INEQUALITIES OF NORDHAUS–GADDUM TYPE FOR CONNECTED DOMINATION

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Abstract. A set S of vertices of a graph G is a *connected dominating set* if every vertex not in S is adjacent to some vertex in S and the subgraph induced by S is connected. The *connected domination number* $\gamma_c(G)$ is the minimum size of a connected dominating set of G. In this paper we prove that $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 4$ for every n-vertex graph G such that G and $\overline{G}$ have diameter 2 and show that the bound is sharp for each value of the right side. Also, $\gamma_c(G) + \gamma_c(\overline{G}) \le \frac{3n}{4}$ if G and $\overline{G}$ are connected, have minimum degree at least 3 and $n \ge 14$. We also prove that $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 2$ if $\gamma_c(G), \gamma_c(\overline{G}) \ge 4$ and show that the bound is sharp when $\min\{\delta(G), \delta(\overline{G})\} = 6$.

Keywords: connected dominating set, connected domination number, Nordhaus-Gaddum inequalities.
MSC 2000: 05C69

1. Introduction

A large part of extremal graph theory studies the extremal values of graph parameters on families of graphs. Results of *Nordhaus–Gaddum type* study the extremal values of the sum (or product) of a parameter on a graph and its complement, following the classic paper of Nordhaus and Gaddum [4] solving these problems for the chromatic number on n-vertex graphs. In this paper we consider such problems for the parameter measuring the minimum size of a connected dominating set.

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For domination problems, multiple edges are irrelevant, so we use the model of "graph" that forbids multiple edges. We use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G)\colon uv \in E(G)\}$ and the *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$. The *open neighborhood* $N(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N(v)$, and the *closed neighborhood* $N[S]$ of S is the set $N(S) \cup S$. The minimum and maximum vertex degrees in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. Given graphs G and H, the *cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and edge set defined by making (u, v) and (u', v') adjacent if and only if either (1) $u = u'$ and $vv' \in E(H)$ or (2) $v = v'$ and $uu' \in E(G)$.

For a graph G, a set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V(G)$, and S is a *connected dominating set* if also the subgraph induced by S, denoted $G[S]$, is connected. The minimum size of a dominating set and a connected dominating set are the *domination number* $\gamma(G)$ and the *connected domination number* $\gamma_c(G)$. Hedetniemi and Laskar [2] observed that $\gamma_c(G) = n - \ell(G)$ when G is a connected n-vertex graph and $\ell(G)$ is the maximum number of leaves in a spanning tree of G.

Our purpose in this paper is to establish a sharp upper bound on the sum $\gamma_c(G) + \gamma_c(\overline{G})$ in terms of the number of vertices and the minimum degrees of G and $\overline{G}$. We prove that if G is an n-vertex graph such that $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, then $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 4$, and also $\gamma_c(G) + \gamma_c(\overline{G}) \le \frac{3n}{4}$ if $n \ge 14$, G and $\overline{G}$ are connected, and $\min\{\delta(G), \delta(\overline{G})\} \ge 3$. We also prove that $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 2$ when $\gamma_c(G), \gamma_c(\overline{G}) \ge 4$.

Note first that the case $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$ is the interesting case; it is forced when $\gamma_c(G), \gamma_c(\overline{G}) \ge 3$, since $\operatorname{diam} G \ge 3$ if and only if $\gamma_c(\overline{G}) \le 2$. When the diameter is larger the sum can be larger. When G is connected, it has a spanning tree with at least two leaves, so $\gamma_c(G) \le n - 2$. Hence $\gamma_c(G) + \gamma_c(\overline{G}) \le n$ when G is connected with diameter at least 3, and equality holds for the path P_n and the cycle C_n. If $\operatorname{diam} G \ge 3$ and $\delta(G) \ge 3$, then $\gamma_c(G) + \gamma_c(\overline{G}) \le 3n/4$ by Theorem A below. The case $k = 3$ of Theorem A was proved independently by many researchers.

Theorem A. [1, 3] If G is a connected n-vertex graph with minimum degree k, where $k \le 5$, then $\gamma_c(G) \le \dfrac{3n}{k+1} - c_k$, where c_k is a small constant (in particular $c_3 = 2$).

2. A bound on $\gamma_c(G) + \gamma_c(\overline{G})$

In this section we prove that if G is an n-vertex graph such that $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, then $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 4$, and also $\gamma_c(G) + \gamma_c(\overline{G}) \le \frac{3n}{4}$ if $n \ge 14$, G and $\overline{G}$ are connected, and $\min\{\delta(G), \delta(\overline{G})\} \ge 3$. We also prove that $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 2$ when $\gamma_c(G), \gamma_c(\overline{G}) \ge 4$. We start by relating $\gamma_c(G) + \gamma_c(\overline{G})$ to the minimum degrees.

