The distance between two separating, reducing slopes is at most 4

Mingxing Zhang, Ruifeng Qiu and Yannan Li*

Abstract

Let M be a simple 3-manifold such that one component of ∂M, say F, has genus at least two. For a slope α on F, we denote by $M(\alpha)$ the manifold obtained by attaching a 2-handle to M along a regular neighborhood of α on F. If $M(\alpha)$ is reducible, then α is called a reducing slope. In this paper, we shall prove that the distance between two separating, reducing slopes on F is at most 4.

Keywords: S-cycle, extended S-cycle, reducing slope.

1 Introduction

Let M be a compact, orientable 3-manifold such that ∂M contains no spherical components. M is said to be simple if M is irreducible, ∂-irreducible, anannular and atoroidal.

Let M be a simple 3-manifold. For a component F of ∂M, a slope γ on F is an isotopy class of essential simple closed curves on F. For a slope γ on F, we denote by $M(\gamma)$ the manifold obtained by attaching a 2-handle to M along a regular neighborhood of γ on F, then capping off a possible 2-sphere component of the resulting manifold by a 3-ball. A slope γ on F is said to be reducing if $M(\gamma)$ is reducible. The distance between two slopes α and β on F, denoted by $\Delta(\alpha, \beta)$, is the minimal geometric intersection number among all

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the curves representing the slopes. Note that if F is a torus, then $M(\gamma)$ is the Dehn filling along γ. Two important results about reducing handle additions on simple 3-manifolds are the following:

(1) Suppose that F is a torus, α and β are two reducing slopes on F. Gordon and Luecke[GL1] proved that $\Delta(\alpha, \beta) \leq 1$. This means that there are at most three reducing slopes on F.

(2) Suppose that $g(F) > 1$. Scharlemann and Wu[SW] proved that there are only finitely many basic degenerating slopes on F. As a corollary of this result, there are only finitely many separating, reducing slopes on F.

In this paper, we shall continue to study reducing handle additions. The main result is the following theorem:

Theorem 1. Suppose that M is a simple 3-manifold, and F is a genus at least two component of ∂M. If α and β are two separating, reducing slopes on F, then $\Delta(\alpha, \beta) \leq 4$.

Comments on Theorem 1.

1. It is possible that $\Delta(\alpha, \beta)$ is arbitrarily large when α and β are two non-separating, reducing slopes on F. For example, one can construct a simple 3-manifold N such that there is a separating, reducing slope γ on ∂N which bounds a punctured torus T in ∂N. Then $N(\gamma)$ is reducible and $\partial N(\gamma)$ contains a toral component T^* such that $T \subset T^*$. By the [GL2] and [SW], there are infinitely many slopes α on T such that $N(\alpha) = N(\gamma)(\alpha)$ is reducible.

2. Let M be a simple 3-manifold containing no essential closed surfaces of genus g. Suppose that α and β are separating slopes on ∂M such that $M(\alpha)$ and $M(\beta)$ contains an essential closed surface of genus g. If $g \leq 1$, then $\Delta(\alpha, \beta) \leq 14$, see [SW]. If $g > 1$, then it is possible that $\Delta(\alpha, \beta)$ is arbitrarily large, see [QW1] and [QW2].

2 Labeled graph

The following Lemma follows from the proof of Lemma 3.3 in [SW].

Lemma 2.1. Suppose M is a simple manifold. If α is a separating, reducing slope and $M(\alpha)$ is ∂-irreducible, then M contains an incompressible and ∂-incompressible planar surface in M with all boundary components having the same slope α.

Proof. Suppose P is a planar surface in M with all boundary components parallel to α. Capping off all such components by mutually disjoint disks in $M(\alpha)$, we get a surface $\hat{P}$ in $M(\alpha)$. P is called a presphere if $\hat{P}$ is a reducing sphere of $M(\alpha)$. Since $M(\alpha)$ is reducible, the prespheres must exist. Assume P is a presphere such that $|\partial P|$ is minimal. Then P must be incompressible.

Now suppose P is ∂-compressible, with D a ∂-compressing disk. Let $\partial D = u \cup v$, where u is an arc in P, and v is an essential arc in ∂M. Since P is incompressible, v is essential on $\partial M - \partial P$.

∂-compressing P along D, we get a new surface, which has one or two new boundary components, depending on whether the two endpoints of v lie on the different components of P. If a new boundary component is trivial in ∂M, we cap off the component by a disk. In this way, we get a new surface denoted by P'. There are two possibilities:

(1) v has endpoints on the different components of ∂P.

Now $\hat{P'}$ is also a reducing 2-sphere and $|\partial P'| < |\partial P|$. It contradicts the assumption that $|\partial P|$ is minimal.

(2) v has endpoints on the same component of ∂P.

$\hat{P'}$ has two components, each of which is a compressing disk of $M(\alpha)$, a contradiction.

□

Suppose that M is a simple 3-manifold, and F is a genus at least two component of ∂M. Assume α and β are separating, reducing slopes on F. If one of $M(\alpha)$ and $M(\beta)$, say $M(\beta)$, is ∂-reducible, then, by Lemma 4.2 of [SW], $\Delta(\alpha, \beta) = 0$. Hence we may assume that $M(\alpha)$ and $M(\beta)$ are ∂-irreducible.

