A high order HODIE finite difference scheme for 1D parabolic singularly perturbed reaction–diffusion problems

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1. Introduction

In this paper we consider the 1D parabolic reaction–diffusion singularly perturbed problem

$$
\begin{align*}
&u_t + L_{x,e}u = f(x, t), \quad (x, t) \in Q = \Omega \times (0, T) \equiv (0, 1) \times (0, T], \\
u(x, 0) = 0, \quad x \in \overline{\Omega}, \quad u(0, t) = u(1, t) = 0, \quad t \in (0, T],
\end{align*}
$$

(1)

where the spatial differential operator is given by

$$
L_{x,e}u = -e u_{xx} + b(x, t)u.
$$

(2)

We assume that $0 < e \ll 1$ and it can be arbitrarily small, the reaction term satisfies $b(x, t) > \beta > 0$ for all $(x, t) \in \overline{Q}$, and that the data are sufficiently smooth functions, concretely $b, f \in C^{(8,4)}(\overline{Q})$. Moreover, we assume sufficient compatibility conditions are satisfied by the data in order that the exact solution $u \in C^{(6,3)}(\overline{Q})$. Sufficient conditions for this regularity are the following (see [13]):

$$
\begin{align*}
f(s, 0) &= 0, \\
(L_{x,e}f)(s, 0) &= f_t(s, 0), \\
(L_{x,e}(f_t - L_{x,e}f))(s, 0) &= f_{tt}(s, 0), \quad s = 0, 1.
\end{align*}
$$

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It is well-known (see [12,20]) that the solution of (1) has a boundary layer at \( x = 0, 1 \) of width \( \mathcal{O}(\sqrt{\varepsilon} \ln \varepsilon) \). Under sufficient compatibility conditions, the classical theory (see [13], p. 352) proves that the solution of (1) satisfies the crude bounds

\[
|u^{(k,m)}(x,t)| \leq Ce^{-k/2}, \quad 0 \leq k + 2m \leq 8.
\]

Nevertheless, these bounds are not useful for the analysis of the uniform convergence of the numerical method. The following result (see [12,21]) gives the asymptotic behavior, with respect to \( \varepsilon \), of the exact solution, showing the presence of two boundary layers, and also a decomposition into a regular and a singular component of the solution of (1).

**Theorem 1.** The solution of (1) satisfies

\[
|u^{(k,m)}(x,t)| \leq C(1 + \varepsilon^{-k/2}B_e(x)), \quad 0 \leq k + 2m \leq 6,
\]

where function \( B_e(x) \) is given by \( B_e(x) = e^{-\sqrt{\varepsilon}/x} + e^{-\sqrt{\varepsilon}/(1-x)} \). Moreover, it can be decomposed as \( u = \phi + \psi \), where the regular component \( \phi \) satisfies

\[
|\phi^{(k,m)}(x,t)| \leq C(1 + \varepsilon^{-k/2}), \quad 0 \leq k + 2m \leq 6
\]

and the singular component \( \psi \) satisfies

\[
|\psi^{(k,m)}(x,t)| \leq C\varepsilon^{-k/2}B_e(x), \quad 0 \leq k + 2m \leq 6.
\]

There are many papers in the literature dealing with the numerical approximation of time dependent singularly perturbed reaction–diffusion problems. In [2,5,14,17] 1D linear or nonlinear problems of reaction–diffusion type were considered, in [3,8] the case of 2D parabolic problems was analyzed and coupled linear systems were solved in [11,15]. In some papers the fully discrete method is directly considered and the analysis of the convergence is based on appropriate bounds for the truncation error and the discrete maximum principle satisfied by the fully discrete operator (see [15]). Nevertheless, this technique cannot be extended to higher order time approximation than the Euler method due to the lack of a discrete maximum principle for the fully discrete method.

To elude this difficulty, in other papers (see [4,5,7,8]) the analysis of the uniform convergence is made in two steps. In the first one, the problem is discretized in time and appropriate estimates for the corresponding local error are proved. In the second step, the set of one dimensional problems resulting from the time discretization are discretized in space and it is necessary to prove the uniform convergence of the corresponding spatial discretization. So far, and motivated by the classical analysis of the convergence for initial value problems, in some previous papers this analysis was based on the uniform convergence of the numerical scheme which discretizes some auxiliary problems arising in the definition of the method. To deduce that uniform convergence, the asymptotic behavior of the solution of these auxiliary problems was needed.

In [6] the uniform convergence of the numerical method, used to discretize the singularly perturbed problems resulting after the time discretization, was proved without the use of any auxiliary problems. This new proof requires detailed *a priori* information on the asymptotic behavior of the solution of these problems, which is based on an inductive argument and the well-known behavior of steady singularly perturbed reaction–diffusion problems (see [19,20]). In [6] the implicit Euler method was considered to approximate the time variable and the central finite difference approximation for the spatial variable. In this paper we show that this strategy can be extended to a higher order approximation in space, namely we consider a HODIE (high order via differential identity expansion) compact fourth order scheme [16], and we prove that the fully discrete method is first order uniformly convergent in time and almost fourth order convergent in space. This increase of order of uniform convergence is the main advantage of the new numerical method due to a lack of high order methods in the literature for singularly perturbed parabolic problems. The basic idea of the analysis is the same as in [6], but now the proof of the uniform convergence of the numerical method is considerably more sophisticated than the one in [6]. First, it requires appropriate bounds for higher derivatives of the solution of the semidiscrete problem. Moreover, a precise decomposition into a regular and a singular part of the solution of the semidiscrete problem is given, which is used to prove the high order uniform convergence of the HODIE scheme.

The paper is structured as follows. In Section 2 we recall that the backward Euler method defined on a uniform mesh is a first order uniformly convergent method. Moreover, the asymptotic behavior of the solution of the semidiscrete problems resulting with respect to the diffusion parameter \( \varepsilon \) for each component. This analysis is one of the most interesting subjects in this paper and it is crucial for the analysis of the uniform convergence of the forthcoming numerical scheme. In Section 3 a compact and stable finite difference method is used to discretize the previous semidiscrete problems. A mesh of Vulanović type (see [22,23]) is used to discretize the space variable, proving that the fully discrete method has first order convergence in time and almost fourth order convergence in space. In Section 4 the numerical results for some test problems are given, illustrating in practice the efficiency of the method and corroborating the order of uniform convergence theoretically proved.