Theorem 1. If G is an n-vertex graph with $n \ge 4$ and $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, then $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 4$. Furthermore, this bound is sharp for each value of the right side.

Proof. Since $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, both G and $\overline{G}$ are connected. Therefore neither has a vertex of degree $n - 1$ or 1 (the neighbor of a vertex of degree 1 would have degree $n - 1$). Hence $\delta(G), \delta(\overline{G}) \ge 2$ and, as noted earlier, $\gamma_c(G), \gamma_c(\overline{G}) \ge 3$.

Let x be a vertex of minimum degree in G, and let $X = V(G) \setminus N[x]$. Since $\gamma_c(G) \geq 3$, we have $X \neq \emptyset$. Since $\operatorname{diam} G = 2$, the set $N(x)$ dominates X. Let S_1 be a largest subset of $N(x)$ that does not dominate X. Let $S_2 = N(x) - S_1$, and let $T_1 = X \cap N(S_1)$ and $T_2 = X - T_1$ (see Figure 1). Since $\gamma_c(G) \geq 3$, both S_1 and T_1 are nonempty, and since S_1 does not dominate X both S_2 and T_2 are nonempty. By the maximality of S_1, we have $T_2 \subseteq N(y)$ for each $y \in S_2$. Now $S_1 \cup \{x, y\}$ is a connected dominating set of G (for any $y \in S_2$), so $\gamma_c(G) \leq 2 + |S_1|$.

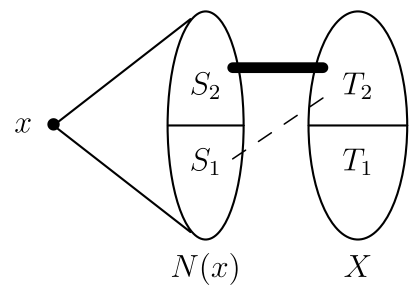

FIGURE 1. The sets S_1, S_2, T_1 and T_2

On the other hand, for any vertex $y \in S_2$, there exists a vertex $y' \in X$ such that $yy' \in E(\overline{G})$, since $\gamma_c(G) \geq 3$. Now for each $z \in T_2$, $\{x, z\} \cup \{y' \colon y \in S_2\}$ is a connected dominating set of $\overline{G}$. This implies that $\{x, z\} \cup \{y' \colon y \in S_2\}$ is a connected dominating set for $\overline{G}$, so $\gamma_c(\overline{G}) \leq 2 + |S_2|$.

Thus $\gamma_c(G) + \gamma_c(\overline{G}) \leq |N(x)| + 4 = \delta(G) + 4$. By symmetry in G and $\overline{G}$, we have $\gamma_c(G) + \gamma_c(\overline{G}) \leq \min\{\delta(G), \delta(\overline{G})\} + 4$. This completes the proof of the bound.

To prove sharpness, we construct for each positive integer r a connected graph G_r with $\delta(G_r) = r$, $\delta(\overline{G_r}) = r^2 - r + 1$, and $\gamma_c(G_r) + \gamma_c(\overline{G_r}) = r + 4$. Form the graph G_r as follows. Let $H_1 = H_2 = K_r$, with $V(H_2) = \{v_1, ..., v_r\}$. To the cartesian product $H_1 \square H_2$ add $r + 1$ new vertices, say $y, x_1, \ldots, x_r$ and add edges joining y to all x_i and edges joining x_i to all vertices of $H_1 \square H_2$ having second coordinate v_i, for $1 \leq i \leq r$. The resulting graph is G_r. Figure 2 shows G_2. Note that $\operatorname{diam}(G_r) = \operatorname{diam}(\overline{G_r}) = 2$ and $\min\{\delta(G_r), \delta(\overline{G_r})\} = r$; in particular, $\deg_G(y) = r$, and the degree in $\overline{G_r}$ of vertices in $H_1 \square H_2$ is $r_r^2 + 1$.

