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The Exact Linearization of Some Classes of Ordinary Differential Equations for Order $n > 2$

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"When integrating the differential equations the most difficult task is the introduction of suitable variables, which may not be found by the general rule. That's why we have to go in reverse order. After finding a splendid substitution, we should look for such problems, where it might be adopted with success."

Karl Jacobi

The method of exact linearization nonlinear ordinary differential equations (ODE) of order n suggested by one of the authors is demonstrated in [1, 2]. This method is based on the factorization of nonlinear ODE through the first order nonlinear differential the operators, and is also based on using both point and nonpoint, local and nonlocal transformations. Exact linearization of autonomous the third, the fourth and the fifth orders ODE is presented in this paper. For the first time general form of autonomous fourth [3] and fifth order equations, admitting exact linearization with using nonpoint transformation is found by second of the authors. We obtain the formulas in quadratures for finding general and partial solutions of investigated classes of equations. For the realization of transformations and construction of the considered equations we used the computer algebra system MAPLE.

1 Preliminary informations

The following result plays an important role in this paper:

Proposition 1.1 [1]. *The equation*

$$y^{(n)} = f\left(y, y', \ldots, y^{(n-1)}\right), \qquad (') = d/dx \tag{1.1}$$

by means of the invertible transformation

$$y = v(y)z, \qquad dt = u(y)dx, \tag{1.2}$$

where $v(y)$ and $u(y)$ are smooth functions in domain (x, y), reduces to linear autonomous form

$$z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) + c = 0, \qquad b_k, c = \text{const}, \tag{1.3}$$

if and only if (1.1) admits the factorization of the form

$$\prod_{k=1}^{n} \left[\frac{1}{u} D - \frac{v^*}{vu} y' - r_k\right] y + cv = 0, \qquad D = d/dx, \quad (*) = d/dy \tag{1.4}$$

via nonlinear first order differential operators (commutative factorization) or

$$\prod_{k=n}^{1}\left[D-\left(\frac{v^*}{v}+(k-1)\frac{u^*}{u}\right)y'-r_k u\right]y+cu^n v=0 \tag{1.5}$$

(noncommutative factorization), where r_k *are distinct roots of the characteristic equation*

$$r^n+\sum_{k=1}^{n}\binom{n}{k}b_k r^{n-k}=0. \tag{1.6}$$

In what follows we shall sequentially consider the corresponding class of ODE for the cases $n=3$, $n=4$, and $n=5$. Although first two cases are known [1–3], they are considered here because they form the base of investigation of 5th order ODE.

For finding the transformation (1.2) we use also the proposition about the structure of basis of solutions of linear ODE with variable coefficients.

Proposition 1.2 [2]. *The second order nonlinear nonautonomous equation*

$$v^{**}-\frac{2}{v}v^{*2}+\left(\frac{2}{y}-\frac{n^2-n+2}{2n}\frac{u^*}{u}-f\right)v^*+\left(\frac{n^2-n+2}{2n}\frac{u^*}{u}+f\right)\frac{1}{y}v=0$$

has the solution

$$v(y)=y\left(\alpha+\beta\int u^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy\right)^{-1}.$$

2 Linearization of the third order equations

Let us consider the equation

$$y'''=F\left(y,y',y''\right). \tag{2.1}$$

By virtue of Proposition 1.1, it can be reduced to the linear ODE

$$\dddot{z}+3b_1\ddot{z}+3b_2\dot{z}+b_3 z+c=0 \tag{2.2}$$

by transformation (1.2), if and only if, it admits the factorization (up to the term cu^3v):

$$\left[D-\left(\frac{v^*}{v}+2\frac{u^*}{u}\right)y'-r_3u\right]\left[D-\left(\frac{v^*}{v}+\frac{u^*}{u}\right)y'-r_2u\right]\left[D-\frac{v^*}{v}y'-r_1u\right]y+cvu^3=0. \tag{2.3}$$

We obtain the differential equation

$$\begin{aligned}
&y'''\left(1-\frac{v^*}{v}y\right)-y''y'\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]+y'^3\left[\left(1-\frac{v^*}{v}y\right)\right.\\
&\quad\times\left(6\frac{{v^*}^2}{v^2}+6\frac{u^*v^*}{uv}+3\frac{{u^*}^2}{u^2}-2\frac{v^{**}}{v}-\frac{u^{**}}{u}\right)+4\frac{v^*v^{**}}{v^2}y+3\frac{u^*v^{**}}{uv}y-\frac{v^{**}}{v}-\left.\frac{v^{***}}{v}y\right]\\
&\quad-u(r_1+r_2+r_3)\left\{y''\left(1-\frac{v^*}{v}y\right)-y'^2\left[\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(2\frac{v^*}{v}+\frac{u^*}{u}\right)\right]\right\}\\
&\quad+(r_1r_2+r_1r_3+r_2r_3)u^2y'\left(1-\frac{v^*}{v}y\right)-r_1r_2r_3u^3y+cvu^3=0.
\end{aligned} \tag{2.4}$$

