Stability of Networked Control Systems in the Presence of Packet Losses

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***Abstract*— In this paper we present a general framework for networked control systems, where all components are assumed to be connected through a communication network. We use the *uncertainty threshold principle* to show that under certain conditions there is a rate for dropped packets for which an undisturbed networked control system is mean square stable.**

I. Introduction

Networked control systems are systems whose sensors, actuators, estimator units, and control units are connected through communication networks [1] [2] [3] [4]. This type of system has the advantage of greater flexibility with respect to traditional control systems. Also, it allows for reduced wiring, as well as a lower installation cost. It also permits greater agility in diagnosis and maintenance procedures. Examples of such systems can be seen in aircrafts or manufacturing plants.

Unlike regular control systems, in networked control systems the synchronization between different sensors, actuators and control units is not guaranteed. Furthermore, there is no guarantee for zero delay or even constant delay in sending information from sensors to the control units and control signals from the control units to the actuators. When there is congestion in the communication network, some packets are dropped to either reduce the queue size in the path or to inform the senders to reduce their transmission rates [5] [6]. In real time systems, particularly control systems, delays or dropped packets may be catastrophic and may cause instability in the control system.

The communication network in the feedback loop makes the analysis and design of networked control systems complex. The tools and methods developed in conventional control theory are not enough for this analysis and should be modified to account for the additional complexity.

In [1] the stability of networked control systems for a continuous plant and a continuous controller is studied. The model of the network in this reference is a bus that all sensors and actuators have access to but only one sensor or actuator can access the bus at a time. There the authors introduce a protocol to schedule the use of shared resources and the modelled delay is due to this scheduling.

To reduce the required bandwidth for the networked control system in [2], the controller uses an estimate of the state of the system. When the error of the estimate is bigger than a certain threshold, the true value of the state of the system (or part of it) is transmitted. Similarly to [1], in [2] the network model is a dedicated network for the control systems and the packet dropping and packet delay is not considered in the analysis.

In [3] the stability of the control system is studied under the networked induced delay and dropped packets. In the analysis of stability in the presence of packet dropping, it is assumed that the full information of the state is transmitted through the network and it may be lost because of the dropped packets in the network. On the other hand, the control command is sent directly to the plant, therefore, the network is not used for transmitting the control signal. With these assumptions in the model of networked control systems, the authors use the stability analysis for asynchronous dynamical systems to find the maximum packet dropping rate under which the overall system is stable.

In [4] the stability of a linear networked control system in the presence of dropped packets is studied. As in [3], the controller is directly connected to the plant, therefore, there are no dropped packets for the control signal. The authors in [4] allow imperfect observation of the state of the system. Stability analysis for the networked control system in this reference is based on the stability of Markov jump linear systems.

In the present paper we give a general framework for networked control systems. Under this framework all components, including sensors, actuators, estimator(s), and controller(s), are assumed to be connected through a communication network. We do not assume that this network is dedicated to the control system, so other types of traffic are allowed to use the same media. We consider network delay in our analysis only inasmuch as it relates to dropping packets due to an extensive delay, and the effect of this on the stability of the system. Here we focus our attention on the packet dropping phenomena caused by network congestion avoidance mechanisms.

In Section II we present the problem setup based on a linear quadratic Gaussian control problem and we state the network model. We argue that the *separation theorem for*

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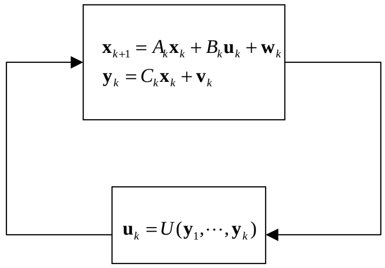

Fig. 1. Control system without network.

linear systems and quadratic cost [7] does not apply to the general framework of networked control systems. Therefore, we use a suboptimal method to simplify the calculation of the estimator and controller. In Section III we show that under certain conditions the existence of the solution to the Riccati-like equation and the mean square stability of the networked control system with perfect state information is guaranteed. In Section IV we give the conditions under which an undisturbed networked control system with imperfect state observation is mean square stable. In Section V we present numerical examples and simulations. Finally, in the last section we present our conclusions for this paper and we lay out future research.