We claim that $\gamma_c(G_r) = r + 1$ and $\gamma_c(\overline{G_r}) = 3$, which yields the desired sum. Since $\operatorname{diam}(G) = 2$, $\gamma_c(\overline{G}) \geq 3$. If $v \in N_{G_r}(x_1) - \{y\}$ and $z \in N_{G_r}(x_2) - \{y\}$, then $\{y, v, z\}$ is a connected dominating set of $\overline{G}$; thus $\gamma_c(\overline{G_r}) = 3$. To show that $\gamma_c(G_r) = r + 1$, first we note that $\{v, x_1, \ldots, x_r\}$ is a connected dominating set, so $\gamma_c(G_r) \leq r + 1$. Now let S be a connected dominating set, and let $T = N[x_i] - \{y\}$. If S does not intersect T, which induces a copy of H_1 along with x_i, then dominating T requires S to have a vertex in each copy of H_2, plus y or x_i. This requires $r + 1$ vertices. Thus $|S| \geq r + 1$ unless S intersects each of r disjoint sets whose union does not dominate y. Thus again $|S| \geq r + 1$, and hence $\gamma_c(G) = r + 1$. This completes the proof. □

Since $d_G(v) + d_{\overline{G}}(v) = n - 1$ for any $v \in V(G)$, we have $\min\{\delta(G), \delta(\overline{G})\} \leq \lfloor (n-1)/2 \rfloor$, and Theorem 1 has the following immediate consequence.

Corollary 2. If G is an n-vertex graph with $n \geq 4$ and $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq \lfloor \frac{n+7}{2} \rfloor$. The bound is sharp for $n = 5$.

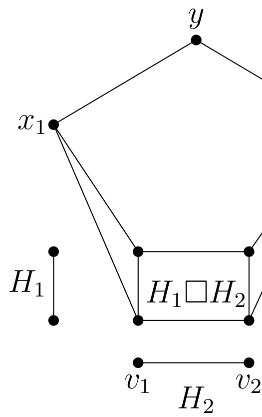

FIGURE 2. The graph G_2 with $\gamma_c(G_2) + \gamma_c(\overline{G_2}) = \delta(G_2) + 4$

For a cycle of order 5 this bound is attained. Since we take the minimum of $\delta(G)$ and $\delta(\overline{G})$, equality in Corollary 2 requires G to be $(n-1)/2$-regular.

Corollary 3. If G is an n-vertex graph, with $n \geq 14$, such that G and $\overline{G}$ are connected and have minimum degree at least 3, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq \frac{3n}{4}$. Furthermore this bound is sharp when 4 divides n.

Proof. If $\gamma_c(G) \leq 2$ or $\gamma_c(\overline{G}) \leq 2$, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq \frac{3n}{4}$ by Theorem A. If $\gamma_c(G), \gamma_c(\overline{G}) \geq 3$ and $n \geq 14$, then Corollary 2 completes the proof of the bound.

To prove sharpness when 4 divides n we use the ring-of-cliques which first was introduced by Sampathkumar et al. in [5]. Form a connected graph with minimum degree 3 by first putting r copies of K_4 in a ring and then deleting one edge $x_i y_i$ from the i th complete graph and replacing these edges with $y_i x_{i+1}$ for $1 \leq i \leq r$ (see Figure 3). Since no spanning tree of this graph has more than $n/4 + 2$ leaves, the equality holds in the upper bound.

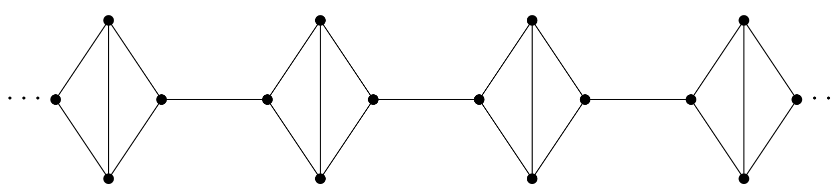

FIGURE 3. Sharpness example for Corollary 3

□

By a closer look at the proof of Theorem 1, we can improve the upper bound when both G and $\overline{G}$ have larger connected domination number. In particular, our next theorem does not contradict the sharpness example of Theorem 1, because $\gamma_c(\overline{G}) = 3$ in that construction.

Theorem 4. If G is an n-vertex graph with $n \geq 4$ and $\gamma_c(G), \gamma_c(\overline{G}) \geq 4$, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq \min\{\delta(G), \delta(\overline{G})\} + 2$. The bound is sharp when $\min\{\delta(G), \delta(\overline{G})\} = 6$.

Proof. As observed earlier, the hypothesis requires $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, so the scenario of Theorem 1 applies. Define x, X, S_1, S_2, T_1, T_2 as in the proof of Theorem 1. Since $\gamma_c(G) \ge 4$, no two vertices in $N(x)$ can dominate X; call this fact $(*)$. By $(*)$, $|S_1| \ge 2$.

