

Low-energy effective action for pions and a dilatonic meson

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Numerical simulations of QCD-like theories in which the number of flavors is adjusted so that the beta function is very small, but confinement and chiral symmetry breaking nevertheless take place, appear to reveal the presence of a flavor-singlet scalar meson which can be as light as the pions. Because the breaking of dilatation symmetry, quantified by the beta function, is small relative to QCD, a possible explanation is that the scalar meson is a pseudo Nambu-Goldstone boson associated with the approximate dilatation symmetry. We use this observation to systematically develop a low-energy effective action that accounts for both the pions and the “dilatonic” scalar meson. In order to justify the power counting that controls the couplings of the dilatonic meson we invoke the Veneziano limit, in which the number of fundamental-representation flavors N_f grows in proportion with the number of colors N_c , while the ratio N_f/N_c is kept close to, but below, the critical value where the conformal window is entered.

I. INTRODUCTION

A non-abelian gauge theory with N_c colors and N_f flavors of massless Dirac fermions can be in one of three phases. For small N_f , the renormalized coupling becomes strong toward the infrared, leading to confinement and chiral symmetry breaking. As N_f is increased the running slows down until eventually an infrared-attractive fixed point (IRFP) develops [1, 2]. The long-distance theory is then conformal, exhibiting neither confinement nor chiral symmetry breaking. Finally, if N_f gets too large, asymptotic freedom is lost.

In this paper we consider “walking” theories, where confinement and chiral symmetry breaking still occur, but N_f has become so large that the theory is on the verge of developing an IRFP. This is manifested in a non-perturbative beta function that is much smaller than in QCD.¹ Recent lattice studies of some candidate walking theories reveal the presence of a flavor-singlet, parity-even scalar meson that can be as light as the pions at the fermion masses used in those simulations, and much lighter than any other hadronic states in the theory. A light scalar meson was found in the $SU(3)$ theory with $N_f = 8$ Dirac fermions in the fundamental representation [4, 5], which current evidence favors to be chirally broken and confining [6–10]. A light scalar was also found in the $SU(3)$ theory with two Dirac fermions in the sextet (two-index symmetric) representation [11]. Because the non-perturbative study of the beta function is very difficult [12], it is still not entirely certain whether this model is chirally broken or infrared conformal. While the results of Ref. [13] favor the chirally broken option, other recent studies are inconclusive [14, 15].

The standard low-energy effective theory for the pions—the pseudo Nambu-Goldstone bosons (pNGBs) of spontaneously broken chiral symmetry—is the chiral lagrangian [16–18].² But when the scalar meson is as light as the pions, a consistent, unitary low-energy effective theory must accommodate this scalar state as well. Moreover, if the scalar is much lighter than all other states apart from the pions, it is natural to ask for an effective field theory description of this scalar, even if fermion masses are taken so small that the pions become lighter than this scalar.

A natural question is whether, like the pions, the light scalar meson can be interpreted as a pNGB associated with the spontaneous breaking of another approximate symmetry. In this paper we will attempt to provide such a framework, where the new approximate symmetry is the invariance under dilatations, or scale transformations. For this reason, we will refer to the scalar meson in this paper as the dilatonic meson.³

It is far from obvious that such an interpretation makes sense. In the case of chiral symmetry, the fermion mass m is a parameter that can be tuned continuously such that, for $m \rightarrow 0$, exact chiral symmetry is recovered, and the pions become massless. By contrast, while scale invariance is broken softly by the fermion mass, its breaking by the scale dependence of the renormalized coupling constitutes a hard breaking. Even if we dial the fermion mass to zero, we do not have another parameter that would allow us to continuously control the explicit breaking of scale invariance encoded in the beta function. Moreover, if ultimately confinement and chiral symmetry breaking are to take place, it is unavoidable

¹ For a recent review of lattice results, see Ref. [3].

² For a recent review of chiral perturbation theory with applications to lattice QCD, see Ref. [19].

³ For a previous treatment of this problem, see Ref. [20]. For a discussion of the effective theory for a light dilatonic scalar in theories with approximate scale invariance, but without spontaneously broken of chiral symmetry, see Ref. [21].

that the coupling will keep growing as the energy scale is lowered. The hard breaking of dilatation symmetry by the running (or walking) of the coupling is therefore of a very different nature than the soft breaking of chiral symmetry by a fermion mass term.