II. Problem Definition

Consider the feedback control system in Figure 1:

$$\begin{aligned} \mathbf{x}_{k+1} &= A_k\mathbf{x}_k + B_k\mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k &= C_k\mathbf{x}_k + \mathbf{v}_k \ , \end{aligned} \tag{1}$$

where $\mathbf{x}_k \in \mathcal{R}^n$ is the state of the system, $\mathbf{y}_k \in \mathcal{R}^m$ is the observation, $\mathbf{u}_k \in \mathcal{R}^d$ is the control, $\mathbf{v}_k \in \mathcal{R}^m$ and $\mathbf{w}_k \in \mathcal{R}^n$ are independent discrete time white Gaussian processes, and A_k, B_k, and C_k are known matrices with proper dimensions. We consider the quadratic cost function

$$\frac{1}{N}E\left\{\mathbf{x}_N^TQ_N\mathbf{x}_N + \sum_{k=0}^{N-1}(\mathbf{x}_k^TQ_k\mathbf{x}_k + \mathbf{u}_k^TR_k\mathbf{u}_k)\right\} \ , \tag{2}$$

where R_K is a positive definite matrix and Q_k is a positive semidefinite matrix. It is well known that with the quadratic cost in (2) and an admissible control $\mathbf{u}_k = \mu_k(I_k)$, where $I_k = \{\mathbf{y}_0, \mathbf{y}_1, \cdots, \mathbf{y}_k, \mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_{k-1}\}$ is the information vector and $I_0 = \{\mathbf{y}_0\}$ is the initial information vector, the optimal control problem has a solution as follows [7]

$$\begin{aligned} \mu_k^*(I_k) &= L_kE\{\mathbf{x}_k|I_k\}, \\ L_k &= -(R_k + B_k^TK_{k+1}B_k)^{-1}B_k^TK_{k+1}A_k \ , \end{aligned} \tag{3}$$

where matrix K_k is given recursively by the Riccati equation

$$\begin{aligned} K_N &= Q_N, \\ P_k &= A_k^TK_{k+1}B_k(R_k + B_k^TK_{k+1}B_k)^{-1} \\ &\qquad B_k^TK_{k+1}A_k \ , \\ K_k &= A_k^TK_{k+1}A_k - P_k + Q_k \ . \end{aligned} \tag{4}$$

Since the system is linear and the noise process is assumed to be Gaussian, the conditional expectation of the state of the system given information vector I_k can be calculated using Kalman filtering. The solution of the Kalman filtering problem can be written as follows [7]:

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= A_k\hat{\mathbf{x}}_k + B_k\mathbf{u}_k + \Sigma_{k+1|k+1}C_{k+1}^TN_{k+1}^{-1} \\ &\qquad (\mathbf{y}_{k+1} - C_{k+1}(A_k\hat{\mathbf{x}}_k + B_k\mathbf{u}_k)) \end{aligned} \tag{5}$$

and

$$\hat{\mathbf{x}}_0 = E\{\mathbf{x}_0\}\Sigma_{0|0}C_0^TN_0^{-1}\left(\mathbf{y}_0 - C_0E\{\mathbf{x}_0\}\right), \tag{6}$$

where the matrices are computed off-line and are given recursively by

$$\begin{aligned} \Sigma_{k+1|k+1} &= \Sigma_{k+1|k} - \Sigma_{k+1|k}C_{k+1}^T \\ &\quad (C_{k+1}\Sigma_{k+1|k}C_{k+1}^T + N_{k+1})^{-1}C_{k+1}\Sigma_{k+1|k} \\ \Sigma_{k+1|k} &= A_k\Sigma_{k|k}A_k^T + M_k, \end{aligned} \tag{7}$$

for $k = 0, 1, \cdots, N-1$, with

$$\Sigma_{0|0} = S - SC_0^T(C_0SC_0^T + N_0)^{-1}C_0S. \tag{8}$$

In these equations, M_k, N_K, and S are the covariance matrices of $\mathbf{w}_k$, $\mathbf{v}_k$, and $\mathbf{x}_0$, respectively, and we assume that $\mathbf{w}_k$ and $\mathbf{v}_k$ have zero means.

