Exercises on the affine Grassmannian and their applications

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This is a working draft and may change without notice. Any feedback (even in the form of small typos) is much appreciated! – BWH

1. Loop Groups

Exercise 1.1. (Loop groups and $\mathrm{Gr}_{\mathbf{G}_m}$)

(i) Let R be a commutative ring with connected spectrum (i.e. R has no idempotents other than 0, 1). Show that every invertible element $f(t)$ of $R((t))$ can be uniquely written in the form

$$f(t) = r \cdot t^n \cdot f_+(t) \cdot f_-(t),$$

for some $r \in R^\times$, some $n \in \mathbf{Z}$, and where

$$f_+(t) = 1 + \sum_{i=1}^{\infty} r_i t^i \in R[\![t]\!]$$

and

$$f_-(t) = 1 + \sum_{i=1}^{\infty} r_{-i} t^{-i} \in R[t^{-1}]$$

where each r_{-i} is nilpotent.

(ii) Let $\mathbb{W}$ be the presheaf that assigns to such a commutative ring R the set of power series of the form $f_+(t)$ in part (i). Show that $\mathbb{W}$ is represented by a group subscheme (no ind!) of the loop group $L\mathbf{G}_m$. (The commutative ring $\mathbb{W}$ is called the ring of big Witt vectors or "Witt vectors in the sense of Cartier," in contrast to the usual or "p-typical" Witt vectors that may be familiar from the theory of local fields.)

Similarly, let $\widehat{\mathbb{W}}$ be the presheaf that assigns to each such R the set of polynomials of the form $f_-(t)$ in part (i). Show that $\widehat{\mathbb{W}}$ is represented by a group sub-ind-scheme of $L\mathbf{G}_m$.

(iii) Show that as group ind-schemes, we have

$$L\mathbf{G}_m \simeq \mathbf{G}_m \times \mathbf{Z} \times \mathbb{W} \times \widehat{\mathbb{W}}.$$

In particular, this implies that $\mathrm{Gr}_{\mathbf{G}_m} \simeq \mathbf{Z} \times \widehat{\mathbb{W}}$ is not reduced.

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(iv) Show that $\mathrm{Gr}_{\mathbf{G}_m}$ is formally smooth. (This also holds for general G.)

(v) Show that the morphism $L\mathbf{G}_m \to L\mathbf{A}^1$ is not an open embedding.

Remark 1. A recurring phenomenon in geometric representation theory is exhibited in (iv) and (v) above: morally speaking, the different connected components of $L\mathbf{G}_m$ (labeled by $\mathbf{Z}$) glue together nicely. In this case, the concrete fact underlying the result is that if k is a field and k' is a field containing k, then there is a canonical isomorphism $L\mathbf{G}_m(k') = L\mathbf{A}^1(k)\backslash\{0\}$ that holds at the level of k' points.

Exercise 1.2. Show that LSL_2 (the functor sending a k-algebra R to $SL_2(R((z)))$) and $L^{poly}SL_2$ (the functor sending a k-algebra R to $SL_2(R[z^{\pm}])$) are group objects in the category of k-spaces and the category of ind-schemes. (*Suggestion*: you can do this categorically, without using equations.)

2. Proto-example: The Bruhat–Tits tree for $SL_2(K)$ for K a non-archimedean local field

A reference for this is Serre's *Trees* (1980), §II. Let K be an arbitrary non-archimedean local field (assume that it has finite residue field) and let X denote the Bruhat–Tits tree of $SL_2(K)$. Recall that its vertices can be identified with $PGL_2(K)/PGL_2(\mathcal{O}_K)$ or equivalently, the space of $\mathcal{O}$-lattices in K^2, up to homothety.

Exercise 2.1. (Symmetry of the Edge Relation on X) Let v and w be vertices in X such that there exist representatives $\Lambda_v \in v$ and $\Lambda_w \in w$ such that $\Lambda_v \supset \Lambda_w \supset \pi\Lambda_v$, where π is a uniformizer of $\mathcal{O}_K$. Show that there exists $\Lambda'_v \in v$ and $\Lambda'_w \in w$ such that $\Lambda'_w \supset \Lambda'_v \supset \pi\Lambda'_w$.