A hint is provided by the effective low-energy treatment of the anomalous axial symmetry $U(1)_A$. In this case, too, there is no way to “turn off” the anomaly within a given theory. In order to overcome this problem one takes the familiar large- N_c limit: the number of colors N_c is taken to infinity, while the number of flavors is held fixed [22–25]. In this limit the axial anomaly becomes vanishingly small, and the flavor-singlet pseudoscalar meson eventually becomes as light as the pions. A large- N_c extension of the standard chiral lagrangian that accommodates the $U(1)_A$ symmetry was systematically developed in Refs. [18, 26].

Here we will invoke a different large- N limit [27]. Considering $SU(N_c)$ gauge theories with N_f Dirac fermions in the fundamental representation, the so-called Veneziano limit is defined as the limit $N \rightarrow \infty$, in which we take $N_c = N$, while the ratio

$$n_f = \frac{N_f}{N_c} , \quad (1.1)$$

is held fixed. Asymptotic freedom imposes the restriction $n_f < 11/2$. For each N_c , we let $N_f^*(N_c)$ be the largest number of flavors such that the theory confines, and chiral symmetry is broken spontaneously in the chiral limit $m \rightarrow 0$. The *conformal window*, in which the coupling runs into an IRFP and the long-distance theory is conformal, thus corresponds to N_f values in the range

$$N_f^*(N_c) < N_f < (11/2)N_c . \quad (1.2)$$

In the Veneziano limit, the conformal window becomes

$$n_f^* < n_f < 11/2 , \quad (1.3)$$

where

$$n_f^* = \lim_{N_c \rightarrow \infty} \frac{N_f^*(N_c)}{N_c} . \quad (1.4)$$

In this paper we are interested in chirally broken theories, and thus we will only consider theories for which

$$n_f < n_f^* . \quad (1.5)$$

Since N_c and N_f are integers, the increments one can make in n_f are steps of size $1/N$. It follows that, in the Veneziano limit, n_f effectively becomes a continuous parameter. For the fundamental representation, the increments in the coefficients of the two-loop beta function also come in steps of size $1/N$, suggesting that $n_f - n_f^*$ is the parameter controlling the smallness of the beta function, or the “walking” nature of the theory.

Our analysis will be based on the conjecture that dilatation symmetry is recovered in a *triple limit*: the chiral limit $m \rightarrow 0$; the Veneziano large- N limit; and finally the limit $n_f \nearrow n_f^*$. We will assume that in this triple limit the trace anomaly becomes vanishingly small towards the infrared, both inside and outside of the conformal window. Correspondingly, the effective low-energy theory developed in this paper aims to provide the broken phase with a systematic expansion in three small parameters: the fermion mass, $1/N$, and $n_f - n_f^*$. As usual, it involves a derivative expansion as well.

This paper is organized as follows. In Sec. II we discuss the microscopic theory. We briefly review the trace anomaly, and show how it can be encoded in the coupling of the

microscopic theory to an external dilaton source $\sigma(x)$. In addition, we introduce the familiar external sources for scalar and pseudoscalar fermion bilinears. As usual, the external fields allow us to match correlation functions between the microscopic and the effective theory, and ultimately, to determine the coupling constants of the effective theory order by order in the low-energy expansion.

In Sec. III we begin with a more detailed discussion of the triple limit which is the starting point of our low-energy expansion. We specify the assumptions, or conjectures, that must be made in order to justify the expansion.

Having introduced the power counting, we proceed to address a complication in the construction of the low-energy theory that was previously encountered in the treatment of the $U(1)_A$ symmetry in large N_c [18, 26]. In principle, each term in the effective low-energy lagrangian can be multiplied by an arbitrary potential $V(\tau - \sigma)$, where $\tau(x)$ is the effective field that represents the dilatonic meson. Like the $U(1)_A$ symmetry, the dilatation symmetry is abelian. As a result, it does not impose any algebraic constraints on these potentials, and the coefficients of their power-series expansion in $\tau - \sigma$ are independent parameters of the low-energy theory. If nothing further could be done, the power-series coefficients would constitute an infinitely large set of low-energy parameters, and our effective field theory ends up having no predictive power. Fortunately, however, the matching of the microscopic and the effective theories for $n_f \nearrow n_f^*$ imposes a power-counting hierarchy on the coefficients of the potentials. As a consequence, out of the four potentials that we encounter *a-priori* in the leading-order lagrangian, three are eliminated altogether. The remaining potential, denoted V_d , gets truncated to a linear function

$$V_d(\tau - \sigma) = c_0 + c_1(\tau - \sigma) . \quad (1.6)$$

In Sec. IV we proceed to study the classical ground state, which turns out to have some delicate features. In the chiral limit, the classical vacuum is controlled by the potential of the dilatonic meson, which in turn is proportional to $e^{4\tau} V_d(\tau)$. The parameter c_1 , introduced in Eq. (1.6) above, is the only parameter of the leading-order lagrangian that accounts for the hard breaking of scale invariance via the walking of the coupling in the microscopic theory. We find that the c_1 term is crucial for the existence of a stable classical vacuum. We also study the tree-level masses, the dilatation current, and the fermion condensate.