Henceforth, \( C \) denotes a generic positive constant independent of the diffusion parameter \( \varepsilon \) and also of the discretization parameters \( N \) and \( M \). We denote by \( \| \cdot \|_D \) the maximum norm over any region \( D \) defined by \( \| g \|_D = \max_{x \in D} |g(x)| \).
2. The time semidiscretization: uniform convergence and asymptotic behavior

The backward Euler method on the uniform mesh \( \Omega^M = \{ t_k = k\tau, 0 \leq k \leq M, \tau = T/M \} \) is used to discretize the time variable. Then, the semidiscrete problem reads

\[
\begin{aligned}
  z^0 &= 0, \\
  (I + \tau L_{x,\cdot}) z(x, t_n) &= \tau f(x, t_n) + z(x, t_{n-1}), \quad 1 \leq n \leq M, \\
  z(0, t_n) &= z(1, t_n) = 0.
\end{aligned}
\]  

(7)

To prove its uniform convergence the following auxiliary problems are defined

\[
\begin{aligned}
  (I + \tau L_{x,\cdot}) \hat{z}(x, t_n) &= \tau f(x, t_n) + u(x, t_{n-1}), \\
  \hat{z}(0, t_n) &= \hat{z}(1, t_n) = 0, \quad 1 \leq n \leq M.
\end{aligned}
\]  

(8)

Lemma 1 (See [8]). The local error for the method \((7)\), defined by \( E_n = u(x, t_n) - \hat{z}(x, t_n) \), satisfies

\[\|E_n\|_{\Omega^n} \leq C \tau^2, \quad 1 \leq n \leq M.\]

Then, using that \( \| (I + \tau L_{x,\cdot})^{-1} \|_{\Omega^n} \leq 1/(1 + \tau\beta) \), it follows that the Euler method satisfies a maximum principle and it is uniformly stable in the maximum norm.

Theorem 2 (See [8]). The global error for the method \((7)\), defined by \( e_n = u(x, t_n) - z(x, t_n) \), satisfies

\[\|e_n\|_{\Omega} \leq C \tau, \quad 1 \leq n \leq M.\]  

(9)

Therefore, the Euler method is a first order uniformly convergent method.

In [6] some technical results, which we will need to deduce the asymptotic behavior of the exact solution of \((7)\) and its derivatives up to sixth order, were proved. We summarize these results in the following Lemma.

Lemma 2.

1. Let \( z(x, t_{n-1}) \) and \( z(x, t_n) \) be the solution of problem \((7)\) at the time levels \( n-1 \) and \( n \), respectively. Then, it holds

\[|z(x, t_n) - z(x, t_{n-1})| \leq C \tau, \quad 1 \leq n \leq M.\]

(10)

2. Let \( u(x, t) \) be the solution of \((1)\) and \( \hat{z}(x, t_n) \) the solution of \((8)\). Then, it holds

\[\|u(x, t_n) - \hat{z}(x, t_n)\|_{\Omega^n} \leq C \tau^3, \quad 1 \leq n \leq M.\]

(11)

Therefore, from the uniform stability of the Euler method, it follows

\[\|u(x, t_n) - u(x, t_{n-1}) - \hat{z}(x, t_n) + \hat{z}(x, t_{n-1})\|_{\Omega^n} \leq C \tau^2, \quad 1 \leq n \leq M.\]

(12)

3. Let \( z(x, t_{n-2}), z(x, t_{n-1}) \) and \( z(x, t_n) \) be the solution of problem \((7)\) at the time levels \( n-2 \), \( n-1 \) and \( n \), respectively. Then, it holds

\[|z(x, t_n) - 2z(x, t_{n-1}) + z(x, t_{n-2})| \leq C \tau^2, \quad 2 \leq n \leq M.\]

(13)

4. Let \( z(x, t_{n-1}) \) and \( z(x, t_n) \) be the solution of problem \((7)\) at the time levels \( n-1 \) and \( n \), respectively. Then, it holds

\[|z''(x, t_n) - z''(x, t_{n-1})| \leq C \tau \tau^{-1}, \quad 1 \leq n \leq M.\]

(14)

Corollary 1. Let \( z(x, t_{n-1}) \) and \( z(x, t_n) \) be the solution of problem \((7)\) at the time levels \( n-1 \) and \( n \), respectively. Then, it holds

\[|z'(x, t_n) - z'(x, t_{n-1})| \leq C \tau \tau^{-1/2}, \quad 1 \leq n \leq M.\]

(15)

Proof. The result is based on the following inequality (see [1])

\[\|g''\| \leq \frac{2}{\mu} \|g\| + \mu \|g''\|,\]

(16)

where \( J = [a, a + \mu] \) is an arbitrary interval with \( \mu > 0 \) and \( g \in C^2(J) \). Taking \( g = z(x, t_n) - z(x, t_{n-1}) \) and \( \mu = \epsilon^{1/2} \), the result trivially follows from \((10)\) and \((14)\).
Lemma 3. Let \(u(x,t)\) be the solution of (1) and \(z(x,t_n)\) the solution of (8). Then, for \(1 \leq n \leq M\) it holds

\[
\|u_n(x,t_n) - Z(x,t_n)\|_{\Pi} \leq C t^2 \varepsilon^{-1/2}, \quad \|u''(x,t_n) - z''(x,t_n)\|_{\Pi} \leq C t^2 \varepsilon^{-1}, 
\]

\[
\|u_n(x,t_n) - Z(x,t_n) - (u_n(x,t_{n-1}) - Z(x,t_{n-1}))\|_{\Pi} \leq C t^3 \varepsilon^{-1/2},
\]

\[
\|u_n(x,t_n) - Z(x,t_n) - (u_n(x,t_{n-1}) - Z(x,t_{n-1}))\|_{\Pi} \leq C t^2 e^{-1/2}.
\]

Therefore, from the uniform stability of the Euler method, it follows

\[
\|u_n(x,t_n) - Z(x,t_n)\|_{\Pi} \leq C t^2 \varepsilon^{-1/2}, \quad \|u''(x,t_n) - z''(x,t_n)\|_{\Pi} \leq C t^2 \varepsilon^{-1},
\]

\[
\|u_n(x,t_n) - Z(x,t_n) - (u_n(x,t_{n-1}) - Z(x,t_{n-1}))\|_{\Pi} \leq C t^3 \varepsilon^{-1/2},
\]

\[
\|u_n(x,t_n) - Z(x,t_n) - (u_n(x,t_{n-1}) - Z(x,t_{n-1}))\|_{\Pi} \leq C t^2 e^{-1/2}.
\]

Proof. Using ideas similar to those in [6], it is straightforward to prove that

\[
(I + \tau L_{xx})(u_n(x,t_n) - Z(x,t_n)) = u_n(x,t_n) - u_{xx}(x,t_{n-1}) - \tau u_{xx}(x,t_n) + 2\tau b_{xx}(x,t_n)(Z(x,t_n) - u_n(x,t_n)) - u_n(x,t_n) + \tau b_{xx}(x,t_n)(Z(x,t_n) - u(x,t_n)).
\]

From Lemma 1 and the bound

\[
\|u_{xx}(x,t_n) - u_{xx}(x,t_{n-1}) - \tau u_{xx}(x,t_n)\| \leq C t^2 / \varepsilon,
\]

it follows that

\[
\|u_n(x,t_n) - Z(x,t_n)\|_{\Pi} \leq C (t^2 / \varepsilon + \tau \|Z(x,t_n) - u_n(x,t_n)\|_{\Pi}).
\]