Exercise 2.2. (Simple connectivity) We saw in class that X is connected. (Work out the details to really nail it down.) Prove that X is also simply connected: that is, between any two vertices $v, w \in X$, there exists a *unique* path from v to w, without assuming the fact that X is a tree.

Exercise 2.3. (Hecke operators)

(i) Fix any vertex $v_0 \in X$. For any integer $n \geq 1$, prove that the number of vertices of X of distance n from v_0 is $q^{n-1}(q+1)$.

(ii) For any $n \geq 0$, let Θ_n denote the operator on complex functions f on X, where for any vertex $v \in X$, we have

$$\Theta_n f(v) = \sum_{\substack{w \in X \\ d(v,w)=n}} f(w).$$

Prove the following identities:

$$\Theta_1\Theta_1 = \Theta_2 + (q+1)\Theta_0$$
$$\Theta_1\Theta_n = \Theta_{n+1} + q\Theta_{n-1} \quad (n \geq 2).$$

(iii) Define

$$T_0 = \Theta_0 = \mathrm{id}$$
$$T_1 = \Theta_1$$
$$T_n = \sum_{0 \le i \le n/2} \Theta_{n-2i} = \Theta_n + T_{n-2}.$$

Show that

$$T_1 T_n = T_{n+1} + q T_{n-1}$$

for any $n \ge 1$.

(iv) Deduce that T_n and Θ_n are polynomials in T_1. Prove the identity

$$\sum_{n=0}^{\infty} T_n x^n = \frac{1}{1 - T_1 x + q x^2},$$

where x denotes an indeterminate variable.

3. Examples

Exercise 3.1. Let $\mathbf{G}_a$ denote the additive group. Show that $\mathrm{Gr}_{\mathbf{G}_a} \simeq \varinjlim \mathbf{A}^n$, where $\mathbf{A}^n \hookrightarrow \mathbf{A}^{n+1}$ is the standard inclusion.

Exercise 3.2. (The affine Grassmannian for $GL_1 = \mathbf{G}_m$) Let R be an arbitrary commutative ring.

(i) Show that an element

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots \in R[\![t]\!]$$

is invertible if and only if a_0 is invertible in R.

(ii) Let $f = a_m t^m + a_{m+1} t^{m+1} + \cdots \in R(\!(t)\!)$ (so $m \in \mathbf{Z}$). Show that $f \in R(\!(t)\!)^\times$ if and only if there exists a k such that $a_m, a_{m+1}, \ldots, a_{m+k}$ are nilpotent and a_{m+k+1} is invertible. This implies in particular that if R is a field, then $R(\!(t)\!)$ is a field.

(iii) Show that as an ind-variety, $\mathrm{Gr}_{\mathbf{G}_m} \cong \mathbf{Z}$; in other words, it is a discrete countable set. Use part (ii) to show that $\mathrm{Gr}_{\mathbf{G}_m}$ is *not* an ind-reduced ind-scheme (i.e. not a colimit of reduced schemes).

Exercise 3.3. Let $K = k(\!(t)\!)$, where k is a field and $\mathcal{O} = k[\![t]\!]$. We say two $\mathcal{O}$-lattices $\Lambda, \Lambda' \subset K^n$ are **homothetic** if there exists an $\alpha \in K^\times$ such that $\Lambda = \alpha\Lambda$. Show that Gr_{PGL_n} is isomorphic to the set of $\mathcal{O}$-lattices up to homothety.

Exercise 3.4. Recall that the **valuation** $\nu(\Lambda)$ of an $\mathcal{O}$-lattice Λ is the minimal $n \in \mathbf{Z}$ such that t^n occurs in the determinant of a basis for Λ.

(i) Prove that the valuation of an $\mathcal{O}$-lattice Λ is well-defined.

(ii) Show that Gr_{SL_n} consist of the lattices of valuation 0. Conclude that Gr_{SL_n} is connected.

Exercise 3.5.

(i) Show that $\mathrm{Gr}_{\mathrm{PGL}_2}$ has two connected components.

(ii) Show that we have an injection $\mathrm{Gr}_{\mathrm{SL}_2} \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_2}$, even though we have a surjection $\mathrm{SL}_2 \twoheadrightarrow \mathrm{PGL}_2$.