For the sake of determination of the explicit form of transformation (1.2), let us introduce the notation

$$-\left[3\frac{v^{**}}{v}y+\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\left(1-\frac{v^*}{v}y\right)\right]=3f(y)\left(1-\frac{v^*}{v}y\right), \tag{2.5}$$

from where we'll obtain the equation for the transformation factor $v(y)$ in (1.2):

$$v^{**}-2\frac{v^{*2}}{v}+\left(\frac{2}{y}-\frac{4}{3}\frac{u^*}{u}-f\right)v^*+\frac{1}{y}\left(\frac{4}{3}\frac{u^*}{u}+f\right)v=0, \qquad f=f(y). \tag{2.6}$$

Proposition 2.1. *Equation (2.6) has general solution, expressed through quadratures*

$$v(y)=\frac{y}{\alpha+\beta\int u^{4/3}\exp\left(\int f dy\right)dy}, \tag{2.7}$$

where α, β are integration constants.

The formulas (2.6) and (2.7) could be obtained from Proposition 1.2 and [2].

Proposition 2.2. *Equation (2.1) can be linearized by means of the transformation of type (1.2), if and only if it has the following form:*

$$\begin{gathered} y'''+3fy'y''+\left(\frac{1}{3}\frac{\varphi^{**}}{\varphi}-\frac{5}{9}\frac{\varphi^{*2}}{\varphi^2}-\frac{1}{3}f\frac{\varphi^*}{\varphi}+f^2+f^*\right)y'^3+3b_1\varphi y''+b_1\varphi\left(3f+\frac{\varphi^*}{\varphi}\right)y'^2 \\ +3b_2\varphi^2y'+\varphi^{5/3}\left[b_3\exp\left(-\int f dy\right)\int\varphi^{4/3}\exp\left(\int f dy\right)dy+\frac{c}{\beta}\right]=0, \end{gathered} \tag{2.8}$$

Such an equation can be linearized by means of the transformation

$$z=\beta\int\varphi^{4/3}\exp\left(\int f dy\right)dy, \qquad dt=\varphi dx, \tag{2.9}$$

where β is normalizing factor.

For $c=0$ we shall obtain the one-parameter solutions sets

$$\frac{\int\varphi^{1/3}\exp\left(\int f dy\right)dy}{\int\varphi^{4/3}\exp\left(\int f dy\right)dy}=r_kx+C, \tag{2.10}$$

where r_k are the simple roots of characteristic equation (2.11):

$$r^3+3b_1r^2+3b_2r+b_3=0. \tag{2.11}$$

Transforming (2.3) with the aid of (2.4), (2.7) (where $\alpha=0$), assuming $u=\varphi(y)$ and assuming that $r_k,\ k=1,2,3$ are the distinct roots of the characteristic equation (2.11), we shall arrive at (2.8).

The special case of equation (2.8) is obtained for $\varphi=\exp\left(-\frac{3}{4}\int f dy\right)$.

Remark 1. Thus, the equations of type

$$y'''+f(y)y'y''+\varphi(y)y''+\sum_{k=0}^{3}f_k(y)y'^k=0 \tag{2.12}$$

may be tested by the method of exact linearization.

Example 1. The equation

$$y''' - \frac{y'y''}{y} + 3b_1 yy'' + 3b_2 y^2 y' + \frac{1}{2} b_3 y^4 + \frac{c}{2} y^2 = 0 \tag{2.13}$$

by the substitution

$$z = y^2, \qquad dt = y dx \tag{2.14}$$

is reducible to the linear equation of the form (2.2) and has one-parameter set of solutions of the form

$$y = -2/(r_k x + C), \tag{2.15}$$

where r_k are simple roots of the characteristic equation (2.11).