Consider now the case where $\mathbf{w}_n$ and $\mathbf{v}_n$ are stationary, the system is time invariant, and the matrices defining the cost in (2) are constant in time. Therefore, we can drop the time index. Assume that the pair (A, B) is controllable and the pair $(A, Q^{1/2})$ is observable, where $Q = (Q^{1/2})^TQ^{1/2}$. Then, for the infinite horizon case the optimal control policy is a steady-sate policy and is given by [7]

$$\mu^*(I_k) = L\hat{\mathbf{x}}_k \ , \tag{9}$$

where

$$L = -(R + B^TKB)^{-1}B^TKA \ , \tag{10}$$

and K is the unique positive semidefinite solution of the algebraic Riccati equation

$$K = A^T(K - KB(R + B^TKB)^{-1}B^TK)A + Q. \tag{11}$$

By a similar argument, if the pair (A, C) is observable and the pair $(A, M^{1/2})$ is controllable, where $M = (M^{1/2})^TM^{1/2}$, then as $k \longrightarrow \infty$, $\hat{\mathbf{x}}_k$ can be generated by a steady state Kalman filter

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} = \ & (A + BL)\hat{\mathbf{x}}_k + \\ & \overline{\Sigma}C^TN^{-1}(\mathbf{y}_{k+1} - C(A + BL)\hat{\mathbf{x}}_k), \end{aligned} \tag{12}$$

where $\overline{\Sigma}$ is given by

$$\overline{\Sigma} = \Sigma - \Sigma C^T(C\Sigma C^T + N)^{-1}C\Sigma, \tag{13}$$

and Σ is the unique positive semidefinite solution of the Riccati equation

$$\Sigma = A(\Sigma - \Sigma C^T(C\Sigma C^T + N)^{-1}C\Sigma)A^T + M. \tag{14}$$

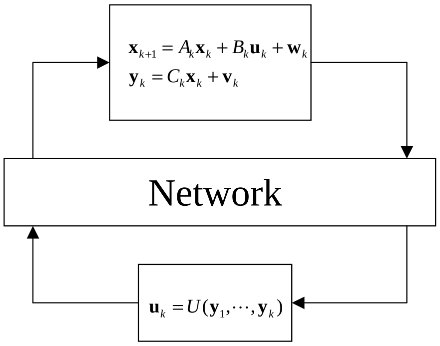

Fig. 2. A networked control system with single connection to the communication network.

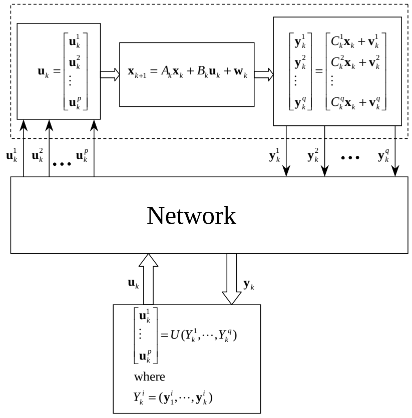

Fig. 3. A networked control system with distributed sensors and actuators.

Now consider system (1) whose feedback loop is closed through a communication network. Here we assume that a single controller uses the observation data $\mathbf{y}_k$, which it receives from the plant through the network, to generate the control command $\mathbf{u}_k$. This control command is sent to the plant through the network. For example, we assume two cases of networked control system. In the first example (Figure 2) the plant collects all the sensor data and transmits it to the controller, then receives the control commands from the controller and sends them to the actuators. The communication between the plant and the controller is through a communication network. In the second example (Figure 3), the plant has a series of sensors and actuators that are distributed throughout the plant and each of these sensors and actuators are connected to the communication network. Therefore, they send and receive their data directly to and from the controller.