Suppose $\hat{P}$(resp. $\hat{Q}$) is a reducing 2-sphere in $M(\alpha)$(resp. $M(\beta)$) such that $p = |\partial P|$(resp. $q = |\partial Q|$) is minimal among all the reducing 2-spheres, where $P = \hat{P} \cap M$(resp. $Q = \hat{Q} \cap M$). By the proof of Lemma 2.1, P and Q are incompressible and ∂-incompressible in M. Isotopy P and Q so that $|P \cap Q|$ is minimal. Then each component of $P \cap Q$ is either an essential arc or an essential circle on both P and Q.

Let Γ_P is a graph in $\hat{P}$ obtained by taking the arc components of $P \cap Q$ as edges and taking the boundary components of P as fat vertices. Similarly, we can define Γ_Q in $\hat{Q}$.

Lemma 2.2. There are no 1-sided disk faces on Γ_P(resp.Γ_Q). □

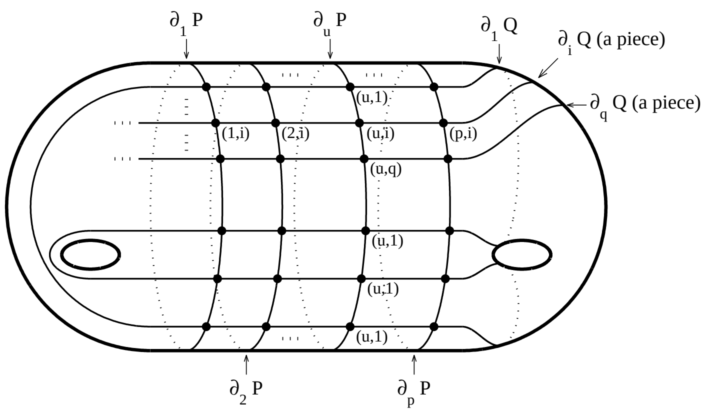


Figure 1

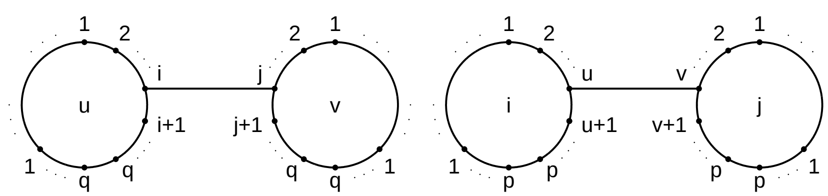


Figure 2: Labels on Γ_P and Γ_Q

Number the components of ∂P with $\partial_1 P, \partial_2 P, \cdots, \partial_u P, \cdots, \partial_p P$ consecutively on ∂M, this means that $\partial_u P$ and $\partial_{u+1} P$ bound an annulus in ∂M with interior disjoint from P. See Figure 1. Similarly, number the components of ∂Q with $\partial_1 Q, \partial_2 Q, \cdots, \partial_i Q, \cdots, \partial_q Q$. These give corresponding labels of the vertices of Γ_P and Γ_Q. For an endpoint x of an edge e in Γ_P, if it belongs to $\partial_u P \cap \partial_i Q$, then we label it as (u, i), or i(resp. u) in Γ_P(resp. Γ_Q) for shortness when u(resp. i) is specified. See Figure 2. Now each edge e of Γ_P has been labeled with $(u, i) - (v, j)$, or $i - j$(resp. $u - v$) in Γ_P(resp. Γ_Q) for shortness. See Figure 2. When we travel around $\partial_u P$, the labels appear in the order $1, 2, \cdots, q, q, \cdots, 2, 1, \cdots$(repeated $\Delta(\alpha, \beta)/2$ times). Note that Γ_Q have the same property.

3 Parity rule

We first sign the endpoints of the edges in Γ_P(and in Γ_Q). Fix the directions on α and β. Then each point in $\alpha \cap \beta$ can be signed "+" or "−" depending on whether the direction determined by right-hand rule from α to β is pointed to the outside of M or to the inside of M. See Figure 3. Since α and β is separating, the signs "+" and "−" appear alternately on both α and β.