Example 2. The elementary nontrivial system of hydrodynamic type (so-called the triplet). It was shown in [4] that corresponding system can be transformed into the set of Euler equations (the Euler–Poinsot case of a rigid body dynamics), which can be written in in terms of energy variables as

$$\dot{u}_1 = a u_2 u_3, \qquad \dot{u}_2 = b u_3 u_1, \qquad \dot{u}_3 = c u_1 u_2, \qquad a + b + c = 0. \tag{2.16}$$

Eliminating the variables of the coupled system (2.16), we obtained decoupled system

$$\begin{aligned} &\dddot{u}_1 - \frac{1}{u_1} \dot{u}_1 \ddot{u}_1 - 4bc u_1^2 \dot{u}_1 = 0, \qquad \dddot{u}_2 - \frac{1}{u_2} \dot{u}_2 \ddot{u}_2 - 4ca u_2^2 \dot{u}_2 = 0, \\ &\dddot{u}_3 - \frac{1}{u_3} \dot{u}_3 \ddot{u}_3 - 4bc u_3^2 \dot{u}_3 = 0. \end{aligned} \tag{2.17}$$

The equations (2.17) are factorizables:

$$\left(D_t - \frac{\dot{u}_i}{u_i} - r_{3i} u_i \right) (D_t - r_{2i} u_i) \left(D_t - \frac{\dot{u}_i}{u_i} - r_{3i} u_i \right) u_i = 0, \qquad i = \overline{1,3},$$

where r_{ki}, $k = \overline{1,3}$, are roots of corresponding characteristic equations. After the transformations $u_i^2 = z_i$, $d\tau_i = u_i dt$ the system (2) reduces to the linear one (see also [1]):

$$z_1'''(\tau_1) - 4bc z_1'(\tau_1) = 0, \qquad z_2'''(\tau_2) - 4ac z_2'(\tau_2) = 0, \qquad z_3'''(\tau_3) - 4ab z_3'(\tau_3) = 0.$$

3 Linearization of the fourth-order equations

Let us consider the autonomous nonlinear fourth–order differential equation

$$y^{\text{iv}} = F\left(y, y', y'', y'''\right). \tag{3.1}$$

Equation (3.1) could be linearized by the transformation (1.2) to the form

$$\ddddot{z} + 4b_1 \dddot{z} + 6b_2 \ddot{z} + 4b_3 \dot{z} + b_4 z + c = 0, \tag{3.2}$$

according to Proposition 1.1, if and only if it admits the factorization (up to the term cvu^4)

$$\begin{aligned} &\left[D - \left(\frac{v^*}{v} + 3\frac{u^*}{u} \right) y' - r_4 u \right] \left[D - \left(\frac{v^*}{v} + 2\frac{u^*}{u} \right) y' - r_3 u \right] \\ &\quad \times \left[D - \left(\frac{v^*}{v} + \frac{u^*}{u} \right) y' - r_2 u \right] \left[D - \frac{v^*}{v} y' - r_1 u \right] y + cvu^4 = 0. \end{aligned} \tag{3.3}$$

Proposition 3.1. *If equation (3.1) can be linearized by means of (1.2), then it admits the factorization*

$$
\begin{aligned}
&\left(1-\frac{v^*}{v}y\right)y^{\mathrm{iv}}-y'y'''\left[4\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(8\frac{v^*}{v}+7\frac{u^*}{u}\right)\right]-y''^2\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\right.\\
&\left.\times\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]+y'^2y''\left[\left(1-\frac{v^*}{v}y\right)\left(36\frac{{v^*}^2}{v^2}-18\frac{v^{**}}{v}+44\frac{u^*v^*}{uv}+25\frac{{u^*}^2}{u^2}+7\frac{u^{**}}{u}\right)\right.\\
&\left.+18\frac{v^{**}v^*}{v^2}y+22\frac{u^*v^{**}}{uv}y-6\frac{v^{***}}{v}y\right]-y'^4\left[\frac{v^{****}}{v}y-\left(2\frac{v^{***}}{v}y-6\frac{v^{**}v^*}{v^2}y\right)\left(2\frac{v^*}{v}+3\frac{u^*}{u}\right)\right.\\
&-6\frac{{v^{**}}^2}{v^2}y+15\frac{v^{**}{u^*}^2}{vu^2}y-4\frac{v^{**}u^{**}}{uv}y+\left(1-\frac{v^*}{v}y\right)\left(4\frac{v^{***}}{v}-24\frac{v^{**}v^*}{v^2}+24\frac{{v^*}^3}{v^3}-18\frac{u^*v^{**}}{uv}\right.\\
&\left.\left.+36\frac{u^*{v^*}^2}{uv^2}+30\frac{{u^*}^2v^*}{u^2v}-8\frac{u^{**}v^*}{uv}-10\frac{u^*u^{**}}{u^2}+15\frac{{u^*}^3}{u^3}+\frac{u^{***}}{u}\right)\right]+4b_1u\left\{y'''\left(1-\frac{v^*}{v}y\right)\right.\\
&-y'^3\left[\frac{v^{***}}{v}y-3\frac{v^{**}u^*}{uv}y-3\frac{v^{**}v^*}{v^2}y+\left(1-\frac{v^*}{v}y\right)\left(\frac{u^{**}}{u}-6\frac{u^*v^*}{vu}+3\frac{v^{**}}{v}-6\frac{u{v^*}^2}{v^2}-3\frac{{u^*}^2}{u^2}\right)\right]\\
&\left.-y'y''\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]\right\}+6b_2u^2\left\{y''\left(1-\frac{v^*}{v}y\right)\right.\\
&\left.-y'^2\left[\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(\frac{u^*}{u}+2\frac{v^*}{v}\right)\right]\right\}+4b_3u^3y'\left(1-\frac{v^*}{v}y\right)+b_4u^4y+cvu^4=0.
\end{aligned}
\tag{3.4}
$$