The network in the feedback loop affects the dynamics of the networked control system in two different ways.

The first impact of the network in a networked control system is the packet loss due to network congestion. The data transmission protocols like TCP guarantee the delivery of data packets. Therefore, when one or more packets are lost the transmitter retransmits the lost packets. However, since a retransmitted packet usually has a long delay, the retransmitted control command packets are outdated by the time they arrive at the actuator. Therefore, in our model, we assume that the system discards the control commands that are retransmitted. The sensor data packets are used in the estimator module to estimate the state of the system, the more data the estimator receives from the sensors the more accurate the estimate of the state is. In other words, if a sensor packet is lost the estimator module can proceed without the lost packet, but when it receives the retransmitted version of the lost sensor data packet, the estimator can use the new information to improve the estimation error. Although in principle this is possible, a time stamping procedure should be used and all the receivers and transmitters should be synchronized in order to implement it. Unlike TCP, in UDP no packet is retransmitted and the dropped packets are considered lost. This results in smaller transmission delay which is very important for the stability of networked control systems. We use UDP for our analysis.

The second impact of the network is the delay of the sensor and the control command data packets. This delay is a combination of a fixed propagation delay and a random delay due to the congestion in the network. In this paper we do not address the impact of the network delay on the control system and we concentrate our attention on the lost packets.

The networked control system can be modelled by the following set of equations:

$$\begin{aligned} \mathbf{x}_{k+1} &= A_k\mathbf{x}_k + B_k\Theta_k\mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k &= C_k\mathbf{x}_k + \mathbf{v}_k \ , \end{aligned} \tag{15}$$

where $\Theta_k = diag(\theta_k^1, \cdots, \theta_k^d)$ is a diagonal binary random matrix. $\theta_k^i = 0$ indicates that the control command u_k^i, i^{th} component of $\mathbf{u}_k$, is lost and $\theta_k^i = 1$ indicates that this component of the control command is delivered. Similarly to the control command, the sensor data that arrives at the estimator can be modelled as follows:

$$\hat{\mathbf{y}}_k = \Phi_k\mathbf{y}_k, \tag{16}$$

where $\Phi_k = diag(\phi_k^1, \cdots, \phi_k^m)$ is a diagonal random matrix.

In this paper we assume that Θ_k and Φ_k are i.i.d. This assumption can be justified if the bandwidth of the communication network is much bigger than the bandwidth of the feedback control system.

The information vector in the networked control system is a combination of observation $\{\mathbf{y}_k\}$, control command $\{\mathbf{u}_k\}$, and control command packet delivery indicator $\{\Theta_k\}$.

Since the random matrices $\Theta_{k-1}, \Theta_{k-2}, \cdots$ are not necessarily observable at time k, the optimal control problem for the networked control system in (15) and (16) and a quadratic cost is not separable into optimal solutions for the estimator part and the controller part. Therefore, there is no known analytical solution for this problem. To avoid the computationally complex numerical solution, we use an approximate solution for the optimal control problem. In this approximate solution, we assume that the optimal controller and the optimal estimator can be calculated separately.