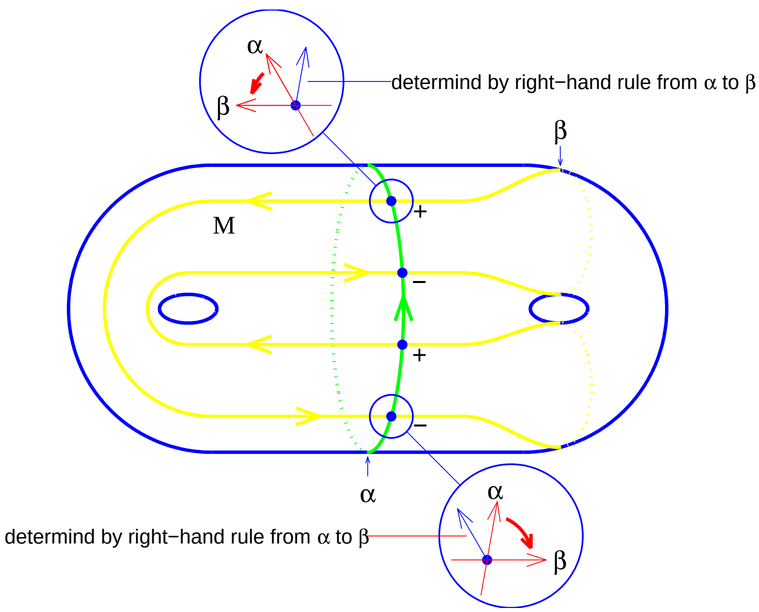


Figure 3

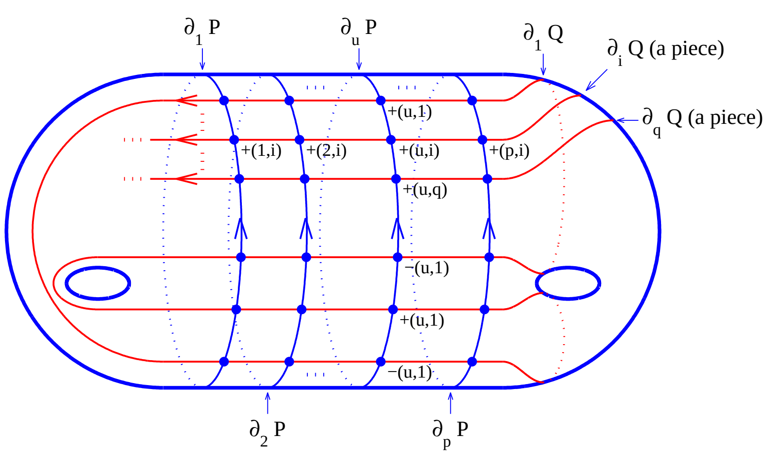


Figure 4: Signs on $\partial P \cap \partial Q$

Give a direction to each boundary components ∂P(resp. ∂Q) such that they are all parallel to α(resp. β) on ∂M. Then each point $x \in \partial P \cap \partial Q$ can be signed as above. We denoted by $c(x)$ the sign of x. See Figure 4. Now the signed labels appear on $\partial_u P$ as $+1, +2, \cdots, +q, -q, \cdots, -2, -1, \cdots,$ (repeated $\Delta(\alpha, \beta)/2$ times). See Figure 5.

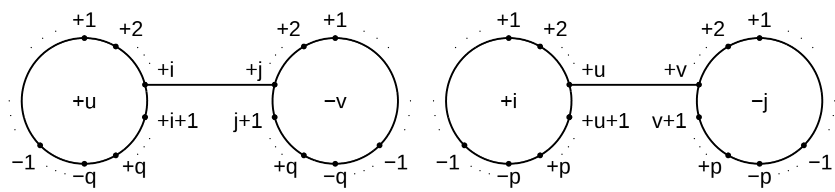


Figure 5

Now we sign the vertices of Γ_P. Suppose $P \times [0, 1]$ be a thin regular neighborhood of P in M. Let $P^+ = P \times 1$ and $P^- = P \times 0$. For some $1 \leq u \leq p, 1 \leq i \leq q$, let c be a component of $\partial_u P \times [0, 1] \cap \partial_i Q$ with the induced direction of $\partial_i Q$. We define the sign of $\partial_u P$ as follows:

(1) Suppose c intersects $\partial_u P$ at a "+" point, we define the sign of $\partial_u P$ is "+"(resp. "−") if the direction of c is from P^+ to P^-(resp. from P^- to P^+).

(2) Suppose c intersects $\partial_u P$ at a "−" point, we define the sign of $\partial_u P$ is "+"(resp. "−") if the direction of c is from P^- to P^+(resp. from P^+ to P^-).

Since each component of ∂Q has the same direction with β on F, the definition as above is independent of the choices of c and i.

For example, in Figure 6 and Figure 7, the signs of $\partial_u P$, $\partial_v P$ and $\partial_w P$ are "+", "−" and "−" respectively.

Since M is orientable, $\partial_u P$ and $\partial_v P$ have the same direction on P when $\partial_u P$ and $\partial_v P$ have the same signs. This means the labels $+1, +2, \cdots, +q,\ -q, \cdots, -1$ of the edge-endpoints appear on both $\partial_u P$ and $\partial_v P$ are in the same direction in Γ_P. Similarly, the labels $+1, +2, \cdots, +q, -q \cdots, -1$ appear in opposite the directions when $\partial_u P$ and $\partial_v P$ have different signs. See Figure 7.

We may define the sign $s(i)$ of $\partial_i Q$ in Γ_Q.

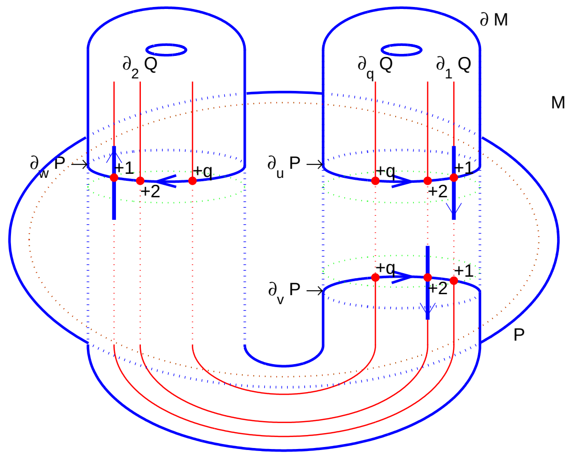


Figure 6: Signs on ∂P

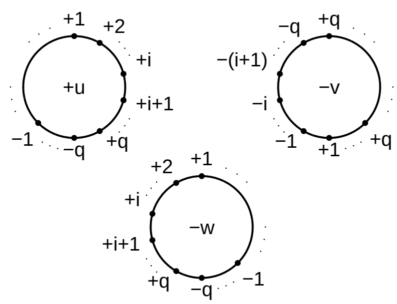


Figure 7: Signs on ∂P

The labels with the signs defined as above are said to be Type A. Now we have Parity rule A:

Lemma 3.1(Parity rule A). For an edge e in Γ_P(and Γ_Q) with its endpoints x labeled (u, i) and y labeled (v, j), the following equality holds:

$$s(i)s(j)s(u)s(v)c(x)c(y) = -1 \qquad (*).$$

Proof. Let $P \times I$ be a thin regular neighborhood of P in M. Then $e \times I \subset Q$ and $x \times I \subset \partial_u P \times I$ and $y \times I \subset \partial_v P \times I$.

Now there are four possibilities:

Case 1. $s(i) = s(j)$ and $c(x) = c(y)$ as in Figure 8(a).

Since $s(i) = s(j)$, $\partial_i Q$ and $\partial_j Q$ have the same direction. In this case, $x \times I$ and $y \times I$ have the opposite directions(as in Figure 8(a). Since $c(x) = c(y)$, by the definitions of $s(u)$ and $s(v)$, $s(u) \neq s(v)$. Hence the equality $(*)$ holds.

Case 2 $s(i) = s(j)$ and $c(x) \neq c(y)$ as in Figure 8(b).

Since $s(i) = s(j)$, $\partial_i Q$ and $\partial_j Q$ have the same direction. In this case, $x \times I$ and $y \times I$ have the opposite directions as in Figure 8(b). Since $c(x) \neq c(y)$, by the definitions of $s(u)$ and $s(v)$, $s(u) = s(v)$. Hence the equality $(*)$ holds.

Case 3 $s(i) \neq s(j)$ and $c(x) = c(y)$ as in Figure 8(c). .

Since $s(i) \neq s(j)$, $\partial_i Q$ and $\partial_j Q$ have opposite directions. In this case, $x \times I$ and $y \times I$ have the same direction as in Figure 8(c). Since $c(x) = c(y)$, by the definitions of $s(u)$ and $s(v)$, $s(u) = s(v)$. Hence the equality $(*)$ holds.

Case 4 $s(i) \neq s(j)$ and $c(x) \neq c(y)$ as in Figure 8(d).

Since $s(i) \neq s(j)$, $\partial_i Q$ and $\partial_j Q$ have opposite directions. In this case, $x \times I$ and $y \times I$ have the same direction as in Figure 8(d). Since $c(x) \neq (y)$, by the definitions of $s(u)$ and $s(v)$, $s(u) \neq s(v)$.

Hence the equality $(*)$ holds. □

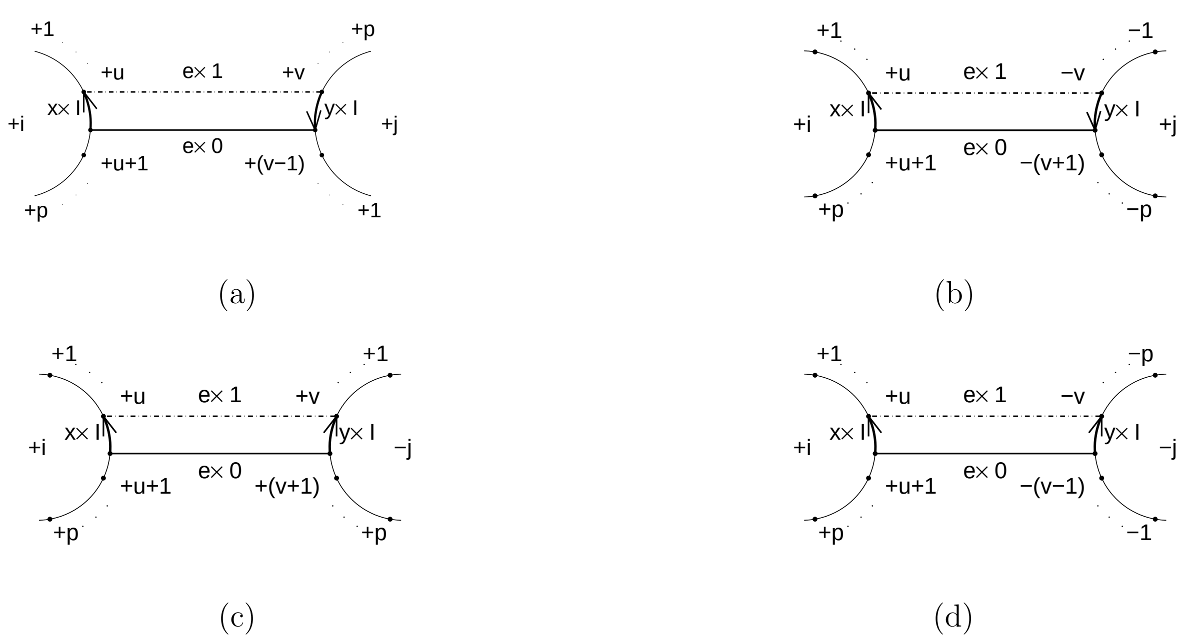


Figure 8

Now suppose that e is an edge of Γ_P with $\partial e = x \cup y$, and x is labeled (u, i). Let $g(x) = c(x) \times s(u)$. Then the signed label $g(x)i$ of x is said to be Type B.