Applying the differential operator $\left[D-\left(\frac{v^*}{v}+3\frac{u^*}{u}\right)y'-r_4u\right]$ to (2.4) and adding to the obtained expression term cvu^4, we shall arrive at the formula (3.4), where r_k satisfy the characteristic equation

$$r^4+4b_1r^3+6b_2r^2+4b_3r+b_4=0. \tag{3.5}$$

Introducing the notation $4\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(8\frac{v^*}{v}+7\frac{u^*}{u}\right)=-4f(y)\left(1-\frac{v^*}{v}y\right)$, we shall arrive at the equation

$$v^{**}-\frac{2}{v}{v^*}^2+\left(\frac{2}{y}-\frac{7}{4}\frac{u^*}{u}-f\right)v^*+\left(\frac{7}{4}\frac{u^*}{u}+f\right)\frac{1}{y}v=0,\qquad f=f(y). \tag{3.6}$$

Proposition 3.2. *Equation (3.6) has general solution of the form*

$$v(y)=\frac{y}{\alpha+\beta\int u^{7/4}\exp\left(\int f\,dy\right)dy}, \tag{3.7}$$

where α, β are arbitrary constants.

The formulas (3.6) and (3.7) could be obtained from Proposition 1.2 and [2].

Proposition 3.3. *The equation (3.1) can be linearized by means of transformation (1.2) if and only if it has the following form:*

$$y^{(\mathrm{iv})}+4f(y)y'y'''+y''^2\left(\frac{5}{4}\frac{\varphi^*}{\varphi}+3f\right)+y'^2y''\left(\frac{7}{2}\frac{\varphi^{**}}{\varphi}-\frac{45}{8}\frac{{\varphi^*}^2}{\varphi^2}-\frac{\varphi^*}{\varphi}f+6\left(f^2+f^*\right)\right)$$

$$
\begin{aligned}
&+y'^4\left(\frac{195}{64}\frac{\varphi^{*^3}}{\varphi^3}-\frac{57}{16}\frac{\varphi^*\varphi^{**}}{\varphi^2}+\frac{3}{4}\frac{\varphi^{***}}{\varphi}+\left(\frac{5}{4}\frac{\varphi^{**}}{\varphi}-\frac{33}{16}\frac{\varphi^{*^2}}{\varphi^2}\right)f+\frac{21}{4}\frac{\varphi^*}{\varphi}\left(f^2+f^*\right)\right.\\
&\left.+f^3+3ff^*+f^{**}\right)+4b_1\varphi\left\{y'''+y'y''\left(3f+\frac{5}{4}\frac{\varphi^*}{\varphi}\right)-y'^3\left(\frac{15}{16}\frac{\varphi^{*^2}}{\varphi^2}-\frac{3}{4}\frac{\varphi^{**}}{\varphi}\right.\right.\\
&\left.\left.-\frac{1}{2}\frac{\varphi^*}{\varphi}f-f^2-f^*\right)\right\}+6b_2\varphi^2\left\{y''+y'^2\left(\frac{3}{4}\frac{\varphi^*}{\varphi}+f\right)\right\}+4b_3\varphi^3y'\\
&+\varphi^{\frac{9}{4}}e^{-\int f dy}\left[b_4\int\varphi^{\frac{7}{4}}e^{\int f dy}dy+\frac{c}{\beta}\right]=0.
\end{aligned}
\tag{3.8}
$$

It can be reduced to (3.2) by means of the transformation

$$
z=\beta\int\varphi^{7/4}\exp\left(\int f(y)dy\right)dy,\qquad dt=\varphi(y)dx. \tag{3.9}
$$

Transforming (3.3) with the aid of (3.4), (3.6) (where $\alpha=0$, assuming that $\varphi=u(y)$ and that r_k, $k=\overline{1,4}$ are distinct roots of the characteristic equation (3.5), we shall arrive at (3.8).