III. Controller

With the assumptions made in the previous section the finite horizon suboptimal control policy for the networked control system can be calculated as follows [7]:

$$\mu_k(E[\mathbf{x}|I_k]) = L_k E[\mathbf{x}|I_k] \tag{17}$$

where the gain matrix L_k is given by

$$L_k = \quad -\Psi_k E[\Theta_k] B_k^T K_{k+1} A_k \ , \tag{18}$$

where

$$\Psi_k = \left(R_k + E[\Theta_k B_k^T K_{k+1} B_k \Theta_k] \right)^{-1} \ ,$$

and where the matrices K_k are given by the recursive equation

$$\begin{aligned} &K_N = Q_N \\ &K_k = A_k^T K_{k+1} A_K - \\ &\qquad A_k^T K_{k+1} B_k E[\Theta_k] \Psi_k E[\Theta_k] B_k^T K_{k+1} A_k + Q_k \ . \end{aligned} \tag{19}$$

In the case of a time invariant system and constant matrices, Q_k and R_k, it is not necessarily true that the above equation converges to a steady-state solution even if the matrices A, B, and Q satisfy the conditions for the existence of the steady-state solution in (11). In fact, if the packet dropping rate is high, i.e. the probability of $\theta_k^i = 0$ is larger than a certain threshold, K_k diverges to infinity starting from any nonnegative initial condition [8]. A possible interpretation for this phenomena is that if there is a lot of uncertainty about the system, i.e. the packet dropping rate is high, optimization over a long period is meaningless. This phenomena has been called the *uncertainty threshold principle* [9] [10].

To give the necessary and sufficient condition for the existence of the steady-state control policy we introduce some notations and definitions. The Moore-Penrose pseudo-inverse of the $n_1 \times n_2$ matrix S is denoted by $(S)^+$ and defined by $S^+ S\mathbf{z} = \mathbf{z}$, $\forall \mathbf{z} \in \mathcal{R}(S^T)$ and $S^+\mathbf{z} = 0$, $\forall \mathbf{z} \in \mathcal{N}(S^T)$, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ denote its range space and null space, respectively [11]. The space of $n \times n$ symmetric matrices is denoted by ψ^n. The mapping $\mathcal{A} : \psi^n \to \psi^n$ is defined by [12]

$$\begin{aligned} \mathcal{A}X &= A^T X A - A^T X B \bar{\Theta} (E[\Theta B^T X B \Theta])^+ \bar{\Theta} X A \\ \mathcal{A}^i X &= \mathcal{A}\mathcal{A}^{i-1} X. \end{aligned}$$

The following theorem gives the necessary and sufficient condition for the existence of the steady-state control policy[12].

Theorem 1: Assume that the pair (A, B) is controllable and the pair $(A, Q^{1/2})$ is observable. Then, for the stationary random matrix Θ_k with $\bar{\Theta} = E[\Theta_k]$ and $E[\Theta_k Y \Theta_k] = E[\Theta Y \Theta]$, where Y is a $d \times d$ matrix, the forward Riccati-like difference equation

$$\begin{aligned} &K_{k+1} = A^T K_k A - \\ &A^T K_k B \bar{\Theta} \left(R + E[\Theta B^T K_k B \Theta] \right)^{-1} \bar{\Theta} B^T K_k A + Q \end{aligned} \tag{20}$$

has a steady state solution as $k \longrightarrow \infty$ if and only if $\tilde{\rho} = \lim_{i\to\infty} \|\mathcal{A}^i I\|^{1/i} < 1$.

One result of Theorem 1 is that an undisturbed system that satisfies the condition $\tilde{\rho} = \lim_{i\to\infty} \|\mathcal{A}^i I\|^{1/i} < 1$ is stabilizable in the mean square sense, i.e. there exists a gain matrix L such that for the closed loop system $\mathbf{x}_{k+1} = (A - B\Theta_k L)\mathbf{x}_k$, $\lim_{i\to\infty} E[\|\mathbf{x}_i\|^2] = 0$ for all initial conditions $\mathbf{x}_0$.