As an application, we study in Sec. V the one-loop effective potential. This provides a concrete example that the renormalization of the effective theory works in essentially the same way as in standard chiral perturbation theory. We summarize our results in Sec. VI. Various technical details are relegated to several appendices.

II. THE MICROSCOPIC THEORY

In this section we discuss the relevant properties of the microscopic theory. We begin in Sec. IIA with a brief review of the dilatation current and the trace anomaly. As a further gearing up, we recall in Sec. IIB the basic steps in the construction of the chiral lagrangian. Apart from the obviously needed dynamical effective fields, a set of source fields with appropriate transformation properties, or spurions, is needed to match the effective theory to the low-energy sector of the microscopic theory. We also review how the anomalous $U(1)_A$ symmetry of QCD can be accommodated in the effective field theory framework, since this provides us with a useful lesson. In Sec. IIC we return to dilatation symmetry, and

discuss how the operatorial trace of the energy-momentum tensor can be encoded in the coupling of the microscopic theory to an external source with the transformation properties of a dilaton.

A. Dilatation current and trace anomaly

The dilatation current is given by

$$S_\mu = x_\nu T_{\mu\nu} . \quad (2.1)$$

where $T_{\mu\nu}$ is the symmetric, gauge-invariant energy-momentum tensor. Being a conserved current, $T_{\mu\nu}$ does not renormalize.⁴ In the asymptotically-free gauge theories that we will study in this paper the dilatation current is not conserved. Its divergence is given by the trace of the energy-momentum tensor,

$$\partial_\mu S_\mu = T_{\mu\mu} \equiv -T . \quad (2.2)$$

On shell, the trace of the energy-momentum tensor is [30]

$$T = T_{cl} + T_{an} , \quad (2.3a)$$

$$T_{cl} = m[\bar{\psi}\psi] , \quad (2.3b)$$

$$T_{an} = \tilde{\beta}[F^2] + \gamma_m m[\bar{\psi}\psi] . \quad (2.3c)$$

The operatorial trace divides into a classical part T_{cl} and a quantum part T_{an} . We use the shorthand $F^2 = F_{\mu\nu}^a F_{\mu\nu}^a$, where $F_{\mu\nu}^a$ is the non-abelian field strength, while $[\dots]$ denotes renormalized operators. Since $T_{\mu\nu}$ is renormalization-group invariant, so is S_μ , and thus T . Moreover, $m[\bar{\psi}\psi]$ is invariant by itself, and hence T_{cl} and T_{an} are separately renormalization-group invariant.⁵ The renormalization-group evolution of the renormalized coupling and fermion mass is controlled by

$$\tilde{\beta} = \frac{\beta(g)}{2g} , \quad \beta(g) = \mu \frac{\partial g}{\partial \mu} , \quad (2.4)$$

$$\gamma_m = \gamma_m(g) = -\frac{\mu}{m} \frac{\partial m}{\partial \mu} . \quad (2.5)$$

B. Chiral symmetry and spurions

The explicit breaking of dilatation symmetry must be incorporated into the low-energy effective theory that we will construct in the following sections. We can learn how to do this from the conceptually similar problem of extending the chiral lagrangian to accommodate the anomalous $U(1)_A$ symmetry in the 't Hooft large- N_c limit.

We first recall the key steps in the construction of the standard chiral lagrangian [16–18]. We introduce a dynamical effective field $\Sigma(x)$ taking values in the Nambu–Goldstone-boson

⁴ This would not be true in the presence of elementary scalar fields, because of the existence of an “improvement term” with which $T_{\mu\nu}$ can mix [28, 29].