Equation (3.8) is recovered as the particular case of the above for $\varphi=\exp\left(-\frac{4}{7}\int f(y)dy\right)$.

Remark 2. Thus, the equations of the type

$$
y^{\text{iv}}+fy'y'''+\varphi_1y''^2+\varphi_2y'^2y''+f_4y'^4+\varphi y'''+\varphi_3y'y''+\varphi_4y''+f_3y'^3+f_2y'^2+f_1y'+f_0=0
$$

may be tested by the method of exact linearization.

Example 3. The equation

$$
y^{\text{iv}}-\frac{3}{y}y'y'''-\frac{1}{y}y''^2+\frac{3}{y^2}y'^2y''+4b_1yy'''-4b_1y'y''+6b_2y^2y''+4b_3y^3y'+\frac{1}{2}b_4y^5+\frac{1}{2}cy^4=0
$$

is reduced to (3.2) by means of the substitution (2.14) and admits for $c=0$ one-parameter set of solutions (2.15), where r_k are different characteristic roots of the equation (3.5).

4 Linearization of the fifth-order equation

Let us consider the autonomous nonlinear fifth-order differential equation

$$
y^{\text{v}}=F\left(y,y',y'',y''',y^{\text{iv}}\right). \tag{4.1}
$$

Equation (4.1) can be reduced by means of transformation (1.2) to the linear equation of the form

$$
z^{(\text{v})}(t)+5b_1z^{(\text{iv})}(t)+10b_2z'''(t)+10b_3z''(t)+5b_4z'(t)+b_5z(t)+c=0, \tag{4.2}
$$

by virtue of Proposition 1.1 if and only if it admits the factorization (up to the term cvu^5)

$$
\begin{aligned}
&\left[D-\left(\frac{v^*}{v}+4\frac{u^*}{u}\right)y'-r_5u\right]\left[D-\left(\frac{v^*}{v}+3\frac{u^*}{u}\right)y'-r_4u\right]\left[D-\left(\frac{v^*}{v}+2\frac{u^*}{u}\right)y'-r_3u\right]\\
&\times\left[D-\left(\frac{v^*}{v}+\frac{u^*}{u}\right)y'-r_2u\right]\left[D-\frac{v^*}{v}y'-r_1u\right]y+cvu^5=0.
\end{aligned}
\tag{4.3}
$$

Proposition 4.1. *If equation (4.1) can be linearized with the aid of (1.2), then there exists the factorization*