At the boundary, using a continuity argument we can deduce

\[u_{xx}(x,t_n) - Z''(x,t_n) = 0, \quad 1 \leq n \leq M, \quad x = 0, 1.\]

Then, using the maximum principle we have

\[
\|u_{xx}(x,t_n) - Z''(x,t_n)\|_{\Pi} \leq C (t^2 / \varepsilon + \tau \|u_n(x,t_n) - Z(x,t_n)\|_{\Pi}).
\]

Following the argument given in [18, Lemma 3], based on the mean value theorem, which permits to deduce simultaneously bounds for the first and the second derivatives, we can obtain

\[
\|u_n(x,t_n) - Z(x,t_n)\|_{\Pi} \leq C t^2 / \varepsilon, \quad \|u''(x,t_n) - z''(x,t_n)\|_{\Pi} \leq C t^2 / \varepsilon.
\]

To obtain sharp bounds we use again (16) taking \(\mu = \varepsilon^{1/2}\) and \(g = u(x,t_n) - Z(x,t_n)\).

In the same way, it can be proved that

\[
(I + \tau L_{xx})(u_n(x,t_n) - Z(x,t_n)) - (u_n(x,t_{n-1}) - Z(x,t_{n-1})) = u_n(x,t_n) - 2u_{xx}(x,t_{n-1}) + u_{xx}(x,t_{n-2}) + \tau (u_{xx}(x,t_{n-1}) - u_{xx}(x,t_n)) + 2\tau b_{xx}(x,t_n)(Z(x,t_n) - u_n(x,t_n)) - (Z(x,t_{n-1}) - u_n(x,t_{n-1})) + \tau (b_{xx}(x,t_n) - b_{xx}(x,t_{n-1}))(Z(x,t_{n-1}) - u_n(x,t_{n-1})) + \tau (b_{xx}(x,t_n) - b_{xx}(x,t_{n-1}))(Z(x,t_{n-1}) - u_n(x,t_{n-1}))
\]

and the same technique as before, taking now \(\mu = \varepsilon^{1/2}\) and \(g = u(x,t_n) - Z(x,t_n) - u(x,t_{n-1}) - Z(x,t_{n-1})\), allows to obtain the required result.

Lemma 4. Let \(z(x,t_{n-2}), z(x,t_{n-1})\) and \(z(x,t_n)\) be the solution of problems (7) at the time levels \(n-2, n-1\) and \(n\), respectively. Then, it holds

\[
|z''(x,t_n) - 2z''(x,t_{n-1}) + z''(x,t_{n-2})| \leq C t^2 \varepsilon^{-1}, \quad 2 \leq n \leq M.
\]

Proof. It is straightforward to see that it holds

\[
z''(x,t_n) - 2z''(x,t_{n-1}) + z''(x,t_{n-2}) = [z''(x,t_n) - u_{xx}(x,t_n)] - [z''(x,t_{n-1}) - u_{xx}(x,t_{n-1})] + [u_{xx}(x,t_n) - 2u_{xx}(x,t_{n-1}) + u_{xx}(x,t_{n-2})].
\]

From (19) and the estimates (4) for the derivatives of \(u\), the result follows.
Lemma 5. Let $z(x, t_{n-1})$ and $z(x, t_n)$ be the solution of problem (7) at the time levels $n-1$ and $n$, respectively. Then, it holds
\[ |z^{(4)}(x, t_n) - z^{(4)}(x, t_{n-1})| \leq C \tau e^{-2}, \quad 1 \leq n \leq M. \]  

Proof. Differentiating twice (7) and subtracting the equations corresponding to the time levels $t_n$ and $t_{n-1}$, we have
\[
\frac{\partial}{\partial t}(z^{(4)}(x, t_n)) - \frac{\partial}{\partial t}(z^{(4)}(x, t_{n-1})) = \frac{\partial}{\partial x}(b(x, t_n)z^{(4)}(x, t_n)) - \frac{\partial}{\partial x}(b(x, t_{n-1})z^{(4)}(x, t_{n-1})) - \frac{\partial}{\partial x}(b(x, t_n)z^{(2)}(x, t_n)) + \frac{\partial}{\partial x}(b(x, t_{n-1})z^{(2)}(x, t_{n-1}))
\]
and using the Cauchy estimate, we obtain
\[
|b(x, t_n)z^{(6)}(x, t_n) - b(x, t_{n-1})z^{(6)}(x, t_{n-1})| \leq C|b(x, t_n) - b(x, t_{n-1})||z^{(4)}(x, t_n) - z^{(4)}(x, t_{n-1})|.
\]

Note that the maximum principle for problem (7) together with the bound (10) prove $|z(x, t_n)| \leq C$. Hence, $|z^{(m)}(x, t_n)| \leq C e^{-1}$ and $|z^{(m)}(x, t_n)| \leq C e^{-1/2}$. From this bound and (15), we have
\[
|b(x, t_n)z^{(6)}(x, t_n) - b(x, t_{n-1})z^{(6)}(x, t_{n-1})| \leq C \tau e^{-1/2}.
\]
Collecting the above estimates we deduce the required result. □

Corollary 2. Let $z(x, t_{n-1})$ and $z(x, t_n)$ be the solution of problem (7) at the time levels $n-1$ and $n$, respectively. Then, it holds
\[ |z^{(m)}(x, t_n) - z^{(m)}(x, t_{n-1})| \leq C \tau e^{-3/2}, \quad 1 \leq n \leq M. \]  

Proof. Using (14), Lemma 4 and (16) with $\mu = \tau^{1/2}$ and $g = z^{(m)}(x, t_n) - z^{(m)}(x, t_{n-1})$, the result follows. □

Theorem 3. Let $z(x, t_n)$ be the solution of problem (7) at the time level $n$. Then, it holds
\[ |z^{(k)}(x, t_n)| \leq C(1 + \tau^{k/2}B_c(x)), \quad 0 \leq k \leq 6, \quad 1 \leq n \leq M. \]  