⁵ For an explicit verification, see Ref. [31].

manifold, and endow it with appropriate transformation properties under chiral symmetry. For N_f Dirac fermions in a complex representation, the chiral symmetry is $SU(N_f)_L \times SU(N_f)_R$. The effective field is unitary, $\Sigma(x) \in SU(N_f)$, and transforms according to

$$\Sigma(x) \rightarrow g_L \Sigma(x) g_R^\dagger, \quad g_{L,R} \in SU(N_f)_{L,R}. \quad (2.6)$$

The order-by-order construction of the effective lagrangian is dictated by the power counting

$$\delta \sim m/\Lambda_{IR} \sim p^2/\Lambda_{IR}^2. \quad (2.7)$$

Here δ is a generic name for the small expansion parameter of the effective theory, m is the fermion mass, and p^2 is a characteristic momentum(-squared) of the pNGBs. Λ_{IR} is the non-perturbative dynamical scale of the theory which we will discuss later on (see Sec. III A). At leading order the effective lagrangian consists of terms of order δ in the power counting, the next to leading order consists of terms of order δ^2 , and so on.

The effective lagrangian at each order is fixed by matching correlation functions calculated in the microscopic theory with the corresponding correlation functions calculated in the effective theory. In order to facilitate this matching, a common set of source fields is introduced in both the microscopic and effective theories. In the matching process we can make use of any correlation function obtained by multiple differentiations of the (microscopic and effective) partition functions with respect to the external sources. For the basic chiral lagrangian, one needs a scalar source $\mathcal{S}_{AB}(x)$, $A, B = 1, 2, \dots, N_f$, which couples to the bilinear $\bar{\psi}_A(x)\psi_B(x)$; and a pseudoscalar source $\mathcal{P}_{AB}(x)$ that couples to the bilinear $\bar{\psi}_A(x)i\gamma_5\psi_B(x)$. The chiral transformation properties are

$$\psi \rightarrow (g_R P_R + g_L P_L)\psi, \quad (2.8a)$$

$$\bar{\psi} \rightarrow \bar{\psi}(P_L g_R^\dagger + P_R g_L^\dagger), \quad (2.8b)$$

$$\chi \rightarrow g_L \chi g_R^\dagger, \quad (2.8c)$$

where we have promoted the external sources to spurions, and

$$\chi = \mathcal{S} + i\mathcal{P}. \quad (2.9)$$

This allows us to add the invariant

$$\mathcal{L}_m = \bar{\psi}(\chi P_R + \chi^\dagger P_L)\psi, \quad (2.10)$$

to the microscopic lagrangian, thus coupling the theory to the sources \mathcal{S} and \mathcal{P} .

The actual theory is recovered by setting the source fields to their ‘‘expectation values,’’

$$\mathcal{S}(x) = m, \quad \mathcal{P}(x) = 0. \quad (2.11)$$

For simplicity, we assume in this paper that all fermions have a common mass m . Thus, chiral symmetry is softly broken for $m \neq 0$. By dialing $m \rightarrow 0$ exact chiral symmetry is recovered, and the pions become massless.

In general, any dependence of the effective theory on the expectation value of a spurion field encodes some explicit breaking of a symmetry, because the effective lagrangian is only invariant under the simultaneous transformation of the dynamical and spurion (source) fields. In the standard chiral lagrangian, this manifests itself as the breaking of chiral

symmetry by the fermion mass, as we have just discussed. Let us next see how this comes into play when dealing with the anomalous $U(1)_A$ symmetry.

The axial anomaly can be formally cancelled by augmenting the lagrangian of the microscopic theory with a new term

$$-cg^2 N_f \theta(x) \text{tr} F(x) \tilde{F}(x) . \quad (2.12)$$

Here $\theta(x)$ is the axion field, which couples to (minus) the anomalous divergence of the singlet axial current. Under a $U(1)_A$ transformation the axion field transforms by a shift, and this allows us to cancel the anomaly generated by the dynamical fermions against the variation of the axion-dependent term, Eq. (2.12).

The low-energy theory will now depend on the axion field $\theta(x)$. As long as we treat the axion field as a spurion, which transforms under the $U(1)_A$ symmetry, the effective lagrangian will maintain $U(1)_A$ invariance. But once we assign to the axion field a fixed expectation value, the axial anomaly will manifest itself in the effective theory.