$$
\begin{aligned}
&\left(1-\frac{v^*}{v}y\right)y^{\mathrm{v}}-y'y^{\mathrm{iv}}\left[5\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(10\frac{v^*}{v}+11\frac{u^*}{u}\right)\right]-5y''y'''\left[2\frac{v^{**}}{v}y\right.\\
&\left.+\left(1-\frac{v^*}{v}y\right)\left(4\frac{v^*}{v}+3\frac{u^*}{u}\right)\right]+y'^2y'''\left[\left(1-\frac{v^*}{v}y\right)\left(60\frac{v^{*2}}{v^2}-30\frac{v^{**}}{v}+90\frac{u^*v^*}{uv}+60\frac{u^{*2}}{u^2}\right.\right.\\
&\left.\left.-14\frac{u^{**}}{u}\right)+30\frac{v^{**}v^*}{v^2}y+45\frac{u^*v^{**}}{uv}y-10\frac{v^{***}}{v}y\right]y'y''^2\left[\left(1-\frac{v^*}{v}y\right)\left(90\frac{v^{*2}}{v^2}-45\frac{v^{**}}{v}\right.\right.\\
&\left.\left.+120\frac{u^*v^*}{uv}+70\frac{u^{*2}}{u^2}-18\frac{u^{**}}{u}\right)+45\frac{v^{**}v^*}{v^2}y+60\frac{u^*v^{**}}{uv}y-15\frac{v^{***}}{v}y\right]-y'^3y''\left[10\frac{v^{****}}{v}y\right.\\
&-10\left(\frac{v^{***}}{v}y-3\frac{v^{**}v^*}{v^2}y\right)\left(4\frac{v^*}{v}+7\frac{u^*}{u}\right)-60\frac{v^{**2}}{v^2}y+195\frac{v^{**}u^{*2}}{vu^2}y-45\frac{v^{**}u^{**}}{uv}y\\
&+\left(1-\frac{v^*}{v}y\right)\left(40\frac{v^{***}}{v}-240\frac{v^{**}v^*}{v^2}+240\frac{v^{*3}}{v^3}-210\frac{u^*v^{**}}{uv}+420\frac{u^*v^{*2}}{uv^2}+390\frac{u^{*2}v^*}{u^2v}\right.\\
&\left.\left.-90\frac{u^{**}v^*}{uv}-125\frac{u^*u^{**}}{u^2}+210\frac{u^{*3}}{u^3}+11\frac{u^{***}}{u}\right)\right]-y'^5\left[\frac{v^{*****}}{v}y-5\left(\frac{v^{****}}{v}y-4\frac{v^{***}v^*}{v^2}y\right.\right.\\
&\left.+12\frac{v^{**}v^{*2}}{v^3}y-6\frac{v^{**}u^{**}}{uv}y\right)\left(\frac{v^*}{v}+2\frac{u^*}{u}\right)+60\frac{v^{**2}v^*}{v^3}y-20\frac{v^{***}v^{**}}{v^2}y-10\frac{v^{***}u^{**}}{vu}y\\
&-5\frac{v^{**}u^{***}}{vu}y+60\frac{v^{**2}u^*}{v^2u}y+45\frac{v^{***}u^{*2}}{u^2v}y-135\frac{v^{**}v^*u^{*2}}{v^2u^2}y-105\frac{v^{**}u^{*3}}{vu^3}y+\left(1-\frac{v^*}{v}y\right)\\
&\times\left(5\frac{v^{****}}{v}-40\frac{v^{***}v^*}{v^2}-30\frac{v^{**2}}{v^2}+180\frac{v^{**}v^{*2}}{v^3}-120\frac{v^{*4}}{v^4}-30\frac{u^{**}v^{**}}{uv}-40\frac{u^*v^{***}}{uv}\right.\\
&+135\frac{v^{**}u^{*2}}{u^2v}+240\frac{v^{**}v^*u^*}{uv^2}+60\frac{u^{**}v^{*2}}{uv^2}-270\frac{u^{*2}v^{*2}}{u^2v^2}+120\frac{u^*u^{**}v^*}{u^2v}-210\frac{u^{*3}v^*}{u^3v}\\
&\left.\left.-10\frac{u^{***}v^*}{uv}-240\frac{v^{*3}u^*}{v^3u}-10\frac{u^{**2}}{u^2}-15\frac{u^*u^{***}}{u^2}+105\frac{u^{*2}u^{**}}{u^3}-105\frac{u^{*4}}{u^4}+\frac{u^{****}}{u}\right)\right]\\
&+5b_1u\left\{y^{(\mathrm{iv})}\left(1-\frac{v^*}{v}y\right)-y'y'''\left[4\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(8\frac{v^*}{v}+7\frac{u^*}{u}\right)\right]+y'^2y''\left[\left(1-\frac{v^*}{v}y\right)\right.\right.\\
&\left.\times\left(36\frac{v^{*2}}{v^2}-18\frac{v^{**}}{v}+44\frac{u^*v^*}{uv}+25\frac{u^{*2}}{u^2}-7\frac{u^{**}}{u}\right)+18\frac{v^{**}v^*}{v^2}y+22\frac{u^*v^{**}}{uv}y-6\frac{v^{***}}{v}y\right]\\
&-y''^2\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]-y'^4\left[\frac{v^{****}}{v}y-\left(2\frac{v^{***}}{v}y-6\frac{v^{**}v^*}{v^2}y\right)\right.\\
&\times\left(2\frac{v^*}{v}+3\frac{u^*}{u}\right)-6\frac{v^{**2}}{v^2}y+15\frac{v^{**}u^{*2}}{vu^2}y-4\frac{v^{**}u^{**}}{uv}y+\left(1-\frac{v^*}{v}y\right)\left(4\frac{v^{***}}{v}-24\frac{v^{**}v^*}{v^2}\right.\\
&\left.\left.\left.+24\frac{v^{*3}}{v^3}-18\frac{u^*v^{**}}{uv}+36\frac{u^*v^{*2}}{uv^2}+30\frac{u^{*2}v^*}{u^2v}-8\frac{u^{**}v^*}{uv}-10\frac{u^*u^{**}}{u^2}+15\frac{u^{*3}}{u^3}+\frac{u^{***}}{u}\right)\right]\right\}\\
&+10b_2u^2\left\{y'''\left(1-\frac{v^*}{v}y\right)-y'y''\left[3\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(6\frac{v^*}{v}+4\frac{u^*}{u}\right)\right]-y'^3\left[\frac{v^{***}}{v}y-3\frac{v^{**}u^*}{uv}y\right.\right.
\end{aligned}
\tag{4.4}
$$

