L_i^- \leq \beta_i^- \cdot g_i^- \leq U_i^- \ (\forall i) \) and/or \( L_r^+ \leq \beta_r^+ \cdot g_r^+ \leq U_r^+ \ (\forall r) \) where \( L_i^- \) and \( U_i^- \) are respectively the lower and upper bounds on the \( i \)th input-spending, and \( L_r^+ \) and \( U_r^+ \) are respectively the lower and upper bounds on the \( r \)th output-earnings. This will enable researchers to explain the economic meaning of the preferences, i.e., why to reduce some specific input-costs more than others; why to utilize capacity better or how to reduce the capacity costs while producing still the same revenue as earlier.

**Remark 2.** Note that the approach presented in Remark 1 is a two-stage approach consisting of solving the models (28) and (37), which can be formulated as a single model:

\[
\begin{align*}
\text{Max} \quad & \epsilon \left( 1 + \frac{1}{s} \sum_{i=1}^{s} \beta_i^+ \right) - \left( 1 - \frac{1}{m} \sum_{i=1}^{m} \frac{g_i^-}{G_i^-} \beta_i^- \right) \\
\text{s.t.} \quad & \sum_{j \in J} \lambda_j x_{ij} \leq \bar{x}_i - \beta_i^- g_i^- , \quad i = 1, \ldots, m, \\
& \sum_{j \in J} \lambda_j y_{jr} \geq y_{ro} + \beta_r^+ g_r^+ , \quad r = 1, \ldots, s, \\
& \sum_{j \in J} \lambda_j = 1, \quad \lambda_j \geq 0, \quad \forall j \in J,
\end{align*}
\]

where \( \epsilon \) is a small non-Archimedean infinitesimal quantity. The model (61) can be equivalently written as:
Comparing (62) with (54), we find out that the above model is a special case of model (54) provided that (a) the output prices are common between firms, and (b) instead of the output weights \( r_j \) in (54), \( (r_j) \) are used.

Note that the profit inefficiency measures based on the director vectors (56)–(60) are all non-radial (non-proportional) in nature. Thus, these measures have the flexibility to better reflect the trade-offs in their efficiency estimates between inputs and/or outputs due to their differed opportunity costs. See, e.g., Sahoo and Tone (2009a), Sahoo and Tone (2009b), Sahoo and Acharya (2010), Sahoo, Luptacik, and Mahlberg (2011), Mahlberg and Sahoo (2011), among others, for the discussion of the debate on the choice of non-radial measures of efficiency over the radial ones.

3.4. Dealing with negative input-output data

In the presence of negative data, the DCE and DRE measures can be superior to their counterparts, i.e., the CE and RE measures, if appropriate direction vectors are considered. For example, the following direction vectors can be chosen for the DCE and DRE measures:

\[
g_i^- = \begin{cases} \bar{x}_{io} & \text{if } i = 1, \ldots, m, \ r = 1, \ldots, s \\
0 & \text{otherwise}
\end{cases}
\]

\[
g_r^+ = \begin{cases} \bar{y}_{ro} & \text{if } r = 1, \ldots, s \\
0 & \text{otherwise}
\end{cases}
\]

Similarly, the following direction vector enables the profit inefficiency measure (54) to deal with negative data:

\[
g_i^- = \begin{cases} \bar{x}_{io} & \text{if } i = 1, \ldots, m, \ r = 1, \ldots, s \\
0 & \text{otherwise}
\end{cases}
\]

\[
g_r^+ = \begin{cases} \bar{y}_{ro} & \text{if } r = 1, \ldots, s \\
0 & \text{otherwise}
\end{cases}
\]

The second possible choice for the direction vector can be argued based on the property of translation invariance, which enables analysis of data sets containing negative values. Since the directional CE, RE and profit (in)efficiency measures can be all translation-invariant, adding any arbitrary constant to their input-spending and output-earning constraints will not affect their optimal solutions. One can add suitable large positive constants to the input-spending and output-earnings.
so as to make them all non-negative. For instance, for the DCE measure one can use the following transformation:

\[ \bar{x}_i + \tau_i \geq 0, \quad i = 1, \ldots, m; \]
\[ y_r + \sigma_r \geq 0, \quad r = 1, \ldots, s. \]  

(66)

Furthermore, one can turn to other direction vectors such as (23), (44) and (59) that consider taking the range of possible improvements in the input-costs and/or the output-earnings. Since the directional CE, RE and profit (in)efficiency measures are all translation-invariant for (23), (44) and (59) respectively, the use of these direction vectors makes the efficiency assessment possible in the presence of negative data.