Theorem 1 provides a numerical method to check the existence of a steady-state solution for the Riccati-like equation (20). In special cases it is possible to reduce the burden of this numerical method to the calculation of the spectral radius of matrix A. As an example, consider the time invariant networked control system that is connected to a communication network similar to the system in Figure 2. The dynamics of the system in this case can be written as follows

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \theta_k B \mathbf{u}_k + \mathbf{w}_k. \tag{21}$$

The Riccati-like equation (20) for the system in (21) has the following form

$$\begin{aligned} &K_{k+1} = A^T K_k A - \\ &\quad \alpha^2 A^T K_k B \left(R + \alpha B^T K_k B] \right)^{-1} B^T K_k A + Q \ , \end{aligned} \tag{22}$$

where $\beta = 1 - \alpha$ is the packet dropping rate, more precisely $P(\theta_k = 1) = \alpha$ and $P(\theta_k = 0) = \beta$. It is easy to see that for (A, B) controllable, B square and full rank, $(A, Q^{1/2})$ observable and, $R > 0$, if $\max_i |\lambda_i(A)| < 1/\beta^{(1/2)}$, then the steady state solution for the Riccati-like equation (22) exists [8]. There is no known analytical result regarding the existence of the steady state solution for (22) when the matrix B is not full rank and square, but we know that for this case if $\max_i |\lambda_i(A)| \geq 1/\beta^{(1/2)}$, then $\lim_{k\to\infty} K_k \to \infty$ [12]. Therefore, the networked control system in Figure 2 is unstable if the packet dropping rate, β, is greater than $\frac{1}{\max_i(\lambda_i(A))^2}$.

IV. Estimator

In (17) the controller uses the output of the estimator to generate the control command. The estimator finds the best estimate through the conditional expectation of the state given the information vector. The information vector consists of the the control command, the system output received at the estimator, and the packet delivery indicators for the control command. To illustrate our main idea, in this section we assume that our networked control system has a structure similar to the system in Figure 2. The extension of the ideas of this section to the general case is straightforward and is given at the end of the section. The system output packets and the control command acknowledgement packets may get dropped passing through different nodes of the communication network. Therefore, there are several different scenarios for the elements of the information vector at time k.

Considering all possible scenarios for the combination of the estimator and the plant, the networked control system has the following structure

$$\mathbf{x}_{k+1} = A_k\mathbf{x}_k + \theta_k B_k L_k \hat{\mathbf{x}}_k + \mathbf{w}_k \ ,$$
$$\hat{\mathbf{x}}_{k+1} = \begin{cases} (A_k + \varphi_k B_k L_k)\hat{\mathbf{x}}_k + \Gamma_k(\mathbf{y}_{k+1} - C_{k+1} \\ (A_k + \varphi_k B_k \Gamma_k)\hat{\mathbf{x}}_k) \quad \text{if } \mathbf{y}_k \text{ delivered} \\ \\ (A_k + \varphi_k B_k L_k)\hat{\mathbf{x}}_k \qquad \text{if } \mathbf{y}_k \text{ lost} \end{cases} \tag{23}$$

where $\Gamma_k = \Sigma_{k+1|k+1} C_{k+1}^T N_{k+1}^{-1}$, $\hat{\mathbf{x}}_k = E[\mathbf{x}|I_k]$, and $\varphi_k = \theta_k + (1-\theta_k)\bar{\theta}$ is equal to θ_k if the control command packet delivery indicator is delivered and is equal to $\bar{\theta}$ otherwise. For LTI systems in the presence of dropped packets the gain in the Kalman filter, Γ_k, does not reach a steady state even if (A, C) is observable and $(A, M^{1/2})$ is controllable (see equations (5) to (14)). To simplify the stability analysis of the networked control system we replace the estimator in (23) with a suboptimal estimator by replacing Γ_k with $\Gamma = \bar{\Sigma} C^T N^{-1}$. Therefore, the overall system has the following structure:

$$\begin{pmatrix} \mathbf{x}_{k+1} \\ \hat{\mathbf{x}}_{k+1} \end{pmatrix} = \begin{bmatrix} A & \theta_k BL \\ \Gamma CA & (I-\Gamma C)A + \varphi_k BL \end{bmatrix} \begin{pmatrix} \mathbf{x}_k \\ \hat{\mathbf{x}}_k \end{pmatrix} + \begin{bmatrix} I \\ \Gamma C \end{bmatrix} \mathbf{w}_k + \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \mathbf{v}_k \tag{24}$$