$$-3\frac{v^{**}v^*}{v^2}y+\left(1-\frac{v^*}{v}y\right)\left(3\frac{v^{**}}{v}-6\frac{{v^*}^2}{v^2}-6\frac{u^*v^*}{uv}-3\frac{{u^*}^2}{u^2}+\frac{u^{**}}{u}\right)\Bigg]\Bigg\}+10b_3u^3\left\{y''\left(1-\frac{v^*}{v}y\right)\right.$$
$$\left.-y'^2\left[\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(2\frac{v^*}{v}+\frac{u^*}{u}\right)\right]\right\}+5b_4u^4y'\left(1-\frac{v^*}{v}y\right)+b_5u^5y+cvu^5=0.$$

Applying the differential operator $\left[D-\left(\frac{v^*}{v}+4\frac{u^*}{u}\right)y'-r_5u\right]$ to (3.3) (up to the term cvu^4) and adding to the obtained expression the term cvu^5, we arrive at (4.4), where r_k satisfy characteristic equation

$$r^5+5b_1r^4+10b_2r^3+10b_3r^2+5b_4r+b_5=0. \tag{4.5}$$

Introducing the notation $\frac{v^{**}}{v}y+\left(1-\frac{v^*}{v}y\right)\left(10\frac{v^*}{v}+11\frac{u^*}{u}\right)=-5f(y)\left(1-\frac{v^*}{v}y\right)$, we shall arrive at the equation

$$v^{**}-\frac{2}{v}{v^*}^2+\left(\frac{2}{y}-\frac{11}{5}\frac{u^*}{u}-f\right)v^*+\left(\frac{11}{5}\frac{u^*}{u}+f\right)\frac{1}{y}v=0,\qquad f=f(y). \tag{4.6}$$

Proposition 4.2. *Equation (4.6) has general solution of the form*

$$v(y)=\frac{y}{\alpha+\beta\displaystyle\int u^{11/5}\exp\left(\int f dy\right)dy}, \tag{4.7}$$

α, β are arbitrary constants.

The formulas (4.6) and (4.7) could be obtained from Proposition 1.2 and [2].

Equation (4.4)is the particular case of the above for $\varphi=\exp\left(-\frac{5}{11}\int f dy\right)$.

Proposition 4.3. *Equation (4.1) can be linearized by means of the transformation of the form (1.2) if and only if it has the form:*

$$y^{(\mathrm{v})}+5f(y)y'y^{(\mathrm{iv})}+y''y'''\left(7\frac{\varphi^*}{\varphi}+10f\right)+y'^2y'''\left[8\frac{\varphi^{**}}{\varphi}-\frac{63}{5}\frac{{\varphi^*}^2}{\varphi^2}-\frac{\varphi^*}{\varphi}f+10\left(f^2+f^*\right)\right]$$
$$+y'y''^2\left[15\frac{\varphi^{**}}{\varphi}-\frac{112}{5}\frac{{\varphi^*}^2}{\varphi^2}+6\frac{\varphi^*}{\varphi}f+15\left(f^2+f^*\right)\right]+y'^3y''\left[\frac{987}{25}\frac{{\varphi^*}^3}{\varphi^3}-\frac{244}{5}\frac{\varphi^*\varphi^{**}}{\varphi^2}+11\frac{\varphi^{***}}{\varphi}\right.$$
$$\left.+\left(21\frac{\varphi^{**}}{\varphi}-\frac{169}{5}\frac{{\varphi^*}^2}{\varphi^2}\right)f-4\frac{\varphi^*}{\varphi}\left(f^2+f^*\right)+10\left(f^3+3ff^*+f^{**}\right)\right]-y'^5\left[\frac{8064}{625}\frac{{\varphi^*}^4}{\varphi^4}\right.$$
$$-\frac{2946}{125}\frac{{\varphi^*}^2\varphi^{**}}{\varphi^3}+\frac{102}{25}\frac{{\varphi^{**}}^2}{\varphi^2}+\frac{186}{25}\frac{\varphi^*\varphi^{***}}{\varphi^2}-\frac{6}{5}\frac{\varphi^{****}}{\varphi}-\left(\frac{1989}{125}\frac{{\varphi^*}^3}{\varphi^3}-\frac{458}{25}\frac{\varphi^*\varphi^{**}}{\varphi^2}+\frac{19}{5}\frac{\varphi^{***}}{\varphi}\right)f \tag{4.8}$$
$$+\left(\frac{129}{25}\frac{{\varphi^*}^2}{\varphi^2}-\frac{16}{5}\frac{\varphi^{**}}{\varphi}\right)\left(f^2+f^*\right)+\frac{6}{5}\frac{\varphi^*}{\varphi}\left(f^3+3ff^*+f^{**}\right)-\left(f^4+6f^2f^*+3{f^*}^2+4ff^{**}\right.$$
$$\left.\left.+f^{***}\right)\right]+5b_1\varphi\left\{y^{(\mathrm{iv})}+y'y'''\left(4f+\frac{9}{5}\frac{\varphi^*}{\varphi}\right)+y'^2y''\left[\frac{31}{5}\frac{\varphi^{**}}{\varphi}-\frac{189}{25}\frac{{\varphi^*}^2}{\varphi^2}+\frac{22}{5}\frac{\varphi^*}{\varphi}f\right.\right.$$
$$\left.+6\left(f^2+f^*\right)\right]+y''^2\left(3f+\frac{13}{5}\frac{\varphi^*}{\varphi}\right)+y'^4\left[\frac{336}{125}\frac{{\varphi^*}^3}{\varphi^3}+\frac{118}{25}\frac{\varphi^*\varphi^{**}}{\varphi^2}+\frac{6}{5}\frac{\varphi^{***}}{\varphi}+\left(\frac{33}{5}\frac{\varphi^{**}}{\varphi}\right.\right.$$