4. An empirical illustration

To demonstrate the ready applicability of our proposed measure, we conduct an illustrative empirical analysis based on a real-life data set of 50 US banks for the year 1996 (which was taken from Ray (2005)), wherein we make meaningful comparison between our proposed approach and Tone’s (2002) approach by considering how the choice of direction vector plays a significant role in efficiency analysis. The data consists of five outputs – commercial and industrial loans, consumer loans, real estate loans, investments and other income, and four inputs – transaction deposits, non-transaction deposits, labor and capital.

We first make our cost and revenue efficiency assessments using the direction vectors (24) and (45), and then do the same for Tone’s (2002) measures, which are in fact derived from our measures by considering the direction vectors (21) and (42). We use GAMS (General Algebraic Modeling System) software code to do the computations. Table 1 exhibits such detailed results.

We now summarize the results of Table 1 as follows.

1. Branches 1, 3, 5, 13, 24, 45, 48, and 49 are all identified to be fully efficient by all of the efficiency measures; and the remaining 42 branches are declared inefficient by at least one of these efficiency measures.

2. The cost inefficiencies of the branches - 2, 6, 7, 9, 11, 14–16, 21, 33, 34 and 47 are all due to their allocative inefficiencies. However, the cost inefficiencies of the other branches are due to both technical and allocative inefficiencies. Similar results hold for the revenue efficiency as well. See the revenue efficiency scores of branches 7, 9, 10, 17–21, 23, 28–30, 38–39, 41–44, 46 and 50.
### Table 1. Cost and revenue efficiencies of 50 bank branches under analysis

<table>
<thead>
<tr>
<th>DMU</th>
<th>Direction vectors (21) and (42)</th>
<th>RE Assessment</th>
<th>Direction vectors (24) and (45)</th>
<th>RE Assessment</th>
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<tr>
<td></td>
<td>CE Assessment</td>
<td></td>
<td>CE Assessment</td>
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<td></td>
<td>$\rho_{\text{vis}}^{\text{CE}}$</td>
<td>$\rho_{\text{vis}}^{\text{RE}}$</td>
<td>$\eta_{\text{vis}}^{\text{CE}}$</td>
<td>$\eta_{\text{vis}}^{\text{RE}}$</td>
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<td>1.0000</td>
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<td>0.7511</td>
<td>0.7511</td>
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<tr>
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</table>
3. There is no positive correlation between cost and revenue efficiencies of banks. On average, the cost efficiency is found to be higher compared to the revenue efficiency, which means that bank management has to take judicious decisions to enhance their profits by producing the right portfolios of product-mixes.

As argued earlier, the cost- and revenue-based efficiency results are not very meaningful when the underlying objective of a firm is profit maximization. Therefore, we now turn to exhibit the results on profit inefficiencies of banks in Table 2 based on the four direction vectors (57)–(60).

One can observe that profit inefficiency scores depend on the choice of the direction vector used. While, one branch, i.e., 50, is found profit-efficient for all the four considered direction vectors, eight branches (1, 3, 5, 13, 43, 45, 46 and 49) are only efficient for the range direction vector (60). The profit inefficiency measure based on the direction vector (57) yields highest level of average inefficiency as compared to profit inefficiency measures based on the other direction vectors.