The expectation value of the axion field is the vacuum angle θ_0 . The effective theory fails to be invariant under $U(1)_A$ even if we set the vacuum angle to zero. The effective theory now includes an additional dynamical field $\eta(x)$ associated with the flavor-singlet pseudoscalar meson, which couples to the $U(1)_A$ current. Algebraically, the $\eta(x)$ field is similar to the axion, in that they both transform by a shift under $U(1)_A$. Any function of the difference $\eta - \theta$ is invariant under the abelian $U(1)_A$ transformations. As discussed in Refs. [18, 26], already at leading order the effective low-energy theory depends on several such arbitrary non-derivative functions, or potentials $V_I(\eta - \theta)$.⁶ When we set $\theta(x) = \theta_0 = 0$, all that happens is that $V_I(\eta - \theta)$ becomes $V_I(\eta)$. The presence in the effective theory of the $V_I(\eta)$ potentials represents an explicit breaking of $U(1)_A$. The question of how this arbitrariness can be controlled is postponed to Sec. III.

C. Encoding the explicit breaking of dilatations

As explained in the previous subsection, the construction of the low-energy effective theory begins with the identification of the dynamical effective fields and the external source fields. The latter are then promoted to spurions. The new effective field that describes the dilatonic meson will be introduced in the next section. Here we discuss how to encode the anomalous divergence of the dilatation current by coupling the microscopic theory to an external dilaton field $\sigma(x)$.

We consider an asymptotically free gauge theory with N_f Dirac fermions in the fundamental representation. After a rescaling of the bare gauge and fermion fields by the bare

⁶ Because of parity, $V_I(\eta - \theta)$ is an even function of its argument.

coupling, we express the bare action in d dimensions as

$$S_{ev}(\sigma) = \int d^d x e^{(d-4)\sigma(x)} \mathcal{L}_0(x) , \quad (2.13)$$

$$\mathcal{L}_0 = \frac{1}{g_0^2} (\mathcal{L}_k + \mathcal{L}_{src}) , \quad (2.14)$$

$$\mathcal{L}_k = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{i=1}^{N_f} \bar{\psi}_i \mathcal{D} \psi_i , \quad (2.15)$$

$$\mathcal{L}_{src} = \bar{\psi} (\mathcal{S} + i\mathcal{P}\gamma_5) \psi , \quad (2.16)$$

where the covariant derivative is $D_\mu = \partial_\mu + iA_\mu$, and the field strength is $F_{\mu\nu} = F_{\mu\nu}^a T^a = -i[D_\mu, D_\nu]$.

The dilaton field $\sigma(x)$ has been introduced in Eq. (2.13) via an overall factor multiplying the entire bare lagrangian. We have denoted the action by $S_{ev}(\sigma)$, with a subscript “ ev ” to indicate that the dependence on $\sigma(x)$ comes from the evanescent part of the bare lagrangian. $S_{ev}(\sigma)$ is manifestly invariant under the d -dimensional dilatation

$$A_\mu(x) \rightarrow \lambda A_\mu(\lambda x) , \quad (2.17a)$$

$$\psi(x) \rightarrow \lambda^{3/2} \psi(\lambda x) , \quad \bar{\psi}(x) \rightarrow \lambda^{3/2} \bar{\psi}(\lambda x) , \quad (2.17b)$$

$$\mathcal{S}(x) \rightarrow \lambda \mathcal{S}(\lambda x) , \quad \mathcal{P}(x) \rightarrow \lambda \mathcal{P}(\lambda x) , \quad (2.17c)$$

$$\sigma(x) \rightarrow \sigma(\lambda x) + \log \lambda , \quad (2.17d)$$

$$g_0 \rightarrow g_0 . \quad (2.17e)$$

With the obvious exception of $\sigma(x)$, which plays a special role, in this setup the (dynamical and external) bare fields transform according to their canonical mass dimension in $d = 4$ dimensions. The bare parameter g_0 is dimensionful in $d \neq 4$ dimensions, but the anticipated transformation of $1/g_0^2$ has been taken up by the factor of $e^{(d-4)\sigma(x)}$.

In order to see how the σ dependence encodes the anomalous trace of the energy-momentum tensor, T_{an} , we proceed to consider the renormalized lagrangian. Differentiating the bare action once with respect to $\sigma(x)$ and then setting it to zero we have

$$\left. \frac{\partial}{\partial \sigma(x)} S_{ev}(\sigma) \right|_{\sigma(x)=0} = (d-4) \mathcal{L}_0(x) . \quad (2.18)$$