$$-\frac{87}{25}\frac{\varphi^{*2}}{\varphi^2}\Bigg)f+\frac{3}{5}\frac{\varphi^*}{\varphi}\left(f^2+f^*\right)+\left(f^3+3ff^*+f^{**}\right)\Bigg]\Bigg\}+10b_2\varphi^2\left\{y'''+y'y''\left(3f+\frac{13}{5}\frac{\varphi^*}{\varphi}\right)\right.$$

$$\left.+y'^3\left[\frac{6}{5}\frac{\varphi^{**}}{\varphi}-\frac{24}{25}\frac{\varphi^{*2}}{\varphi^2}+\frac{7}{5}\frac{\varphi^*}{\varphi}f+\left(f^2+f^*\right)\right]\right\}+10b_3\varphi^3\left\{y''+y'^2\left(\frac{6}{5}\frac{\varphi^*}{\varphi}+f\right)\right\}+5b_4\varphi^4y'$$

$$+\varphi^{14/5}\exp\left(-\int f dy\right)\left[b_5\int\varphi^{11/5}\exp\left(\int f dy\right)dy+\frac{c}{\beta}\right]=0.$$

By means of the transformation

$$z=\beta\int\varphi^{11/5}\exp\left(\int f dy\right)dy, \qquad dt=\varphi dx \tag{4.9}$$

it reduces to the linear equation

$$z^v+5b_1z^{iv}+10b_2z'''+10b_3z''+5b_4z'+b_5z+c=0. \tag{4.10}$$

The equation (4.8) admits for $c=0$ *one-parameter set of solutions*

$$\frac{\int\varphi^{6/5}\exp\left(\int f dy\right)dy}{\int\varphi^{11/5}\exp\left(\int f dy\right)dy}=r_kx+C, \tag{4.11}$$

where r_k *are roots of the characteristic equation*

$$r^5+5b_1r^4+10b_2r^3+10b_3r^2+5b_4r+b_5=0. \tag{4.12}$$

Remark 4. Thus, the equations of the type

$$y^{\rm v}+fy'y^{\rm iv}+\varphi_1y''y'''+\varphi_2y'^2y'''+\varphi_3y'y''^2+\varphi_4y'^3y''+\varphi y^{\rm iv}+\varphi_5y'y'''+\varphi_6y''^2$$

$$+\varphi_7y'^2y''+\varphi_8y'''+\varphi_9y'y''+\varphi_{10}y''+\sum_{k=0}^{5}f_ky'^k=0$$

may be tested by the method of exact linearization.

Example 4. Equation

$$\begin{aligned}&y^{\rm v}-\frac{6}{y}y'y^{\rm iv}-\frac{5}{y}y''y'''+\frac{15}{y^2}y'^2y'''+\frac{10}{y^2}y'y''^2-\frac{15}{y^3}y'^3y''+5b_1yy^{\rm iv}-15b_1y'y'''-5b_1y''^2\\&+\frac{15}{y}b_1y'^2y''+10b_2y^2y'''-10b_2yy'y''+10b_3y^3y''+5b_4y^4y'+\frac{1}{2}b_5y^6+\frac{1}{2}cy^5=0\end{aligned} \tag{4.13}$$

by the substitution (2.14) is reduced to (4.2) and admits for $c=0$ one-parameter set of solutions (2.15), where r_k are distinct characteristic roots of the equation (4.5).

References

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