As regards the target setting (benchmarking exercise) by firms as to which costs to reduce or which revenues to increase, one can see in Table 3 how an average DMU can attain its full profit efficiency by changing its input-spending and output-earnings. The improvements in input-spending and outputs-earnings of a branch depend precisely on the choice of the direction vector used. For example, the profit inefficiency model based on the direction vector (57) requires an average branch to attain its full efficiency by, reducing its spending on two inputs - transaction cost and non-transaction deposits cost, increasing its spending on other two inputs - labor cost and capital cost, reducing its earnings on one output, i.e., commercial and industrial loans, and increasing its earnings on the other four outputs - consumer loans, real estate loans, investments and other income. Similarly, the profit inefficiency models based on the other direction vectors can be interpreted in an analogous manner. For lack of space, the detail results on the improvements in input-spending and output-earnings by the individual branches based on different direction vectors are not reported here, but are provided as supplementary materials (see Tables 4-7).
Table 2. Profit efficiency of 50 bank branches under analysis

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Direction vector (57)</th>
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Furthermore, it is apparent that the rates of improvements in individual input-spending and individual output-earnings are different, implying that a branch need not reduce its inputs, and increase its outputs, equiproportionately so as to improve its profit efficiency. This establishes the empirical reality that the opportunity costs of consumption (production) of one input (output) over the other are not the same.

Table 3. Average improvements in input-spending and output-earnings across models

<table>
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<th>Direction vector</th>
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<th>$\beta_3$</th>
<th>$\beta_4$</th>
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<th>$\beta_6$</th>
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<td>(57)</td>
<td>0.032</td>
<td>0.031</td>
<td>-0.584</td>
<td>-1.586</td>
<td>-0.794</td>
<td>4.027</td>
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<tr>
<td>(58)</td>
<td>0.201</td>
<td>0.049</td>
<td>0.012</td>
<td>-0.085</td>
<td>-0.068</td>
<td>-0.081</td>
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<tr>
<td>(59)</td>
<td>-4.351</td>
<td>0.829</td>
<td>0.332</td>
<td>0.431</td>
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<td>-0.082</td>
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<tr>
<td>(60)</td>
<td>0.205</td>
<td>0.060</td>
<td>0.012</td>
<td>-0.087</td>
<td>-0.068</td>
<td>-0.081</td>
<td>-0.050</td>
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5. Concluding remarks

Under the condition of no prior unit price information, the current study develops new directional measures of value-based cost and revenue efficiencies, which all satisfy several desirable properties of an ideal efficiency measure. These new value-based cost/revenue efficiencies apply when output quantities/input quantities are homogenous and known. This study then decomposes these new value-based efficiencies into their respective value-based TE and AE components. Our proposed efficiency measures have several advantages over the existing ones: 1) they all satisfy the most important property of translation invariance; 2) they all allow freedom of choice over the use of suitable direction vectors in handling negative data; 3) they can be directly applied as they can be easily implemented in any standard DEA software due to flexible computer programming; 4) they provide the decision makers with the option of specifying the preferable
direction vectors to incorporate their decision-making preferences; and finally, 5) they include the Tone’s (2002) value-based measures as special cases.

Finally, under the condition that when both physical input and output quantities are heterogeneous across firms, or when the precise input and output price data are frequently unavailable, a directional measure of profit inefficiency is developed for firms whose underlying objectives are profit maximization. This measure allows empirical researchers to potentially deal with value judgments (preferences) as to which specific input-cost to reduce or which specific output-earnings to increase to improve the profit efficiency of a firm.

Note that the directional value-based measures developed in this paper are based on the assumption that the firms’ heterogeneity is reflected through their unit prices. However, when prices vary according to the action by firms, i.e., when a firm with higher market power exhibits higher prices relative to a firm without market power, efficiency estimates are exaggerated. Therefore, when price differences are due to different competitive environments, it seems that these environmental factors should be taken into account instead of trying to directly compare the efficiencies of firms. This concerns especially the case when we try to define their efficiency improvement potential with respect to each other. We suggest this as a future research subject.

Acknowledgement

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References


**End Notes**

1 Note that Silva Portela, Thanassoulis, and Simpson (2004) used the direction vector (23) to develop the range directional distance function model to measure the TE in the presence of negative data in DEA.

2 Mehdiloozad, Sahoo, and Roshdi (2014) also used this direction vector in log form to develop a generalized multiplicative directional measure of efficiency.

3 In the presence of negative data, Kerstens and Van de Woestyne (2011) used this direction vector in factor form to develop a generalized proportional distance function measure of efficiency.