Error-Block Codes and Poset Metrics

Marcelo Muniz S. Alves
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Poset metric

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- $J \subset P$ ideal in $P$:
  $x \in J$ and $y \leq x \implies x \in P$
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- $J \subset P$ ideal in $P$:
  - $x \in J$ and $y \leq x \implies x \in P$
- $P$-weight on $\mathbb{F}_q^n$:
  $$\omega_P(v) = |\langle \text{supp}(v) \rangle|$$
Poset metric

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- Niederreiter: generalization of the search for good linear codes via parity check matrices. This corresponds to the search for good linear codes when $P$ is a union of chains.
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Some motivations:

- Niederreiter: generalization of the search for good linear codes via parity check matrices. This corresponds to the search for good linear codes when $P$ is a union of chains.

- Rosenbloom and Tsfasman - codes where some entries overrule others. The Rosenbloom-Tsfasman metric corresponds to $P$ being a union of chains of the same size.

- New problems: given a code $C$, which are the posets that make $C$ “best” in some way? I.e., perfect or MDS.
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- hence $\rho_H(C) \leq \rho_P(C)$
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the \( \pi \)-metric on \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_n \):

\[
\omega_\pi(v) = |\text{supp}_\pi(v)|
\]

where \( v = v_1 + \cdots + v_n, v_i \in V_i, \) and

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\text{supp}_\pi(v) = \{ i; v_i \neq 0 \} \]
in this case,

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- in general
  \[\rho_\pi(C) \leq \rho_H(C) \leq \rho_P(C)\]
\((P, \pi)\)-metric

Balancing (and combining) the previous constructions...

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- $(P, \pi)$-weight of $v \in V$:

$$\omega_{(P, \pi)}(v) = |\langle \text{supp}(v) \rangle|$$
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- one has two “parameters” to deal with: the poset $P$ and the dimensions $k_i = \dim V_i$. 

$(P, \pi)$-metric

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Perfect Linear codes over chains

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► (Brualdi et al) \( C \) is \( r-P \)-perfect \( \iff \) there exists a linear function \( f : \mathbb{F}_q^{n-r} \rightarrow \mathbb{F}_q^r \) such that

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C = \{(f(c)|c); ~ c \in \mathbb{F}_q^{n-r}\}
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- (__, Panek, Firer) $C$ is $(P, \pi)$-$r$-perfect $\iff$ there exists a linear function $f : V_{r+1} \oplus \ldots \oplus V_n \to V_1 \oplus \ldots \oplus V_r$ such that

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The extended Hamming codes

- Hyun, Kim (2004) - classification of the posets $P$ over which the extended Hamming code $H_m$ is a double or triple-error-correcting $P$-perfect code.
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- (__, Panek, Firer) - classification of the posets $P$ for which $H_3$ is one-$\pi$-perfect, in terms of the minimal elements of $P$. (using the design structure of the set of minimal (Hamming) codewords)
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H_3 \text{ is 1-perfect in } V = V_1 \oplus \cdots \oplus V_s \iff
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\[ H_3 \text{ is } 1\text{-perfect in } V = V_1 \oplus \cdots \oplus V_s \iff \]

- $P$ has only one minimal element $i$
- $V_i$ is four-dimensional
- $V_i$ does not contain the support of a minimal codeword
The extended binary Golay code

- Jang, Kim, Oh, Rho (2007) - classification of posets $P$ such that the extended Golay code $G_{24}$ is a four or five-$P$-perfect code.
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- Jang, Kim, Oh, Rho (2007) - classification of posets $P$ such that the extended Golay code $G_{24}$ is a four or five-$P$-perfect code.
- (__, Panek, Firer) - description of posets (not all) $P$ such that $G_{24}$ is one or two-$(P, \pi)$-perfect.
Two (sub)groups of automorphisms

- For each $V_i$, fix a basis $\beta_i = \{v_{i,1}, \ldots, v_{i,k_i}\}$
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- Automorphism of $(P, \pi)$: permutation $\sigma: P \to P$ such that $\pi(\sigma(k)) = \pi(k)$. 
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- Induced automorphism in $V$:
  \[ T_\sigma(e_{i,j}) = e_{\sigma(i),j}. \]
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- $A = \text{group of the } T_\sigma \text{'s.}$
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     where $v \in V_i$ and $\text{supp}(u) \subset \langle i \rangle$;
- $\mathcal{T}$ = group of these “triangular” maps.
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matrix form of an element of $\mathcal{T}$:

$$[T]_B = \begin{pmatrix}
[T]_{B_1}^1 & [T]_{B_2}^1 & [T]_{B_3}^1 & \cdots & [T]_{B_3}^1 \\
0 & [T]_{B_2}^2 & [T]_{B_3}^2 & \cdots & [T]_{B_3}^2 \\
0 & 0 & [T]_{B_3}^3 & \cdots & [T]_{B_3}^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & [T]_{B_n}^n
\end{pmatrix}$$

where each block $[T]_{B_k}^k$ is invertible.
This is the group of units of the associated incidence algebra (over $(P, \pi)$).
Automorphisms

\[ GL(P,\pi)(V) = \text{automorphisms of } (V, \omega_{P,\pi}). \]

If \( T \in GL(P,\pi)(V) \) then \( \exists! F \in \mathcal{T} \) and \( \sigma \in Aut(P,\pi) \) such that

\[ T = F \circ T_\sigma \]
Automorphisms

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$$T = F \circ T_{\sigma}$$

- Moreover,

$$GL_{(P, \pi)}(V) \cong \mathcal{T} \rtimes \mathcal{A}$$
Some special cases

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  - $U \cong S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l} \subset S_n$. 

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