SHORT CYCLES OF LOW WEIGHT IN NORMAL
PLANE MAPS WITH MINIMUM DEGREE 5

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Abstract

In this note, precise upper bounds are determined for the minimum degree-sum of the vertices of a 4-cycle and a 5-cycle in a plane triangulation with minimum degree 5: \( w(C_4) \leq 25 \) and \( w(C_5) \leq 30 \). These hold because a normal plane map with minimum degree 5 must contain a 4-star with \( w(K_{1,4}) \leq 30 \). These results answer a question posed by Kotzig in 1979 and recent questions of Jendrol’ and Madaras.

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$w(C_5) \leq 45$ and $w(K_{1,4}) \leq 39$ for each $T_5$ and $w(K_{1,3}) \leq 23$, which bound is best possible, and $w(K_{1,4}) \leq 45$ for each $M_5$.

Our main result is:

**Theorem 1.** Each normal plane map with minimum degree 5 contains a 4-star with weight at most 30 with a 5-vertex as its centre.

This clearly implies:

**Corollary 2.** Each plane triangulation with minimum degree 5 contains a 4-cycle with weight at most 25 and a 5-cycle with weight at most 30.

The bounds in Theorem 1 and Corollary 2 are all precise, as the following examples show. Take any polyhedron in which every vertex is of type 5.6$^2$ or 6$^3$, such as the Archimedean solid in which every vertex is incident with a 5-face and two 6-faces. Truncate all the vertices to obtain a graph in which every vertex has type 3.10.12 or 3.12$^2$. Cap each 10-face and 12-face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with minimum degree 5 in which the neighbours of every 5-vertex $v$ have degrees (in cyclic order round $v$) (5, 5, 10, 5, 12) or (5, 5, 12, 5, 12). This graph clearly has $w(C_4) = 25$ and $w(C_5) = w(K_{1,4}) = 30$.

It follows that our results above completely solve the problems raised by Kotzig [5] and Jendrol’ and Madaras [3]. In the proof below, we use some ideas from our unpublished manuscript [2].

We shall use the following terminology. The number of edges incident with a vertex $v$ or $r(f)$ respectively, and $v_1, \ldots, v_{d(v)}$ denote the neighbours of $v$, in cyclic order round $v$. If $d(v_i) = 5$ then $v_i$ is a **strong**, **semiweak** or **weak** neighbour of $v$ according as none, one or both of $v_{i-1}, v_{i+1}$ have degree 5, and $v_i$ is **twice weak** if $d(v_j) = 5$ whenever $|j-i| \leq 2$ (modulo $d(v)$). A **k-vertex** is a vertex $v$ with $d(v) = k$, and a **>k-vertex** has $d(v) > k$, etc.

**Proof of Theorem 1.** It suffices to prove the theorem for triangulations, since adding an extra edge to a normal plane map with minimum degree 5 cannot create a new 4-star with a 5-vertex as its centre, nor can it reduce the weight of any existing 4-star. So suppose that $G = (V, E, F)$ is a triangulation that is a counterexample to Theorem 1. Since $G$ is a triangulation, $2|E| = 3|F|$, and so Euler’s formula $|V| - |E| + |F| = 2$ implies

$$
\sum_{v \in V} (d(v) - 6) = -12.
$$
Assign a charge $\mu(v) = d(v) - 6$ to each vertex $v \in V$, so that only 5-vertices have negative charge. Using the properties of $G$ as a counterexample, we define a local redistribution of charges, preserving their sum, such that the new charge $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is, by (1), equal to $-12$. The technique of discharging is often used in solving structural and colouring problems on plane graphs.

Our discharging rules are as follows.

**Rule 1.** (a) Each vertex $v$ of degree 7 sends $\frac{1}{7}$ to each strong neighbour and $\frac{1}{6}$ to each semiweak neighbour.

(b) Each vertex $v$ with degree 8, 9 or $\geq 12$ first gives a “basic” contribution of $\frac{d(v) - 6}{d(v)}$ to each neighbouring vertex $v_i$. Then each neighbour $v_i$ with $d(v_i) > 5$ shares the charge just received equally between $v_i - 1$ and $v_i + 1$.

(c) Each 10-vertex or 11-vertex $v$ first gives a “basic” $\frac{2}{5}$ to each neighbour. Then, whenever $d(v_i) > 5$, $v_i$ transfers $\frac{1}{10}$ of $v$’s donation to each 5-vertex in $\{v_i - 2, v_i - 1, v_i + 1, v_i + 2\}$.