It was shown in Ref. [30] that the evanescent operator on the right-hand side occurs in the d -dimensional trace of the energy-momentum tensor, alongside with the classical (four-dimensional) trace, and with terms proportional to equations of motion. Taking the $d \rightarrow 4$ limit and discarding terms proportional to equations of motion then yields $T_{an}(x)$. Hence

$$S_{ev}(\sigma) = S_{ren}(\sigma) + O(\sigma^2) , \quad (2.19)$$

$$S_{ren}(\sigma) = \int d^d x \left(\mathcal{L}_k^{ren} + T_{cl}(\mathcal{S}, \mathcal{P}) + \sigma T_{an}(\mathcal{S}, \mathcal{P}) \right) , \quad (2.20)$$

where \mathcal{L}_k^{ren} is equal to $g_0^{-2} \mathcal{L}_k$ reexpressed in terms of the renormalized fields and coupling, and the renormalization scale μ . Here (compare Eq. (2.3))

$$T_{cl}(\mathcal{S}, \mathcal{P}) = [\bar{\psi} \mathcal{S} \psi] + [\bar{\psi} i\mathcal{P} \gamma_5 \psi] , \quad (2.21a)$$

$$T_{an}(\mathcal{S}, \mathcal{P}) = \tilde{\beta} [F^2] + \gamma_m \left([\bar{\psi} \mathcal{S} \psi] + [\bar{\psi} i\mathcal{P} \gamma_5 \psi] \right) . \quad (2.21b)$$

Chiral symmetry constrains the renormalization of the sources $\mathcal{S}(x)$ and $\mathcal{P}(x)$ to be the same as that of the fermion mass m . The operators $T_{cl}(\mathcal{S}, \mathcal{P})$ and $T_{an}(\mathcal{S}, \mathcal{P})$ are therefore renormalization-group invariant, too.

The $O(\sigma^2)$ terms in Eq. (2.19) are needed to ensure exact dilatation invariance of the regularized action in d dimensions. However, our derivation of the low-energy action will not depend on these terms, and therefore we do not consider them any further.

III. POWER COUNTING AND THE LEADING-ORDER LAGRANGIAN

In QCD, exact chiral symmetry is restored when the fermion mass m tends to zero. The chiral lagrangian is a low-energy expansion around the chiral limit. The systematic nature of the low-energy expansion derives from the fact that m is a parameter of the microscopic theory that can be varied continuously. When the fermion mass vanishes, so does the mass of the pions.

As mentioned in the introduction, the anomalous $U(1)_A$ symmetry, and its associated flavor-singlet pseudoscalar meson, can be incorporated into the framework of the low-energy expansion by invoking the 't Hooft large- N_c limit, in which the number of colors N_c tends to infinity, while the number of flavors is held fixed. For $m = 0$, the anomalous divergence of the singlet axial current is

$$\partial_\mu J_{5\mu} = cg^2 N_f \text{tr} F \tilde{F} , \quad (3.1)$$

where the numerical constant c depends on the representation in which the fermions transform. The axial anomaly is proportional to g^2 , which scales as $1/N_c$, because in large- N_c counting the product $g^2 N_c$ is $O(1)$. As a result, for $N_c \rightarrow \infty$ (and fixed N_f) the mass of the flavor-singlet pseudoscalar meson tends to zero.⁷

The starting point of the systematic low-energy expansion then involves a double limit: the chiral limit $m \rightarrow 0$, together with the limit $N_c \rightarrow \infty$. The power-counting suppression of the axial anomaly in the microscopic theory is communicated to the low-energy effective theory by matching correlation functions obtained by differentiating both partition functions with respect to the axion source $\theta(x)$. Using this, one can show that the coefficients of the power-series expansion of the $V_I(\eta - \theta)$ potentials, introduced in Sec. II B, adhere to a natural power-counting hierarchy. This limits the number of linearly independent low-energy constants occurring at each order of the expansion to be finite [18, 26].

In our case, we are interested in the different limit already introduced in Sec. I, in which the dilatation current is effectively conserved in the infrared, while the axial current of Eq. (3.1) remains anomalous. In Sec. III A we will discuss this triple limit and the assumptions on which it depends in more detail, and use it to control the low-energy expansion. We then proceed in Sec. III B to write down the leading-order lagrangian of the effective theory. Much like in the case of the $U(1)_A$ symmetry, using only symmetry considerations and the usual chiral power counting allows the leading-order lagrangian to depend on four arbitrary potentials $V_I(\tau - \sigma)$. Using a strategy similar to that followed in Refs. [18, 26], we then match correlation functions obtained by differentiating with respect to the dilaton source $\sigma(x)$ between the microscopic and the effective theories, and use this to establish a power-counting hierarchy for the power-series coefficients of these potentials.