Proof. The proof proceeds by induction. In [6], bounds (23) for $0 \leq k \leq 4$ were proved; then, here we only show the details to obtain the bounds corresponding to the fifth and sixth order derivatives. Let $n = 1$. Using the bounds
\[ |z^{(3)}(x, t_1) - z^{(3)}(x, 0)| \leq C \tau e^{-3/2}, \quad |z^{(4)}(x, t_1) - z^{(4)}(x, 0)| \leq C \tau e^{-2} \]
and following the same technique as in [6], we can deduce that
\[ |z^{(k)}(x, t_1)| \leq C(1 + \tau^{k/2}B_c(x)), \quad k = 5, 6, \]
which is the required result for the first time level. We assume now the induction hypothesis
\[ |z^{(k)}(x, t_n)| \leq C(1 + \tau^{k/2}B_c(x)), \quad 0 \leq k \leq 6, \quad 1 \leq m \leq n \]
and we prove that similar bounds also hold for the time level $n + 1$. The proof can be found in [6] for the derivatives up to fourth order. We know that $z(x, t_{n+1})$ is the solution of the problem
\[
\begin{cases}
\left( \frac{\partial}{\partial t} + L_x \right) z(x, t_{n+1}) = f(x, t_{n+1}) + \frac{z(x, t_{n+1}) - z(x, t_n)}{\tau} = g_1(x, \tau), \\
z(0, t_{n+1}) = z(1, t_{n+1}) = 0, 
\end{cases}
\]
with $|g_1(x, \tau)| \leq C/\tau$. This problem can be also written as
\[
\begin{cases}
-\tau^2 z^{(3)}(x, t_{n+1}) + b(x, t_{n+1})z^{(2)}(x, t_{n+1}) = f(x, t_{n+1}) + \frac{z(x, t_{n+1}) - z(x, t_n)}{\tau} = g_2(x, \tau), \\
z(0, t_{n+1}) = z(1, t_{n+1}) = 0, 
\end{cases}
\]
with $|g_2(x, \tau)| \leq C$ from (10). Differentiating four times w.r.t. $x$ the differential equation in (25), we obtain
\[
\begin{cases}
-\tau^2 z^{(6)}(x, t_{n+1}) + b(x, t_{n+1})z^{(5)}(x, t_{n+1}) = f^{(4)}(x, t_{n+1}) + \frac{z^{(5)}(x, t_{n+1}) - z^{(5)}(x, t_n)}{\tau} = g_3(x, \tau), \\
\end{cases}
\]
and
\[
\begin{cases}
-(b(x, t_{n+1})z^{(6)}(x, t_{n+1}))^{(4)} + b(x, t_{n+1})z^{(5)}(x, t_{n+1}) = g_4(x, \tau), \\
z^{(6)}(0, t_{n+1}) = z^{(6)}(1, t_{n+1}) \quad \text{given} 
\end{cases}
\]
and it is not difficult to deduce that it holds \( |g_2(x, t)| \leq C \epsilon^{-2}, \ |z^{(5)}(0, t_{n+1})| \leq C \epsilon^{-5/2}, \ |z^{(5)}(1, t_{n+1})| \leq C \epsilon^{-5/2}, \ |z^{(6)}(0, t_{n+1})| \leq C \epsilon^{-3}, \ |z^{(6)}(1, t_{n+1})| \leq C \epsilon^{-3}. \) Also, differentiating five times the differential equation in (24), we obtain
\[
\left( \frac{1}{\tau} + L_{x, x} \right) z^{(5)}(x, t_{n+1}) = f^{(5, 0)}(x, t_{n+1}) - (b(x, t_{n+1}) z^{(5)}(x, t_{n+1}))^{(5)} + b(x, t_{n+1}) z^{(5)}(x, t_{n+1}) + \frac{z^{(5)}(x, t_{n})}{\tau} = g_3(x, \tau).
\]
From the inductive hypothesis and the bounds for \( |z^{(k)}(x, t_{n+1})|, \ 0 \leq k \leq 4, \) it follows that \( |g_3(x, \tau)| \leq \frac{C}{\epsilon^5} \left( 1 + \epsilon^{-5/2} B_1(x) \right). \) Then, using the barrier function \( \psi_1(x) = C(1 + \epsilon^{-5/2} B_1(x)) \) and the maximum principle for \( \left( \frac{1}{\tau} + L_{x, x} \right) \) we obtain
\[
|z^{(5)}(x, t_{n+1})| \leq C \left( 1 + \epsilon^{-5/2} B_1(x) \right).
\]
In a completely similar way, it can be proved that \( |z^{(6)}(x, t_{n+1})| \leq C \left( 1 + \epsilon^{-3} B_1(x) \right), \) which is the required result. \( \Box \)

The bounds proved in Theorem 3 are not sufficient to analyze the uniform convergence of the finite difference scheme defined in the next section. Therefore, we consider a decomposition of the solution of the semidiscrete problem (7) given by
\[
z(x, t_n) = \nu(x, t_n) + \approx \psi_1(x, t_n) + \epsilon^2 \psi_2(x, t_n), \quad 0 \leq n \leq M.
\]
where the regular component \( \nu \) is defined by
\[
\nu(x, t_n) = \nu_0(x, t_n) + \epsilon \nu_1(x, t_n) + \epsilon^2 \nu_2(x, t_n), \quad 0 \leq n \leq M.
\]
with
\[
\nu_0(x, t_n) = \frac{\tau f(x, t_n) + \nu_0(x, t_{n-1})}{\tau b(x, t_n) + 1}, \quad \nu_0(x, t_0) = z(x, t_0) = 0, \quad x \in [0, 1],
\]
\[
\nu_1(x, t_n) = \frac{\tau \nu_0'(x, t_n) + \nu_1(x, t_{n-1})}{\tau b(x, t_n) + 1}, \quad \nu_1(x, t_0) = 0, \quad x \in [0, 1],
\]
\[
\begin{cases}
- \epsilon \nu_0''(x, t_n) + \frac{1}{\tau} (b(x, t_n) + 1) \nu_2(x, t_n) = \nu_0''(x, t_n) + \frac{\nu_2(x, t_{n-1})}{\tau}, & x \in (0, 1), \\
\nu_0(0, t_n) = \nu_0(1, t_n) = 0, \\

\nu_0(x, 0) = 0, & x \in [0, 1].
\end{cases}
\]
Note that this regular component is the solution of the problem
\[
\begin{cases}
\nu(x, t_0) = 0, \\
\frac{\nu(x, t_{n-1}) - \nu(x, t_n)}{\tau} + L_{x, x} \nu(x, t_n) = f(x, t_n), & x \in (0, 1), \\
\nu(0, t_n) = \nu_0(0, t_n) + \epsilon \nu_1(0, t_n), \quad \nu(1, t_n) = \nu_0(1, t_n) + \epsilon \nu_1(1, t_n), \quad 1 \leq n \leq M.
\end{cases}
\]
On the other hand, the singular component \( w \) is the solution of the problem
\[
\begin{cases}
w(x, t_0) = 0, \\
\frac{w(x, t_{n-1}) - w(x, t_n)}{\tau} + L_{x, x} w(x, t_n) = 0, & x \in (0, 1), \\
w(0, t_n) = -\nu(0, t_n), \quad w(1, t_n) = -\nu(1, t_n), \quad 1 \leq n \leq M.
\end{cases}
\]
We are interested in obtaining appropriate bounds of these two components and their derivatives up to sixth order.