Remark (∗) Under Type B labels, the labels $+1$, $+2 \cdots$, $+q, -q, \cdots$ appear in the same direction on all the vertices of Γ_P. See Figure 9.

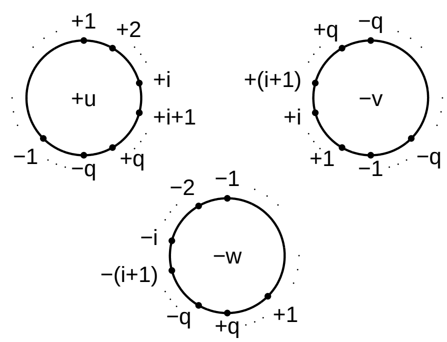


Figure 9: Type B labels

By Lemma 3.1, we have the parity rule for Type B labels.

Lemma 3.2(Parity rule B). Let e be an edge e in Γ_P with its endpoints x labeled (u, i) and y labeled (v, j), then $s(i)s(j)g(x)g(y) = -1$. □

Lemma 3.3. Let e be an edge e in Γ_P with its endpoints x labeled (u, i) and y labeled (v, i). Then $g(x) \neq g(y)$. □

4 S-cycles

In this section, the definitions of a cycle, the length of a cycle, a disk face and parallel edges are standard, see [GL1], [SW] and [W].

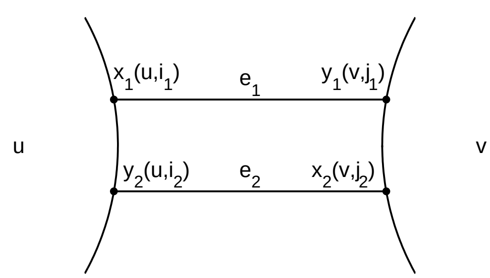
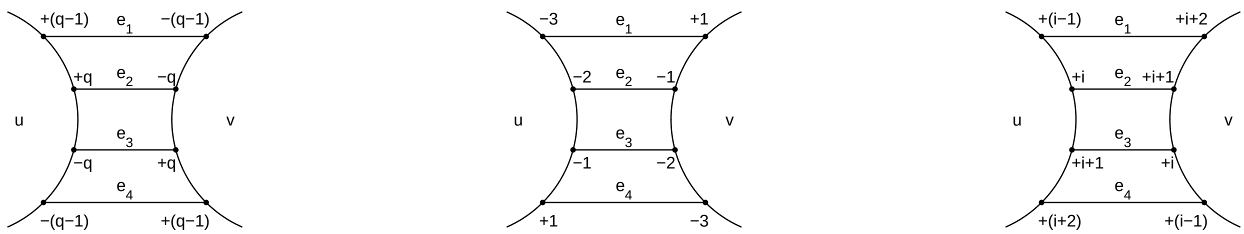


Figure 10:

Suppose a length two cycle $C = \{e_1, e_2\}$ bounds a disk-face in Γ_P, where $\partial e_1 = x_1 \cup y_1$ with x_1 labeled (u, i_1) and y_1 labeled (v, j_1), and $\partial e_2 = x_2 \cup y_2$ with x_2 labeled (v, j_2) and y_2 labeled (u, i_2). See Figure 10. C is said to be a virtual S-cycle if $g(x_1)i_1 = g(x_2)j_2$ and $g(y_2)i_2 = g(y_1)j_1$. In this case, $\{i_1, j_1\}$ is called the label pair of C. Furthermore if $i_1 \neq j_1$, then C is called an S-cycle.



(a) (b) (c)

Figure 11

Lemma 4.1. A virtual S-cycle is either an S-cycle, or its label pair is one of $\{1, 1\}$ and $\{q, q\}$.

Proof Let $\{e_1, e_2\}$ be an S-cycle defined as above. If $i_1 \neq i_2$, then it is an S-cycle. If $i_1 = i_2$, then $i_1 = j_1 = i_2 = j_2$. Hence either $i_1 = 1$ or $i_1 = q$. □

A set of four adjacent parallel edges, say $\{e_1, e_2, e_3, e_4\}$, in Γ_P is called a virtual extended S-cycle if $\{e_2, e_3\}$ is an S-cycle.

A virtual extended S-cycle $\{e_1, e_2, e_3, e_4\}$ is called an extended S-cycle if $\{e_2, e_3\}$ is not an S-cycle labeled $\{1, 2\}$ or $\{q, q-1\}$.

For examples, in Figure 11(a), $\{e_2, e_3\}$ is a virtual S-cycle rather than an S-cycle, and $\{e_1, e_2, e_3, e_4\}$ is a virtual extended S-cycle rather than an extended S-cycle; in Figure 11(b), $\{e_2, e_3\}$ is an S-cycle, but $\{e_1, e_2, e_3, e_4\}$ is a virtual extended S-cycle rather than an extended S-cycle; in Figure 11(c), $\{e_1, e_2, e_3, e_4\}$ is an extended S-cycle.