if $\mathbf{y}_k$ is delivered,

$$\begin{pmatrix} \mathbf{x}_{k+1} \\ \hat{\mathbf{x}}_{k+1} \end{pmatrix} = \begin{bmatrix} A & \theta_k BL \\ 0 & A + \varphi_k BL \end{bmatrix} \begin{pmatrix} \mathbf{x}_k \\ \hat{\mathbf{x}}_k \end{pmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \mathbf{w}_k, \tag{25}$$

if $\mathbf{y}_k$ is lost.

Assume the estimator in (24) has the full information of control commands, observations, and packet delivery status for control commands. Therefore, if (A, C) is observable and $(A, M^{1/2})$ is controllable then the covariance of the error of the estimate is $\bar{\Sigma}$ (see equation (13)). Also, if the conditions of Theorem 1 are satisfied then the system in (24) is stable in the mean square sense. The question for which we seek an answer is under what conditions is the networked control system, with packet dropping rate β in forward and feedback loops, a stable system. We know that a networked control system with perfect state information that satisfies the conditions of Theorem 1 is mean square stable. Therefore, the possible instability of the system in (24-25) is due to the imperfect observation of the state of the system. In other words, because of the error in $\hat{\mathbf{x}}_k$ the control command cannot stabilize the system.

We define the transition matrices $F^0, F^1, \cdots, F^7$ for different modes of operation for the network control system of equations (24) and (25) as follows:

$$F^0 = \begin{bmatrix} A & BL \\ \Gamma CA & (I-\Gamma C)A + BL \end{bmatrix},$$

in this mode $\theta_k = 1$, $\mathbf{y}_{k+1}$ and θ_k are delivered.

$$F^1 = \begin{bmatrix} A & 0 \\ \Gamma CA & (I-\Gamma C)A \end{bmatrix},$$

in this mode $\theta_k = 0$, $\mathbf{y}_{k+1}$ and θ_k are delivered.

$$F^2 = \begin{bmatrix} A & BL \\ \Gamma CA & (I-\Gamma C)A + \bar{\theta}BL \end{bmatrix},$$

in this mode $\theta_k = 1$, $\mathbf{y}_{k+1}$ is delivered, and θ_k is lost.

$$F^3 = \begin{bmatrix} A & 0 \\ \Gamma CA & (I-\Gamma C)A + \bar{\theta}BL \end{bmatrix},$$

in this mode $\theta_k = 0$, $\mathbf{y}_{k+1}$ is delivered, and θ_k is lost.

$$F^4 = \begin{bmatrix} A & BL \\ 0 & A + BL \end{bmatrix},$$

in this mode $\theta_k = 1$, $\mathbf{y}_{k+1}$ is lost, and θ_k is delivered.

$$F^5 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

in this mode $\theta_k = 0$, $\mathbf{y}_{k+1}$ is lost, and θ_k is delivered.

$$F^6 = \begin{bmatrix} A & BL \\ 0 & A + \bar{\theta}BL \end{bmatrix},$$

in this mode $\theta_k = 1$, $\mathbf{y}_{k+1}$ and θ_k are lost.

$$F^7 = \begin{bmatrix} A & 0 \\ 0 & A + \bar{\theta}BL \end{bmatrix},$$

in this mode $\theta_k = 0$, $\mathbf{y}_{k+1}$ and θ_k are lost.

The mean square stability of the networked control system is equivalent to the existence of P_i, $i = 0, 1, \cdots, 7$ such that [13]

$$P_i > 0 \quad , \quad F^i\left(\sum_{j=0}^{7} p_i P_i\right) F^{i^T} < P_i, \quad i = 0, 1, \cdots, 7, \tag{26}$$

where $p_i = \Pr(F = F^i)$. Therefore, to check the mean square stability of the system, one should solve for the feasibility of Linear Matrix Inequalities (LMIs) in (26).