**Lemma 6.** Let assume that \( b, f \in C^{8,4}(\Omega); \) then, the regular component \( \nu \) satisfies
\[
|\nu^{(k)}(x, t_n)| \leq C(1 + \epsilon^{2-k/2}), \quad 0 \leq k \leq 6, \quad 0 \leq n \leq M.
\]
**Proof.** From the definition of \( \nu(x, t_n), \) it immediately follows that \( |\nu^{(k)}(x, t_0)| \leq C, \ 0 \leq k \leq 8, \ |\nu^{(k)}(x, t_n)| \leq C, \ 0 \leq k \leq 6. \)

Problem (29) is similar to problem (7) and therefore Theorem 3 proves that the component \( \nu_0 \) satisfies the crude bounds \( |\nu_0^{(k)}(x, t_n)| \leq C \epsilon^{-k/2}, \ 0 \leq k \leq 6. \) Then, the result trivially follows from the previous bounds for the derivatives of \( \nu_0, \nu_1 \) and \( \nu_2. \) \( \Box \)

**Lemma 7.** The singular component \( w \) satisfies on the boundary
\[
|w^{(k)}(0, t_n) - w^{(k)}(0, t_{n-1})| \leq C \epsilon^{k-1/2}, \quad 0 \leq k \leq 4, \quad 1 \leq n \leq M,
\]
\[
|w^{(k)}(1, t_n) - w^{(k)}(1, t_{n-1})| \leq C \epsilon^{k-1/2}, \quad 0 \leq k \leq 4, \quad 1 \leq n \leq M.
\]
**Proof.** Note that the function \( w(x, t_n) \) satisfies the same bounds as \( z(x, t_n). \) Then, from (10, 14, 15, 21 and 22), the required bounds at \( x = 0 \) and \( x = 1 \) follow. \( \Box \)
Lemma 8. The singular component $w$ satisfies

$$|w^{(k)}(x, t_n)| \leq C \varepsilon^{-k/2} B_{C}(x), \quad k = 0, 1, 2, \quad 0 \leq n \leq M.$$  \hspace{1cm} (32)

Proof. Let $k = 0$. Note that $w(x, 0) \equiv 0$ and the result trivially follows at the time level $t_1$ applying the barrier function technique to the problem

$$(I + \tau L_{x,x})w(x, t_1) = 0, \quad x \in (0, 1), \quad w(0, t_1), \quad w(1, t_1) \text{ given}.$$ 

For $1 \leq n < M$ assume that $|w(x, t_n)| \leq C B_{C}(x), \quad x \in [0, 1]$, and define the auxiliary problem

$$\begin{align*}
(I + \tau L_{x,x})w(x, t_{n+1}) &= w(x, t_n), \quad x \in (0, 1), \\
(w(0, t_{n+1}), \ &w(1, t_{n+1}) \text{ given}.
\end{align*}$$

Using the induction and the maximum principle for $(I + \tau L_{x,x})$, then (32) follows for $k = 0$.

Next, we prove by induction that

$$|w(x, t_n) - w(x, t_{n-1})| \leq C t B_{C}(x), \quad 1 \leq n \leq M.$$  \hspace{1cm} (33)

For $n = 1$, at the interior points it holds

$$(I + \tau L_{x,x})(w(x, t_1) - w(x, t_0)) = -\tau L_{x,x}w(x, t_0) = 0$$

and on the boundary we dispose of the bounds given in Lemma 7. Hence, the maximum principle proves that $|w(x, t_1) - w(x, t_0)| \leq C r B_{C}(x)$. Let assume that $|w(x, t_n) - w(x, t_{n-1})| \leq C t B_{C}(x)$ for $1 \leq n < M$. The semidiscrete problems at the time levels $n$ and $n + 1$ are given by

$$(I + \tau L_{x,x})w(x, t_n) = w(x, t_{n-1}), \quad \text{and} \quad (I + \tau L_{x,x})w(x, t_{n+1}) = w(x, t_n),$$

respectively. Subtracting these two equations, we have

$$\left[I + \tau \left(-\varepsilon \frac{d^2}{dx^2} + b(x, t_{n+1})I\right)\right] (w(x, t_{n+1}) - w(x, t_n)) = (w(x, t_n) - w(x, t_{n-1})) + \tau w(x, t_n)(b(x, t_n) - b(x, t_{n-1})) \leq C t B_{C}(x),$$

using in the last inequality the inductive hypothesis, Taylor expansions and (32) for $k = 0$. Lemma 7 provides the bounds $|w(x, t_{n+1}) - w(x, t_n)| \leq C \varepsilon^{-1/2} B_{C}(x)$. Using (16) with $\mu = \varepsilon^{1/2}$ and $g = w(x, t_n)$ it follows $|w(x, t_{n+1})| \leq C \varepsilon^{-1/2} B_{C}(x)$.

Lemma 9. For $1 \leq n \leq M$, the singular component $w$ satisfies

$$|w'(x, t_n) - w'(x, t_{n-1})| \leq C \varepsilon^{-1/2} B_{C}(x), \quad |w''(x, t_n) - w''(x, t_{n-1})| \leq C \varepsilon^{-1} B_{C}(x).$$  \hspace{1cm} (34)

Proof. Again we proceed by induction. Differentiating Eq. (30) we have

$$(I + \tau L_{x,x})(w'(x, t_1) - w'(x, t_0)) = \tau b_{x}(x, t_1)w(x, t_1) \equiv g_4(x, t_1)$$

and differentiating twice Eq. (30) we have

$$(I + \tau L_{x,x})(w''(x, t_1) - w''(x, t_0)) = 2\tau b_{xx}(x, t_1)w'(x, t_1) + \tau b_{x}(x, t_1)w(x, t_1) \equiv g_5(x, t_1).$$

Then, using (32), it follows that $|g_4(x, t_n)| \leq C \varepsilon B_{C}(x)$ and $|g_5(x, t_n)| \leq C \varepsilon^{-1/2} B_{C}(x)$. On the boundary, using Lemma 7 it holds $|w'(x, t_1) - w'(x, t_0)| \leq C \varepsilon^{-1/2}$ and $|w''(x, t_1) - w''(x, t_0)| \leq C \varepsilon^{-1}$ for $x = 0, 1$. Hence, from the maximum principle we can deduce

$$|w'(x, t_1) - w'(x, t_0)| \leq C \varepsilon^{-1/2} B_{C}(x), \quad |w''(x, t_1) - w''(x, t_0)| \leq C \varepsilon^{-1} B_{C}(x).$$