We show next that $S_2 \cup \{x\}$ is a connected dominating set of G. Otherwise, some vertex $z \in T_1$ is not dominated by S_2. For any $z' \in T_2$, observe that $\{x, z, z'\}$ is a connected dominating set of $\overline{G}$, contradicting $\gamma_c(\overline{G}) \ge 4$. Hence $S_2 \cup \{x\}$ is a connected dominating set, and $\gamma_c(G) \ge 4$ yields $|S_2| \ge 3$.

Since S_2 dominates X, we may let S_3 be a maximal subset of S_2 that does not dominate X. By $(*)$, $|S_3| \ge 2$. Choose a vertex $z \in X$ not dominated by S_3, and choose $z' \in T_2$. For any $y \in S_2 \setminus S_3$, there is some $y' \in X$ such that $yy' \in E(\overline{G})$. Now $\{x, z, z'\} \cup \{y' \colon\ y \in S_2 \setminus S_3\}$ is a connected dominating set of $\overline{G}$, which yields $\gamma_c(\overline{G}) \le |S_2 - S_3| + 3$.

The maximality of S_3 yields $\gamma_c(G) \le |S_3| + 2$. Thus

$$\gamma_c(G) + \gamma_c(\overline{G}) \le 5 + |S_3| + |S_2 - S_3| \le 5 + |S_2| \le \delta(G) + 3,$$

where in the final inequality we have used $|S_1| \ge 2$. By symmetry, we also have $\delta(\overline{G}) + 3$ as an upper bound. Thus $\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 3$.

To further improve the bound, suppose that equality holds. Equality then must hold throughout the display above. In particular, $|S_1| = 2$. Since $S_1 \cup \{x, y\}$ is a connected dominating set of G for any $y \in S_2$, we obtain $\gamma_c(G) = 4$. By symmetry, $\gamma_c(\overline{G}) = 4$. It follows that $\min\{\delta(G), \delta(\overline{G})\} = 5$. By symmetry, we may assume that $\delta(G) = 5$. Thus $|S_1| = 2$ and $|S_2| = 3$. Since S_1 was chosen to be a largest subset of $N(x)$ that does not dominate X, any three vertices of $N(x)$ dominate X, and by $(*)$ no two vertices of $N(x)$ dominate X.

If $N(x)$ has a vertex z with three nonneighbors in $N(x)$, then let z' be the remaining vertex in $N(x)$. Since $\{z, z'\}$ does not dominate X, we may choose $y \in X$ such that y is a common nonneighbor of z and z'. Now $\{x, y, z\}$ is a connected dominating set in $\overline{G}$, contradicting $\gamma_c(\overline{G}) \ge 4$.

Hence, the subgraph of G induced by $N(x)$ has minimum degree at least 2. Let P be a 3-vertex path in the subgraph of G induced by $N(x)$. Since each remaining vertex in $N(x)$ has at least two neighbors in $N(x)$, $V(P)$ dominates $N(x)$. Since any three vertices of $N(x)$ dominate $X \cup \{x\}$, $V(P)$ is a connected dominating set in G, contradicting $\gamma_c(G) \ge 4$.

We conclude that equality is impossible, so

$$\gamma_c(G) + \gamma_c(\overline{G}) \le \min\{\delta(G), \delta(\overline{G})\} + 2.$$

To prove sharpness, we construct a graph G with $\min\{\delta(G), \delta(\overline{G})\} = 6$ and $\gamma_c(G) = \gamma_c(\overline{G}) = 4$. Let $X = \{x, x_1, x_2, \ldots, x_6\}$ and let $Y = \{A_1, \ldots, A_{15}\}$ be the set consisting of all 4-subsets of $X \setminus \{x\}$. Without loss of generality we may assume $A_{13} = \{x_3, x_4, x_5, x_6\}, A_{14} = \{x_1, x_2, x_5, x_6\}$ and $A_{15} = \{x_1, x_2, x_3, x_4\}$. Let $r = 15\binom{12}{2} = 15 \cdot 66$. For $1 \le i \le 15$, let $Z_i = \{z_1^i, ..., z_r^i\}$; each Z_i will be an independent set. For $i \in \{13, 14, 15\}$, let $C_1^i, ..., C_{66}^i$ be a partition of Z_i into sets of size 15. Let $\mathcal{Y}$ be the set of unordered pairs $\{Z_s, Z_t\}$ such that $1 \le s < t \le 12$. For $i \in \{13, 14, 15\}$, let f_i be a bijection mapping from $\mathcal{Y}$ onto $\{C_1^i, \ldots C_{66}^i\}$. Let G be the graph with vertex set $X \cup \bigcup_{i=1}^{15} V(Z_i)$ and with edge set constructed as follows:

(1) add edges joining x to x_i for $i = 1, \ldots, 6$;
(2) add edges $x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6$ and x_6x_1 ;
(3) add edges joining x_i to all vertices of Z_j if and only if $x_i \in A_j$;
(4) add edges joining z_s^i to z_t^j for $1 \leq s \neq t \leq r$ and $1 \leq i \neq j \leq 12$;
(5) if $1 \leq j \leq 66$, $13 \leq i \leq 15$ and $1 \leq s \leq 12$, then add edges joining the vertices of C_j^i to the vertices of Z_s unless either $Z_s \in f_i^{-1}(C_j^i)$ or the vertices have the same subscript.

Obviously $\min\{\delta(G), \delta(\overline{G})\} = 6$ and $\deg(x) = 6$. We claim that $\gamma_c(G) = \gamma_c(\overline{G}) = 4$. Clearly $\{x, x_1, x_2, x_3\}$ is a connected dominating set of G and so $\gamma_c(G) \leq 4$. Let, to the contrary, $\gamma_c(G) < 4$ and S be a $\gamma_c(G)$-set. First let $x \in S$. Since $G[S]$ is connected we may assume $x_1 \in S$. If $|S \cap \{x_1, \ldots, x_6\}| = 2$, then obviously S does not dominate the vertices of Z_i in which $A_i = \{x_1, \ldots, x_6\} \setminus S$ by (3). If $|S \cap \{x_1, \ldots, x_6\}| = 1$, then S does not dominate some vertices of Z_{13} by (4) and (5). Let $x \notin S$. To dominate x, without loss of generality we may assume $x_1 \in S$. If $|S \cap \{x_1, \ldots x_6\}| = 3$, then $S = \{x_1, x_2, x_3\}, \{x_1, x_2, x_6\}$, or $\{x_1, x_6, x_5\}$ since $G[S]$ is connected. Obviously S does not dominate the vertices of the set $\{x_1, \ldots, x_6\}$. If $|S \cap \{x_1, \ldots x_6\}| = 2$, then we may assume $S = \{x_1, x_2\}$ (the case $S = \{x_1, x_6\}$ is similar). Then S does not dominate some vertices of Z_{13}. Finally let $S \cap \{x_1, \ldots x_6\} = \{x_1\}$. Then obviously S does not dominate the vertices of $\{x_1, \ldots, x_6\}$ or the vertices of Z_{13}. Therefore all cases leads to a contradiction. Thus $\gamma_c(G) = 4$.

Finally, we show that $\gamma_c(\overline{G}) = 4$. For any $y \in Z_{13}, z \in Z_{14}$ and $w \in Z_{15}$, $\{x, y, z, w\}$ is a connected dominating set of $\overline{G}$ which implies $\gamma_c(\overline{G}) \leq 4$. Let, to the contrary, $\gamma_c(\overline{G}) < 4$ and S be a $\gamma_c(\overline{G})$-set. First let $x \in S$. Since $G[S]$ is connected, $|S \cap \{x_1, \ldots, x_6\}| \leq 1$. If $|S \cap \{x_1, \ldots, x_6\}| = 1$, then obviously S does not dominate all vertices of $\{x_2, \ldots, x_6\}$. Suppose $|S \cap \{x_1, \ldots, x_6\}| = 0$ and $S = \{x, y, z\}$. Then each of y and z does not dominate exactly four vertices of $\{x_1, \ldots, x_6\}$. Thus S does not dominate at least two vertices of $\{x_1, \ldots, x_6\}$. Now let $x \notin S$. In order to dominate x, we have $S \cap (\cup_{i=1}^{15} Z_i) \neq \emptyset$. Therefore $|S \cap \{x_1, \ldots, x_6\}| \leq 2$. If $|S \cap \{x_1, \ldots, x_6\}| = 2$, then $S \cap \{x_1, \ldots, x_6\}$ does not dominate exactly the vertices of six of Z_is and obviously S does not dominate the vertices of at least one of these. A similar argument leads to a contradiction when $|S \cap \{x_1, \ldots, x_6\}| \leq 1$. Thus $\gamma_c(\overline{G}) = 4$. This completes the proof. □

We conclude this section with an open problem.

Problem 1. Does the bound in Theorem 4 hold with equality when $\min\{\delta(G), \delta(\overline{G})\} > 6$?

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