V. Examples and Simulations

In this section, via numerical examples, we confirm the theoretical results that were presented in the previous sections.

Example 1: Consider the following linear system

$$A = \begin{bmatrix} -0.2247 & -1.5304 & -0.0972 & -1.6739 \\ 0.0272 & 1.3501 & 0.9175 & -0.8232 \\ -0.5781 & 0.1585 & 0.1147 & 1.2752 \\ 0.1480 & -1.6346 & -0.4308 & 0.0089 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.3664 & 0.3370 \\ 0.2062 & -0.8046 \\ -0.9816 & 0.2143 \\ 0.3829 & -0.6150 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.7464 & -1.9967 & -0.7602 & 0.8304 \\ -0.7324 & -0.0821 & -0.6108 & 0.5653 \end{bmatrix}.$$

Matrices Q, R, M, and N are identity matrices of proper dimension. We assume that the packet dropping rate for all packets is the same and it is equal to $\beta = 0.1$. It can be easily checked that this system is controllable and observable. Using Theorem 1 we see that the difference Riccati-like equation has a steady state solution and it is equal to

$$K = \begin{bmatrix} 1.7743 & -3.0179 & -1.8719 & 0.5037 \\ -3.0179 & 33.4406 & 15.3191 & -2.8411 \\ -1.8719 & 15.3191 & 9.3066 & -3.0382 \\ 0.5037 & -2.8411 & -3.0382 & 7.5901 \end{bmatrix}.$$

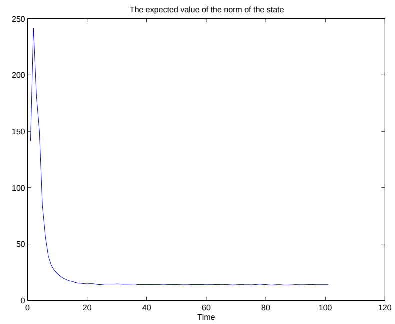

Fig. 4. The expected value of the norm of the state of the system in Example 1.

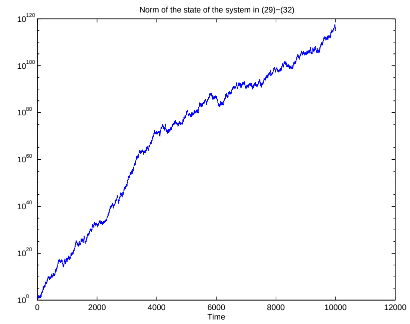

Fig. 5. 2 norm of the state of the system in Example 2.

Therefore, this system with full state information is stable in mean square sense. Furthermore, by checking the condition in (26) we see that the set of LMIs are feasible and therefore, the networked control system is stable. In Figure 4, the expected value of the norm of the state is plotted. In this plot, because of the noise and dropped packets the expected value of the norm of the state does not approach zero and instead it goes to a non-zero steady state value.

Example 2: Consider the system in Example 1 with packet dropping rate $\beta = 0.35$. In this case the difference Riccati-like equation has solution:

$$K = \begin{bmatrix} 12.1872 & -63.0028 & -38.4361 & -7.1466 \\ -63.0028 & 404.1250 & 237.2488 & 28.4469 \\ -38.4361 & 237.2488 & 144.0814 & 18.0860 \\ -7.1466 & 28.4469 & 18.0860 & 22.8959 \end{bmatrix}$$

It can be shown that the LMIs in this case don't have a solution. In Figure 5 it is shown that the networked control system is unstable.

VI. Conclusions and Future Work

In this paper we presented the conditions under which undisturbed networked control systems are mean square stable. These conditions can be checked rather easily if the parameters of the non-networked control system and the packet dropping rate in the network is given. If the dropped packets are not retransmitted these conditions are necessary and sufficient for stability.

In future we will study the stability of the networked control systems with the presence of network delay and packet dropping due to propagation delay and network congestion.

VII. References

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