Let assume that $|w'(x, t_n) - w'(x, t_{n-1})| \leq C \varepsilon^{-1/2} B_{C}(x)$ and $|w''(x, t_n) - w''(x, t_{n-1})| \leq C \varepsilon^{-1} B_{C}(x)$ for $1 < n < M$. Note that

$$(I + \tau L_{x,x})(w'(x, t_{n+1}) - w'(x, t_n)) = (w'(x, t_n) - w'(x, t_{n-1})) + \tau b_{x}(x, t_{n+1})w(x, t_{n+1}) - w(x, t_n) + \tau w(x, t_n)(b(x, t_{n+1}) - b(x, t_n))$$

$$- b(x, t_n) + \tau w'(x, t_n)(b(x, t_n) - b(x, t_{n-1})) \equiv g_6(x, t_{n+1})$$
and
\[
(I + \tau L_{xx})(w'(x, t_{n+1}) - w'(x, t_n)) = (w''(x, t_n) - w''(x, t_{n-1})) + 2\tau b_x(x, t_{n+1})(w'(x, t_{n+1}) - w'(x, t_n)) \\
+ \tau b_{xx}(x, t_{n+1})(w(x, t_{n+1}) - w(x, t_n)) + 2\tau w'(x, t_n)(b(x, t_{n+1}) - b(x, t_n)) \\
+ \tau w'(x, t_n)(b_{xx}(x, t_n) - b_{xx}(x, t_n)) + \tau w'(x, t_n)(b(x, t_n) - b(x, t_{n-1}))
\]

with \(|g_k(x, t_{n+1})| \leq C \tau e^{1/2} B_{c}(x)\) and \(|g_k(x, t_{n+1})| \leq C \tau e^{-1/2} B_{c}(x)\). Using the maximum principle the result follows. \(\square\)

**Lemma 10.** The singular component \(w\) satisfies
\[
|w^{(k)}(x, t_n)| \leq C e^{-k/2} B_{c}(x), \quad 3 \leq k \leq 6, \quad 0 \leq n \leq M.
\]

**Proof.** Differentiating twice Eq. (30), we can write
\[
-\varepsilon w^{(3)}(x, t_n) = -\frac{w''(x, t_n) - w''(x, t_{n-1})}{r} - b_{xx}(x, t_n)w(x, t_n) - 2b_x(x, t_n)w'(x, t_n) - b(x, t_n)w''(x, t_n).
\]

Then, from the estimates for \(w'(x, t_n)\) and \(w''(x, t_n)\), and (34), it directly follows \(|w^{(4)}(x, t_n)| \leq C e^{-2} B_{c}(x)\). Using (16) with \(\mu = \varepsilon^{1/2}\) and \(r = w'(x, t_n)\) it follows \(|w''(x, t_n)| \leq C e^{-3/2} B_{c}(x)\).

Using similar ideas as in the proof of Lemma 9 we can obtain
\[
|w^{(k)}(x, t_n) - w^{(k)}(x, t_{n-1})| \leq C \tau e^{-k/2} B_{c}(x), \quad k = 3, 4, \quad 1 \leq n \leq M.
\]

Then, using the same technique as the one used for \(k = 3, 4\) permit us to deduce the required result for \(k = 5, 6\). \(\square\)

### 3. The fully discrete method: analysis of the uniform convergence

To discretize (7) in space, we use an hybrid finite difference scheme which combines the central difference and a HODIE type operator [10] constructed on a special mesh of Vulanovic type (see [22,23]). This mesh is a generalized Shishkin mesh (see [19,20]) constructed by using a suitable generating function \(\eta\), which also depends on two transition points. Let \(N = 4k\), where \(k\) is a positive integer; then, we divide \([0, 1]\) into three intervals \([0, \eta]\), \([\eta, 1 - \eta]\) and \([1 - \eta, 1]\), where \(\eta\) is
\[
\eta = \min \left\{ \frac{1}{4}, \sqrt[3]{4}, \frac{1}{2} \right\}.
\]

The grid points are defined by \(x_j = \eta j/N, j = 0, 1, \ldots, N/2\), with \(N \in C^2 [0, 1/2]\) and
\[
\eta(z) = \begin{cases} 
4z, & z \in [0, 1/4], \\
4z - 4\sigma(1 - 2z)^2 + 4\sigma(z - 1/4)^2 + \sigma, & z \in [1/4, 1].
\end{cases}
\]

The coefficient \(\sigma\) is such that \(\eta(1/2) = 1/2\) and the mesh is symmetric with respect to the point \(x = 1/2\) (see Fig. 1 where \(N\) is plotted versus \(\eta\)).

Note that in \([0, \eta]\) and \([1 - \eta, 1]\) the mesh points are the same than in the usual piecewise uniform Shishkin mesh for reaction–diffusion problems (see [9,19,20]). Otherwise, in \([\eta, 1 - \eta]\) it is nonuniform but the step sizes, \(h_j = x_j - x_{j-1}, j = 1, \ldots, N\), satisfy
\[
|h_{j+1} - h_j| \leq CN^{-1}, \quad j = N/4, \ldots, 3N/4.
\]

Note that if \(\eta = 1/4\) the mesh is uniform and therefore a classical analysis can be applied; so, here we are only interested in the case \(\eta = 4\sqrt[3]{4}/1\). In this case, the following compact finite difference scheme is defined (see [5]).

\[
L^N_{\varphi} U_j^n = \frac{U_j^n - U_{j-1}^{n-1}}{\tau} - \varepsilon \alpha^2 U_j^n + \Gamma[b_j^n U_j^n] = \Gamma[f_j^n], \quad 0 < j < N, \quad U_0^n = U_N^n = 0.
\]

where
\[
\Gamma[f_j^n] = \begin{cases} 
\nu_j^n, & \text{if } N/4 < j \leq 3N/4, \text{ and } h_{\text{max}}^2 \left( \| b_j^n \|_Q + \frac{1}{r} \right) \geq 6\varepsilon, \\
\nu_j^n - \frac{4 + 2\gamma_j}{6} \nu_{j+1}^n + \frac{4 + 2\gamma_j}{6} \nu_j^n, & \text{otherwise}.
\end{cases}
\]

with
\[
\gamma_j^+ = \frac{h_j^2}{2h_{j+1} h_j}, \quad \gamma_j^- = \frac{h_{j+1}^2}{2h_j h_{j+1}}.
\]
Remark 1. In the case that the mesh is uniform \((\sigma = 1/4)\), we have chosen

\[
\Gamma[v^n] = \frac{1}{12} v^n_{j+1} + \frac{5}{6} v^n_j + \frac{1}{12} v^n_{j-1}, \quad 0 < j < N,
\]
to perform the numerical experiments given in the next section.

Lemma 11. Let \( N \geq N_0 \) be, where \( N_0 > 0 \) is a positive integer independent of \( \varepsilon \) such that

\[
64 \left( \|b\|_\infty + \frac{1}{\tau} \right) < \frac{3\beta N_0^2}{\ln N_0}, \tag{41}
\]
Then, the scheme (40) is of positive type, it satisfies a discrete maximum principle and it is \( \varepsilon \)-uniform stable in the maximum norm.