**Rule 2.** If $d(v) = 11$ then $v$ gives a “supplementary” $\frac{1}{10}$ to each twice weak neighbour.

**Rule 3.** If $v$ is 5-vertex adjacent to an 11-vertex $w$, say $w = v_5$, and if $d(v_1) = d(v_4) = 5$, then $v$ gives back to $v_5$ the following:

(a) $\frac{1}{7}$ if both $v_2$ and $v_3$ have degree $\geq 9$;

(b) $\frac{1}{7}$ if at least one of $v_2, v_3$ has degree exactly 8.

We must prove that $\mu'(v) \geq 0$ for each vertex $v$. If $d(v) \notin \{5, 7, 11\}$, then, by Rule 1 (b) and (c), $v$ distributes its own original charge of $\mu(v) = d(v) - 6$ to its neighbours in equal shares, and possibly participates in transferring the others’ charges, so that $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{d(v) - 6}{d(v)} = 0$. We deal with the remaining values of $d(v)$ in three cases.

**Case 1.** $d(v) = 11$. Then $\mu(v) = d(v) - 6 = 5$. If $v$ has a neighbour $v_i$ with $d(v_i) \geq 6$, then none of $v_i - 2, \ldots, v_i + 2$ is twice weak and so none of them receives a supplementary $\frac{1}{10}$ from $v$ by Rule 2. Thus $\mu'(v) \geq 5 - 11 \times \frac{2}{5} - 6 \times \frac{1}{10} = 0$. So we may assume that all neighbours of $v$ have degree 5.

Each edge $v_i v_{i+1}$ lies in two triangles, say $v_i v_{i+1} v$ and $v_i v_{i+1} w_i$. If $d(w_i) = 8$ for some $i$, then $v$ receives $\frac{1}{7}$ by Rule 3(b) from each of $v_i$ and
\(v_{i+1}\), so that \(\mu'(v) \geq 5 + 2 \times \frac{1}{3} - 11 \times \frac{1}{2} = 0\). So we may assume that 
\(d(w_i) \neq 8\), for each \(i\).

If \(d(w_{i-1}) \geq 9\) and \(d(w_i) \geq 9\) for some \(i\), then \(v_i\) gives back \(\frac{1}{2}\) to \(v\) by Rule 3(a), and we are done. Also, it is impossible that \(d(w_{i-1}) \leq 7\) and 
\(d(w_i) \leq 7\) for any \(i\), since by hypothesis there is no 4-star with weight \(\leq 30\) centered at \(v_i\). Therefore, for each \(i\), one of \(d(w_{i-1})\) and \(d(w_i)\) is at most 7 and the other is at least 9. But this cannot hold for all \(i\) modulo 11, since 11 is odd.

**Case 2.** \(d(v) = 7\). Then \(\mu(v) = d(v) - 6 = 1\). By Rule 1(a), no weak neighbour receives anything from \(v\), and so there are at most four receivers. If there are exactly four, then at least two are semiweak and so receive \(\frac{1}{6}\) each, with a total expenditure by \(v\) of at most \(2 \times \frac{1}{6} + 2 \times \frac{1}{2} = 1\). Otherwise, \(v\) gives at most \(3 \times \frac{1}{3} = 1\).

\[
\begin{array}{cccc}
\text{neighbour:} & \text{strong} & \text{semiweak} & \text{weak} \\
7: & 1/3 & 1/6 & 0 \\
8: & 1/2 & 3/8 & 1/4 \\
9: & 2/3 & 1/2 & 1/3 \\
10: & \geq 3/5 & \geq 1/2 & \geq 2/5 \\
11: & \geq 3/5 & \geq 1/2 & 1/2 \\
\geq 12: & \geq 1 & \geq 3/4 & \geq 1/2 \\
\end{array}
\]

Table 1. Donations to 5-vertices by Rules 1 and 2

**Case 3.** \(d(v) = 5\). Then \(\mu(v) = d(v) - 6 = -1\). The amounts of charge received by \(v\) from its neighbours by Rules 1 and 2 are summarized in Table 1. However, \(v\) may give back charge to some 11-vertices by Rule 3.

Suppose Rule 3(a) applies to \(v\), so that \(v\)'s neighbours \(v_1, \ldots, v_5\) have degrees \((5, \geq 9, \geq 9, 5, 11)\). Then \(v\) is a semiweak neighbour of each of \(v_2\) and \(v_3\), so that it receives at least \(\frac{1}{2}\) from each of them by Table 1, and gives nothing back to either of them by Rule 3. It also receives at least \(\frac{1}{2}\) from \(v_5\) by Table 1, and gives back exactly \(\frac{1}{2}\) to \(v_5\) by Rule 3(a). We deduce that \(\mu'(v) \geq 0\).