⁷ The actual argument involves a comparison of the large- N_c scaling of $\langle J_{5\mu}(x) J_5(y) \rangle$ and of $\langle \text{tr} F \tilde{F}(x) J_5(y) \rangle$ where $J_5 = \bar{\psi} \gamma_5 \psi$.

The result is a leading-order lagrangian that depends only on a finite (and rather small) number of low-energy parameters. In Sec. III C we show that dilatation invariance can be maintained even after we set $\sigma(x) = 0$, at the price of promoting the power-series coefficients of the $V_I(\tau)$ potentials to (global) spurions. We also determine the transformation properties of the renormalized chiral sources, which fixes the dependence of the leading-order lagrangian on the mass anomalous dimension.

A. Nature of the expansion

As explained in the introduction, we conjecture that the effective low-energy theory can be constructed as a systematic expansion around a triple limit. One of these limits is just the chiral limit, $m \rightarrow 0$. The other two limits, namely, the Veneziano large- N limit and the limit $n_f \nearrow n_f^*$, are more subtle.

First, we need to define a reference scale against which to measure dimensionful quantities in the effective theory. We will set the reference infrared scale using one of the decay constants, the pion decay constant \hat{f}_π or the dilatonic-meson decay constant \hat{f}_τ , in the chiral limit. The decay constants are defined by⁸

$$\langle 0 | J_{5\mu}^a(x) | \pi^b \rangle = i p_\mu \hat{f}_\pi \delta^{ab} e^{ipx} , \quad (3.2)$$

$$\langle 0 | T_{\mu\nu}(x) | \tau \rangle = \frac{\hat{f}_\tau}{3} (-\delta_{\mu\nu} p^2 + p_\mu p_\nu) e^{ipx} , \quad (3.3)$$

which implies (see Sec. IV C below)

$$\langle 0 | S_\mu(x) | \tau \rangle = i p_\mu \hat{f}_\tau e^{ipx} . \quad (3.4)$$

Here $|\cdot\rangle$ denotes a single-particle state of momentum p . These states can be labeled by the effective fields of the low-energy theory: $\pi^a(x)$ is the pion field, which is related to the non-linear field as

$$\Sigma = \exp(2i\pi/\hat{f}_\pi) = \exp(2i\pi^a T^a / \hat{f}_\pi) , \quad (3.5)$$

while $\tau(x)$ is a new effective field for the dilatonic meson. The indices $a, b = 1, \dots, N_f^2 - 1$ label the adjoint representation of $SU(N_f)$. Both the non-singlet axial currents $J_{5\mu}^a$ and the energy-momentum tensor $T_{\mu\nu}$ are renormalization-group invariant, and the same applies to the decay constants \hat{f}_π and \hat{f}_τ .⁹

The large- N scaling of the decay constants follows from considering the two-point functions of $J_{5\mu}^a$ or $T_{\mu\nu}$, which gives

$$\hat{f}_\pi \sim \sqrt{N_c} , \quad (3.6)$$

$$\hat{f}_\tau \sim N_c . \quad (3.7)$$

Equation (3.6) is recognized as the familiar scaling law of the decay constant of any meson made of fundamental-representation fermions.¹⁰ With these scaling laws in mind we may

⁸ To avoid confusion, we note that Eqs. (3.2) and (3.3) are used to define the decay constants for any m . However, we will reserve the notation $\hat{f}_\pi, \hat{f}_\tau$ for the decay constants in the chiral limit.

⁹ By contrast, the $U(1)_A$ current is not renormalization-group invariant because of the mixing of $\partial_\mu J_{5\mu}$ with $\text{tr} F \tilde{F}$. This complicates the dependence of the effective theory on the renormalization scale of the microscopic theory [26].

¹⁰ \hat{f}_π is independent of N_f because the flavor index in Eq. (3.2) is held fixed.

thus set the dynamical infrared scale Λ_{IR} to be

$$\Lambda_{IR} \sim \frac{4\pi\hat{f}_\pi}{\sqrt{N_c}} \sim \frac{4\pi\hat{f}_\tau}{N_c}. \quad (3.8)$$

Here we use that $4\pi\hat{f}_\pi$ is usually taken to be the ultraviolet cutoff of the chiral lagrangian, it being the typical scale for the masses of non-Nambu–Goldstone mesons. The \sim symbol in Eq. (3.8) serves to indicate that either expression can be used as the definition of Λ_{IR} . The two options are equally good, because in the Veneziano limit we expect the dimensionless ratio $\sqrt{N_c}\hat{f}_\pi/\hat{f}_\tau$ to smoothly approach a non-zero value. We recall that meson masses are $O(1)$ in the large- N counting, and hence, with the exception of pNGBs, we expect all meson masses to scale like Λ_{IR} .