Proof. The proof is trivial using that the matrix associated with the discrete operator is an M-matrix (see [20]). \( \square \)

Theorem 4. Let assume that \( N \geq N_0 \). Then, the error associated with the hybrid finite difference scheme (40) satisfies

\[
\left| U - z^n \right| \leq \begin{cases} C(N^{-2} \varepsilon + (N^{-1} \ln N)^4), & \text{if } \sigma \neq 1/4 \text{ and } 6\varepsilon \leq h_{\text{max}}^2 \left( \|b\|_\infty + M \right), \\ C(N^{-1} \ln N)^4, & \text{otherwise.} \end{cases}
\]

Proof. The truncation error is given by

\[
L_c^{N,M}[U - z^n] = L_c^{N,M} U^n_j - L_c^{N,M} z^n_j = \Gamma[f_j^n] - L_c^{N,M} z^n_j = \Gamma \left[ \frac{z_j^n - z_{j-1}^n}{\tau} + L_c[z_j^n] \right] - L_c^{N,M} z^n_j = -\varepsilon \left( \Gamma \left( z^n_j \right) - \delta^2 z^n_j \right),
\]
and therefore we must analyze the contribution to the error of the approximation of the second order space derivative. To have an appropriate bound of the error, we consider a decomposition of the discrete solution \( U \) into a regular and a singular component. These components are defined as the solution of the following discrete problems:

\[
L_c^{N,M} V^n_j = \Gamma \left[ \frac{V_j^n - V_{j-1}^n}{\tau} - \varepsilon \delta^2 V_j^n + \Gamma [b_j^n V_j^n] = \Gamma [f_j^n], \quad 0 < j < N, \quad V_0^n = \nu(0, t_n), \quad V_N^n = \nu(1, t_n)
\]
and
\[ L^{NM}_\varepsilon W_j^n = \Gamma \left[ \frac{W^n_j - W^{n-1}_j}{\tau} \right] - \varepsilon \delta^2 W^n_j + \Gamma [b^n_j W^n_j] = 0, \quad 0 < j < N, \quad W_0^n = w(0,t_n), \quad W_N^n = w(1,t_n), \]
respectively, where the initial condition for both problems is zero.

The regular and singular components of the continuous (given in Theorem 1) and semidiscrete problems satisfy similar estimates, and therefore the analysis given in [5] for the steady problem can be extended to the parabolic problem. Hence, taking Taylor expansions and using that (39) holds at the transition points of the Vulanović mesh, we can deduce the following estimates for the regular component at the transition points
\[ |r^{NM}_\varepsilon [V - v^n_j]| \leq \begin{cases} C(\varepsilon N^{-2} + N^{-4}), & \text{if } j = N/4, \ldots, 3N/4, \ \text{and} \ h_{\max}^2 \left( \|b\| \mathcal{Q} + M \right) \geq 6\varepsilon. \\ CN^{-4}, & \text{otherwise}. \end{cases} \]
Note that the estimate \( C\varepsilon N^{-2} \) of the truncation error associated with the regular component at the transition points \( \sigma \), and \( 1 - \sigma \), come from the central difference operator.

So, from the parameter uniform bounds for the truncation error and the uniform stability of the hybrid scheme, it easily follows
\[ ||V - v^n_j|| \leq \begin{cases} C\varepsilon N^{-2}, & \text{if } h_{\max}^2 \left( \|b\| \mathcal{Q} + M \right) \geq 6\varepsilon, \\ CN^{-4}, & \text{otherwise}. \end{cases} \]
On the other hand, for the singular component, using a truncation error argument, based on the exponential character of this component in \( |\sigma, 1 - \sigma| \) and on the width of the mesh step size in \( (0,\sigma) \cup (1 - \sigma, 1) \), we can deduce
\[ |r^{NM}_\varepsilon [W - w^n_j]| \leq \begin{cases} Ce^{-\varepsilon/\sigma} \leq CN^{-4}, & 0 < j < N/4, \\ Ce^{-\varepsilon/2}h_{\max}^2 \leq C(N^{-1} \ln N)^4, & 0 < j < N/4, \\ CN^{-4}, & 3N/4 < j < N, \end{cases} \]
From these bounds and the uniform stability of the hybrid scheme (40), the result follows. \( \square \)

**Remark 2.** If we consider the standard piecewise uniform Shishkin mesh for reaction–diffusion problems (see [19,20]) instead the Vulanović mesh, we can only obtain the estimate
\[ |r^{NM}_\varepsilon [V - v^n_j]| \leq C\varepsilon N^{-1} \]
for the regular component at the transition points \( \sigma \), and \( 1 - \sigma \). So, following the ideas of this paper, we will achieve only almost third order of uniform convergence. Nevertheless, for both the Shishkin and Vulanović meshes, the same maximum errors and orders of uniform convergence have been obtained for all the examples that we have considered in our research.

**Theorem 5.** Let be \( u(x,t) \) the solution of problem (1) and \( U^n_j \) the solution of the discrete problem (40). If we assume that \( N \geq N_0 \) where \( N_0 \) is defined in (41), then it holds
\[ ||u - U^n_j|| \leq \begin{cases} C(\tau + N^{-2} \varepsilon + (N^{-1} \ln N)^4), & \text{if } \sigma \neq 1/4 \text{ and } 6\varepsilon \leq h_{\max}^2 \left( \|b\| \mathcal{Q} + M \right), \\ C(\tau + (N^{-1} \ln N)^4), & \text{otherwise}. \end{cases} \]

**Proof.** The global error satisfies
\[ ||u - U^n_j|| \leq ||u - z^n_j|| + ||z - U^n_j|| \]
and therefore the result follows from Theorems 2 and 4. \( \square \)

**Remark 3.** From Theorem 5, it follows that the error satisfies the bound
\[ ||u - U^n_j|| \leq \begin{cases} C(\tau + N^{-4} M), & \text{if } \sigma \neq 1/4 \text{ and } 6\varepsilon \leq h_{\max}^2 \left( \|b\| \mathcal{Q} + M \right), \\ C(\tau + (N^{-1} \ln N)^4), & \text{otherwise}. \end{cases} \]
If the discretization parameters \( N \) and \( M \) have the same order of magnitude then we deduce third order of uniform convergence in space for scheme (40). Moreover, if they satisfy the restriction
then the almost fourth order convergence in space is obtained.