Lemma 4.2. (1) Γ_P can not contain two S-cycles with distinct label pairs.

(2) Γ_P contains no extended S-cycles.

Proof The proof follows from Lemma 2.2 and Lemma 2.3 of [W]. □

5 Proof of Theorem 1

In this section, we assume $\Delta(\alpha,\beta) \geq 6$ and the endpoints of edges Γ_P are with Type B labels.

Lemma 5.1. There are not two edges which are parallel in both Γ_P and Γ_Q.

Proof The proof follows from Lemma 2.1 of [SW]. □

Lemma 5.2. Γ_P can not have $2q$ parallel edges.

Proof Suppose $S = \{e_1, e_2, \cdots, e_{2q}\}$ is a collection of $2q$ parallel edges joining $\partial_u P$ and $\partial_v P$ in Γ_P, where $\partial e_i = x_i \cup y_i$.

Let $x \in \{x_1, x_2, \cdots, x_{2q}, y_1, y_2, \cdots, y_{2q}\}$, give new labels on x as follows:

(1) label x with i if x is labeled $+i$.

(2) label x with $2q+1-i$ if x is labeled $-i$.

These labels of S give a permutation π of $\{1, 2, \cdots, 2q\}$ defined $\pi(a) = b$ if (a, b) is a label pair of an edge in S. One can see that $\pi(a) = -a + s(mod2q)$, where s is a constant. It follows that $\pi^2(a) = a$. This means if there is an edge e_i with label pair (a, b), then there is a dual edge in S with label pair with (b, a). By Lemma 3.3, $a \neq b$. Then S can be divided into q pairs, each of them consists a pair edges of e_k and e'_k in S such that they have the same label pair, that is they form a length 2 cycle in Γ_Q. Suppose e_{k_0} and e'_{k_0} is a pairs such that they form an innermost length 2 cycle in Γ_Q. Then e_{k_0} and e'_{k_0} are parallel in both Γ_P and Γ_Q, contradicting Lemma 5.1. □

Lemma 5.3. Let Γ be a graph embedded in a 2-sphere with $V(V \geq 3)$ vertices and E edges, if Γ contains no 1-sided disk faces and no 2-sided disk faces, then $E \leq 3V - 6$.

Proof Suppose Γ contains F faces, and all of them have at least 3 sides, then $2E \geq 3F$. Hence $V - E + (2/3)E \geq 2$ and $E \leq 3V - 6$. □

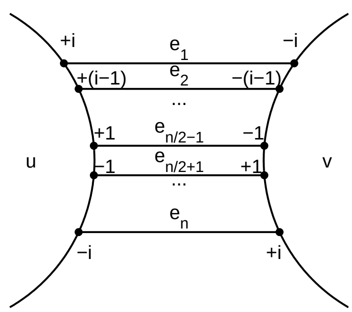
Lemma 5.4. $p \geq 5$.

Proof Suppose, otherwise, that $p \leq 4$. Let $\bar{\Gamma}_P$ be a reduced graph of Γ_P. Then $\bar{\Gamma}_P$ has no 1-sided and no 2-sided disk-faces. Since M is simple, $p > 2$. By Lemma 5.3, there are at most 6 edges in $\bar{\Gamma}_P$. Hence there is at least one vertex of $\bar{\Gamma}_P$ which has valency at most 3. Since $\Delta \geq 6$, Γ_P contains $2q$ parallel edges, contradicting Lemma 5.2. $\square$

An i-collection is a collection $S = \{e_1, e_2, \cdots, e_n\}$ of adjacent parallel edges in Γ_P such that each of e_1 and e_n has $+i$ as a signed label.

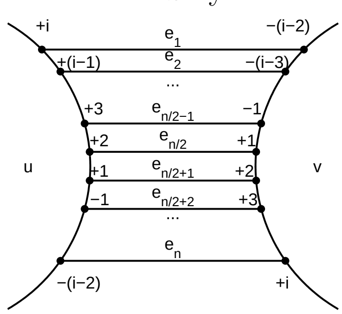
Lemma 5.5. Suppose S is an i-collection, then the signed labels of the endpoints of S must appear as one of the following six types.

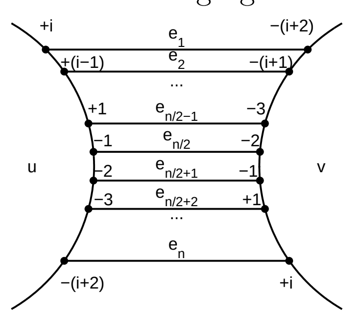
Type I: Each edge in S has the same labels with opposite signs. In this case, S contains a virtual S-cycle labeled $\{1, 1\}$ but no S-cycle. See the following figure.



Type I

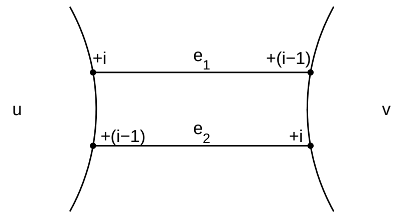
Type II: S contains an S-cycle labeled $\{1, 2\}$. See the following figures.