From now on, we may assume that Rule 3(a) does not apply to \(v\). Suppose Rule 3(b) applies. Because \(v\) is not the centre of a 4-star with weight \(\leq 30\), \(v\)'s neighbours have degrees \((5, 8, \geq 8, 5, 11)\). Thus \(v\) is a semiweak neighbour of \(v_2\) and \(v_3\) and so it receives at least \(\frac{3}{8}\) from each of them by Table 1, and gives nothing back. It also receives at least \(\frac{1}{2}\) from \(v_5\) by Table 1, and gives \(\frac{1}{2}\) back. Thus \(\mu'(v) \geq 0\).
Thus we have proved that Rule 3 does not apply to \( v \) at all, and the amount that \( v \) receives from its neighbours is at least that given in Table 1. Because of the absence of 4-stars with weight \( \leq 30 \), the degree-sequence of \( v \)'s neighbours, in nondecreasing order, must be one of the following.

(5, 5, 5, \( \geq 11 \), \( \geq 11 \)): Then each \( \geq 11 \)-vertex gives \( \geq \frac{1}{2} \) to \( v \) by Table 1.

(5, 5, 6, \( \geq 10 \), \( \geq 10 \)): If each of the two \( \geq 10 \)-neighbours gives \( \geq \frac{1}{2} \) to \( v \), we are done.

Suppose there is a 10-vertex, say \( v_1 \), giving \( \frac{7}{6} \) to \( v \). Then \( v \) must be a twice weak neighbour of \( v_1 \) by Rule 1(c). W.l.o.g., suppose that \( d(v_2) = d(v_3) = 5 \) and \( d(v_4) = 6 \). If \( d(v_4) \geq 12 \) then \( v_4 \) gives \( \geq \frac{2}{5} \) to \( v \) by Table 1, so that \( \mu'(v) \geq -1 + \frac{2}{5} + \frac{2}{5} > 0 \). So we may assume \( 10 \leq d(v_4) < 11 \); note that \( v \) is not a twice weak neighbour of \( v_4 \). Let \( u \) be the vertex (other than \( v \)) adjacent to \( v_1 \) and \( v_5 \). Since \( v_5 \) has two 5-neighbours other than \( u \) (because \( v \) is a twice weak neighbour of \( v_1 \)), and also has a 10-neighbour \( v_1 \), it follows that \( d(u) > 5 \). Then Rule 1(c) ensures that \( v \) receives \( \frac{1}{10} \) from \( v_4 \) via each of \( v_3 \) and \( u \), so that \( \mu'(v) \geq -1 + \frac{2}{5} + \frac{2}{5} + \frac{1}{10} = 0 \).

(5, 5, 7, \( \geq 9 \), \( \geq 9 \)): If \( v \) is weak for neither of the \( \geq 9 \)-neighbours then each of them gives \( \geq \frac{1}{2} \), and we are done by Table 1. Otherwise, \( v \) is weak for a \( \geq 9 \)-neighbour, giving \( \geq \frac{1}{3} \), and semiweak for the other two neighbours of degree \( 7 \) and \( \geq 9 \), giving \( \geq \frac{1}{6} + \frac{1}{3} \) in total.

(5, 5, \( \geq 8 \), \( \geq 8 \), \( \geq 8 \)): If \( v \) is weak for none of the three \( \geq 8 \)-neighbours, then \( v \) receives \( \geq 3 \times \frac{2}{5} > 1 \) in total. Otherwise, \( v \) is weak for one of them and semiweak for the other two, so that receives \( \geq \frac{1}{3} + 2 \times \frac{3}{8} = 1 \) in total.

(5, 6, 6, \( \geq 9 \), \( \geq 9 \)): Each \( \geq 9 \)-neighbour gives \( \geq \frac{1}{5} \).

(5, 6, \( \geq 7 \), \( \geq 8 \), \( \geq 8 \)): For each of the three \( \geq 7 \)-neighbours, \( v \) is semiweak or strong; for at least one of them, \( v \) is strong. By Table 1, \( v \) thus receives either \( \geq \frac{1}{7} + \frac{2}{5} + \frac{1}{3} > 1 \) or \( \geq \frac{3}{5} + \frac{3}{8} + \frac{1}{7} > 1 \).

(5, 7, \( \geq 7 \), \( \geq 7 \), \( \geq 7 \)): For at least two \( \geq 7 \)-neighbours, \( v \) is strong; for the others, semiweak. Thus, \( \mu'(v) \geq -1 + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 0 \).

\((\geq 6, \geq 6, \geq 6, \geq 8, \geq 8)\): \( \mu'(v) \geq -1 + 2 \times \frac{1}{4} = 0 \).

\((\geq 6, \geq 6, \geq 7, \geq 7, \geq 7)\): \( \mu'(v) \geq -1 + 3 \times \frac{3}{8} = 0 \).

Thus we have proved \( \mu'(v) \geq 0 \) for every \( v \in V \) and \( f \in F \), which contradicts (1) and completes the proof of Theorem 1.

References


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