4. Numerical experiments

The first test problem is:

\[
\left\{ \begin{array}{l}
  u_t - \varepsilon u_{xx} + (1 + xe^{-t})u = f(x,t), \quad (x,t) \in (0,1) \times (0,1], \\
  u(x,0) = 0, \quad x \in \Omega, \\
  u(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,1],
\end{array} \right.
\]

where \( f \) is taken such that the exact solution is

\[
u(x,t) = \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1 + e^{-1/\varepsilon}} - \cos^2(\pi x).
\]

Fig. 2 shows the numerical solution for \( \varepsilon = 10^{-6} \); from it we clearly see the boundary layers at both edges \( x = 0 \) and \( x = 1 \). We consider this problem to analyze the influence of the space discretization on the global error. From the error \( E_{e,N,M}^{N,M} = |U_{e,N,M} - u(x_i,t_n)| \), at each grid point, we compute the maximum global errors and the numerical orders of convergence by

\[
E_{e,N,M}^{N,M} = \max_{i=1}^{N} E_{e,N,M}^{i}, \quad p = \log \left( \frac{E_{e,N,M}^{N,M}}{E_{e,2N,2M}^{N,M}} \right) / \log 2.
\]

We denote the \( e \)-uniform errors and the \( e \)-uniform orders of convergence by

\[
E_{e,N,M}^{N,M} = \max_{i=1}^{N} E_{e,N,M}^{i}, \quad p_{uni} = \log \left( \frac{E_{e,N,M}^{N,M}}{E_{e,2N,2M}^{N,M}} \right) / \log 2,
\]

where we have chosen \( S_e = \{ 2^{-6}, 2^{-8}, \ldots, 2^{-30} \} \) in the numerical experiments performed. Table 1 displays the results of method (40) on the Vulanović mesh for some values of the discretization and singular perturbation parameters, varying simultaneously, with the same ratio, the values of \( N \) and \( M \); note that these values satisfy condition (41), which is imposed in Theorem 5, and also condition (42) given in Remark 3. From Table 1 we can deduce the almost fourth order convergence of the method in agreement with Theorem 5. Note that the contribution to the global error of the error corresponding to the time discretization is negligible in this case. To analyze the efficiency of this method we compare the maximum errors associated to method (40) with those ones for the basic method given in [6]. This method combines the implicit Euler method in time with central difference approximation in space. Table 2 displays the results obtained with that method and we observe that the maximum errors are considerably larger than in Table 1. So, we can conclude that the hybrid method (40) is more efficient than the method used in [6].

To study the influence of the two discretization parameters \( N \) and \( M \) on the global error of the method, we calculate the numerical errors fixing one of these parameters and varying the other one. Table 3 displays the \( e \)-uniform errors and the
This problem is similar to the first example, but now we want to show that the error associated to the time discretization dominates over the global error of the hybrid scheme.

The second example is given by

\[
\begin{cases}
  u_t - \varepsilon u_{xx} + (1 + x e^{-t})u = f(x, t), & (x, t) \in (0, 1) \times (0, 1), \\
  u(x, 0) = 0, & x \in \Omega, \\
  u(0, t) = 0, & u(1, t) = 0, & t \in (0, 1),
\end{cases}
\]

where \( f \) is taken such that now the exact solution is

\[
u(x, t) = (1 - e^{-t}) \left( \frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x) \right).
\]

This problem is similar to the first example, but now we want to show that the error associated to the time discretization dominates over the global error of the hybrid scheme.

### Table 1

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 64 )</th>
<th>( N = 128 )</th>
<th>( N = 256 )</th>
<th>( N = 512 )</th>
<th>( N = 1024 )</th>
<th>( N = 2048 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M = 8 )</td>
<td>0.136E–6, 4.011</td>
<td>0.846E–8, 4.005</td>
<td>0.527E–9, 4.003</td>
<td>0.329E–10, 4.001</td>
<td>0.205E–11, 3.934</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( M = 16 )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
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<tr>
<td>( M = 32 )</td>
<td>( \ldots )</td>
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<td>( M = 64 )</td>
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<td>( M = 128 )</td>
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<tr>
<td>( M = 256 )</td>
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<td>( \ldots )</td>
<td>( \ldots )</td>
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<td>( \ldots )</td>
</tr>
</tbody>
</table>

\( \varepsilon \)-uniform orders of convergence taking \( M = 64 \) with the same values of \( N \) and \( \varepsilon \) as before. From these results the almost fourth order of uniform convergence is observed. Table 4 displays the \( \varepsilon \)-uniform errors and the \( \varepsilon \)-uniform orders of convergence taking \( N = 512 \) with the same values of \( M \) and \( \varepsilon \) as before, but now there is no convergence, even for this large value of \( N \). So, these results corroborate that, in this example, the main contribution to the global error comes from the spatial discretization.

### Table 2

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N = 64 )</th>
<th>( N = 128 )</th>
<th>( N = 256 )</th>
<th>( N = 512 )</th>
<th>( N = 1024 )</th>
<th>( N = 2048 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M = 8 )</td>
<td>0.104E–3, 2.013</td>
<td>0.258E–4, 2.007</td>
<td>0.642E–5, 2.004</td>
<td>0.160E–5, 2.002</td>
<td>0.399E–6, 2.001</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( M = 16 )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
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<td>( \ldots )</td>
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<tr>
<td>( M = 32 )</td>
<td>( \ldots )</td>
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<td>( \ldots )</td>
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<tr>
<td>( M = 64 )</td>
<td>( \ldots )</td>
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<tr>
<td>( M = 128 )</td>
<td>( \ldots )</td>
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<td>( \ldots )</td>
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<tr>
<td>( M = 256 )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
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<td>( \ldots )</td>
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</tr>
<tr>
<td>( p^{NM} )</td>
<td>0.521E–3, 3.084</td>
<td>0.615E–4, 3.193</td>
<td>0.672E–5, 3.311</td>
<td>0.677E–6, 3.392</td>
<td>0.645E–7, 3.450</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( p^{MM} )</td>
<td>0.521E–3, 3.084</td>
<td>0.615E–4, 3.193</td>
<td>0.672E–5, 3.311</td>
<td>0.677E–6, 3.392</td>
<td>0.645E–7, 3.450</td>
<td>( \ldots )</td>
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</table>

\( \varepsilon \)-uniform errors and uniform orders of convergence for problem (43) using the hybrid method (40).
Table 5 displays the results of the compact scheme (40) on the Vulanović mesh where the space and time discretization parameters $N$ and $M$ are both multiplied by 2. From these results we only observe the first global order of uniform convergence.

As in the first example, we give the results fixing either $M$ or $N$, taking now $M = 300$ in Table 6 and $N = 64$ in Table 7, respectively. Now, in spite of the chosen values of $N$ and $M$, the opposite effect is observed, i.e., the method does not converge when $M$ is fixed but first order convergence is observed when $N$ is fixed.

Tables 5–7 confirm that for this example the global error is dominated by the errors corresponding to the time discretization and we cannot observe the errors corresponding to the space discretization. So, in general, the hybrid method gives global errors similar to the ones of the basic scheme when the time discretization errors are large.
We can conclude that the hybrid method is better than the basic scheme. First, both methods have the same asymptotic computational cost; second, the hybrid method never gives worse errors than the basic scheme and almost fourth order convergence is observed for problems where the main contribution to the global error is due to the space discretization. Besides, the hybrid scheme can be combined with other techniques (as, for example, defect-correction or Richardson extrapolation) to increase the order of convergence in time, providing high order convergent methods in both the time and space variables.

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