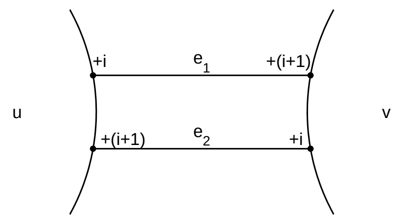
Type II(a) Type II(b)

Type III: $n = 2$, and $\{e_1, e_2\}$ is an S-cycle labeled $\{i, i-1\}$, where $i > 1$. See the following figure.



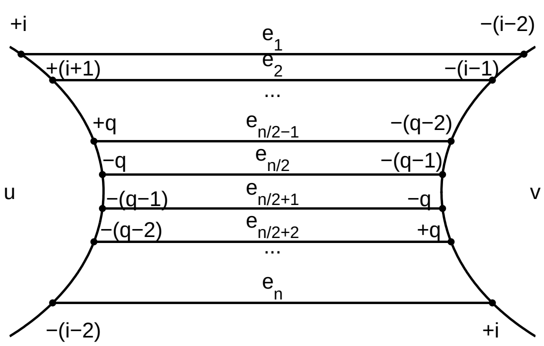
Type III

Type IV: $n = 2$, and $\{e_1, e_2\}$ is an S-cycle labeled $\{i, i+1\}$, where $i < q$. See the following figure.

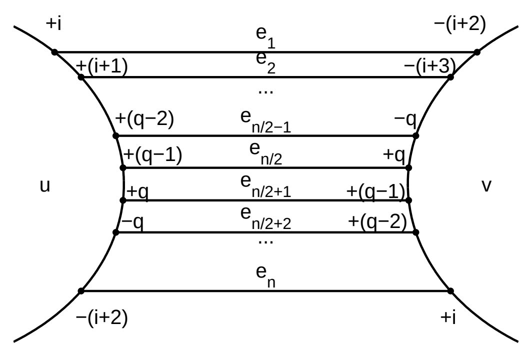


Type IV

Type V: S contains an S-cycle labeled $\{q, q-1\}$. See the following figures.

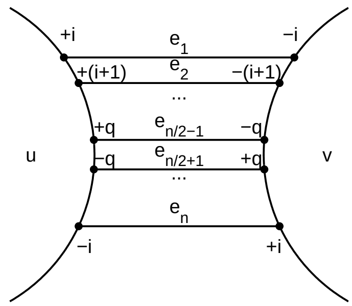


Type V(a)



Type V(b)

Type VI: Each edge in S has the same labels with opposite signs. In this case, S contains a virtual S-cycle labeled $\{q, q\}$ but no S-cycle. See the following figure.



Type VI

Proof Assume that $\partial e_k = x_k \cup y_k$ such that $x_k \in \partial_u P$ and $y_k \in \partial_v P$, and x_1 is labeled with $+i$. Since S is an i-collection, by definition, one of x_n and y_n is labeled with $+i$. If x_n is labeled with $+i$, then $n \geq 2q$, contradicting lemma 5.2. Hence y_n is labeled with $+i$. By remark $(*)$ the signed labels $\{1, 2, \cdots, q, -q, \cdots\}$ appear in the same direction in Γ_P. Hence the signed labels of x_{1+k} is the same with the one of y_{n-k} for all $k = 0, 1, \cdots, n$. It follows that n is even; otherwise, $x_{(1+n)/2} = y_{(1+n)/2}$, contradicting Lemma 3.3.

As signed labels, we assume that $-1 < +1$ and $+q < -q$.

Case 1. The signed label of x_2 is smaller than the one of x_1.

Case 1.1 $n = 2$.

Now S is a virtual S-cycle. If x_1 and x_2 are labeled with $+1$ and -1, then S is of type I. If x_1 and x_2 are with $+i$ and $+(i-1)$ for some $2 \leq i \leq q$, then S is of type III.

Case 1.2 $n \geq 4$.

Now $\{e_{n/2-1}, e_{n/2}, e_{n/2+1}, e_{n/2+2}\}$ is a virtual extended S-cycle. By Lemma 4.2(2), it is not an extended S-cycle. Hence $\{e_{n/2}, e_{n/2+1}\}$ is labeled with one of $(1,1)$,$(1,2)$,$(q, q-1)$,(q, q). Since the signed label of x_2 is smaller than the one of x_1, S contains at least $2q$ edges when $\{e_{n/2}, e_{n/2+1}\}$ is labeled with one of $(q, q-1)$ and (q, q). Hence S is one of type I and type II.

Case 2 the signed label of x_2 is bigger than the one of x_1.

By the same argument as above, S is one of type IV, type V and type VI. □

The proof of Theorem 1

For each $1 \leq i \leq q$, let B_P^{+i} be a subgraph of Γ_P consisting all the vertices of Γ_P and all the edges e such that one endpoint of e is labeled with $+i$.

Since $\Delta(\alpha, \beta) \geq 6$, by Lemma 3.3, there are at least $3p$ edges in B_P^{+i}. By Lemma 5.3, B_P^{+i} contains at least one 2-sided face. Hence there is at least one i-collections in Γ_P for each i.

Claim 1 For each $1 \leq s \leq q-1$, Γ_P contains no an s-collection of type I and an $(s+1)$-collection of type VI simultaneously.

Proof Suppose, otherwise, that S_1 is an s-collection of type I and S_2 is an $(s+1)$-collection of type VI. By the definitions of type I and type VI, for each $i \leq s$, there are two edges in S_1 with both two endpoints incident to $\partial_i Q$; for each $j \geq s+1$, there are two edges in S_2 with both two endpoints incident to $\partial_j Q$. Hence each edge in $S_1 \cup S_2$ is a length 1 cycle in Γ_Q. This means that Γ_Q contains a 1-sided disk-face, contradicting Lemma 2.2. $\square$(Claim 1)

Claim 2 For each $1 \leq s \leq q-1$, Γ_P contains no an s-collection of type I (resp. II) and an $(s+1)$-collection of type V(resp. VI) simultaneously.

Proof Suppose, otherwise, that S_1 is an s-collection of type I and S_2 is an $(s+1)$-collection of type V. By the definition of type V, all the vertices of $\partial_i Q (i \geq s+1)$ be connected by the edges in S_2. By the definition of type I, each edge in S_1 is a length 1 cycle which bounds two disks in $\hat{Q}$, say D_1 and D_2. We may assume that D_1 is disjoint from $\partial_i Q$ for each $i \geq s+1$. Hence Γ_Q contains a 1-sided disk-face, a contradiction.

Similarly, one can prove that, Γ_P contains no an s-collection of type II and an $(s+1)$-collection of type VI simultaneously. $\square$(Claim 2)

Claim 3 For each $1 \leq s \leq q$, Γ_P contains neither s-collections of type II nor s-collections of type V.

Proof Suppose, otherwise, that there is an s-collection of type II for some $1 \leq s \leq q$. Then Γ_P contains an S-cycle labeled $\{1, 2\}$. By Lemma 4.3, each i-collection is one of type

I, type II and type VI for each $i > 2$.

By Claim 1 and Claim 2, the q-collection is one of type I and type II.

If a q-collection is of type I, then Γ_P contains $2q$ edges, contradicting Lemma 5.2.

Assume that there is a q-collection of type II. Then there are two edges connecting $\partial_1 Q$ to $\partial_2 Q$, and two edges connected $\partial_k Q$ to $\partial_{k+2} Q$ for all $1 \leq k \leq q-2$. Each pair of the two edges as above is a length 2 cycle in Γ_Q. Let c be an innermost one of all such cycles. Then the two edges in c are parallel in both Γ_P and Γ_Q, contradicting Lemma 5.1.

Similarly, Γ_P contains no s-collections of type V. □(Claim 3)

Claim 4 For each $1 \leq s \leq q$, Γ_P contains neither s-collections of III nor s-collections of type IV.

Proof Suppose, otherwise, that there is an s-collection S of type IV, then $1 \leq s \leq q-1$. Note that S is also an $(s+1)$-collection of type III. By Lemma 4.3, there is no i-collections of type III and VI for $i \neq s, s+1$. By Claim 3, each i-collection is either of type I or type VI for $i \neq s, s+1$. If $s = 1$, then S is also a 2-collection of type II, contradicting Claim 3. Hence $2 \leq s \leq q-1$.

Since a 1-collection of type VI contains $2q$ edges, contradicting Lemma 5.2. Hence all the 1-collections are of type I. By Claim 1, all the i-collections are of type I for $i < s$. By Lemma 5.2, all the q-collections are of type VI. By Claim 1, all the j-collections are of type VI for $j > s+1$.

Suppose S_1 is an $(s-1)$-collection of type I, and S_2 is an $(s+2)$-collection of type VI. By the definitions of type I and type VI, there is a length 1 cycle incident to $\partial_i Q$ for each $i \neq s, s+1$. Since two edges in S connect $\partial_s Q$ to $\partial_{s+1} Q$, So Γ_Q contains a 1-sided face in Γ_Q, it is a contradiction to Lemma 2.2. □(Claim 4)

By the Claim 3 and Claim 4. all the s-collections are of type I or type VI for $1 \leq s \leq q$. since all 1-collections are of type I, and all q-collections are of type VI. Then we can find some k such that there are a k-collection of type I and a $k+1$-collection of type VI, contradicting Claim 1. □ (Theorem 1)

References

[GL1] C. Gordon, J. Luecke, Reducible Manifolds and Dehn Surgery, Tolology, **35**(1996), 385–409.

[GL2] C. Gordon and J. Luecke, Knots are determined by their complements. Journal of American Mathematical Society, **2**(1989), 371–415.

[QW1] R. Qiu, S. Wang, Small knots and large handle additions, Communications in Analysis and Geometry, **13**(2005), 936-961.

[QW2] R. Qiu, S. Wang, Handle additions producing essential closed surfaces, Pacific Journal of Mathematics, to appear.

[SW] M. Scharlemann, Y. Wu, Hyperbolic manifolds and degenerating handle additions, Journal of Australian Mathematical Society, **Ser**A, **55**(1993), 72-89.

[W] Y. Wu, The reducibility of surgered 3-manifolds, Topology and its Applications, **43**(1992), 213-218.

Mingxing Zhang
Department of Applied Mathematica, Dalian University of Technology,
Dalian, China, 116022
Email: star.michael@263.net

Ruifeng Qiu
Department of Applied Mathematica, Dalian University of Technology,
Dalian, China, 116022
Email: qiurf@dlut.edu.cn

Yannan Li
Department of Applied Mathematica, Dalian University of Technology,
Dalian, China, 116022
Email: yn_lee79@yahoo.com.cn