Turning to the light-meson sector, the three small parameters of the low-energy expansion will be

$$m/\Lambda_{IR}, \quad (3.9a)$$

$$1/N, \quad (3.9b)$$

$$n_f - n_f^*. \quad (3.9c)$$

As already mentioned, while m/Λ_{IR} and $1/N$ (equivalently, $1/N_c$) can be taken arbitrarily small independent of each other, the same is not true for $n_f - n_f^*$, which cannot be parametrically smaller than $1/N$. In the case where the expansion parameters are comparable in size, we will again use δ as a generic name for any one of them. For example, the behavior of the pion mass m_π is

$$m_\pi^2/\Lambda_{IR}^2 \sim (m/\Lambda_{IR})^1 (1/N)^0 (n_f - n_f^*)^0 \sim m/\Lambda_{IR} \sim \delta. \quad (3.10)$$

In addition to the expansion in the input parameters of Eq. (3.9), the effective theory involves, as usual, a derivative expansion, with $p_i \cdot p_j/\Lambda_{IR}^2 \sim \delta$, where $p_i \cdot p_j$ is the product of any two external momenta.

We next reexamine the anomalous divergence of the dilatation current, which, in the chiral limit, reads

$$\partial_\mu S_\mu = -\tilde{\beta}[F^2], \quad (3.11)$$

where, in terms of the 't Hooft coupling

$$\alpha = \frac{g^2 N_c}{4\pi}, \quad (3.12)$$

we have (compare Eq. (2.4))

$$\tilde{\beta} = \frac{\mu}{4\alpha} \frac{\partial\alpha}{\partial\mu}. \quad (3.13)$$

Both α and $\tilde{\beta}(\alpha)$ are $O(1)$ in large- N counting.

We will assume that the trace anomaly can be made arbitrarily small by approaching the sill of the conformal window from below. More specifically, we will assume that, in the chiral limit (compare Eq. (2.3c)),

$$\tilde{\beta}[F^2] \sim (n_f - n_f^*)^\eta, \quad n_f \nearrow n_f^*, \quad (3.14)$$

for some $\eta > 0$, in the sense that all matrix elements of the operator on the left-hand side vanish like $(n_f - n_f^*)^\eta$ when probed on the infrared scale Λ_{IR} . The main motivation for conjecturing Eq. (3.14) is the ever slower running of the coupling as N_f is increased towards the conformal window, evidence for which comes from numerous lattice studies (see Ref. [3] and references therein).

Further heuristic support for Eq. (3.14) comes from modeling chiral symmetry breaking by the gap equation. According to this analysis, chiral symmetry breaking in a walking theory sets in when the coupling has reached a critical value $g_c^2 = 4\pi^2/(3C_2)$ [32–35]. For the fundamental representation, $C_2 = (N_c^2 - 1)/(2N_c)$, and in the large- N_c limit the critical value of the 't Hooft coupling (3.12) is $\alpha_c = 2\pi/3$. In order to determine when the conformal window is entered, α_c is compared with the (putative) value of the IRFP, α_* [36–38].¹¹ The latter is inferred, for example, from the two-loop beta function, while ignoring the possible onset of spontaneous symmetry breaking (see App. A). It follows that, if $\alpha_* > \alpha_c$, the IRFP is not reached, and the theory exists in a chirally broken (and confining) phase; whereas if $\alpha_* < \alpha_c$, the running comes to a halt as $\alpha(\mu) \nearrow \alpha_*$, and the infrared theory is conformal. In the Veneziano limit, α_* is a function of n_f , and the conformal window is entered when $\alpha_*(n_f) = \alpha_c$. Therefore, when the boundary separating the broken and conformal “phases” is reached from within the broken phase, chiral symmetry breaking occurs at a coupling where the beta function tends to zero.

We stress that we have no way of rigorously establishing the correctness of Eq. (3.14). This is the main weakness of our treatment.¹² In addition, unlike the parameters m and $1/N_c$, in which we expand around zero, we do not know the critical value n_f^* around which we are to expand in n_f . In practice, one would have to determine both η and n_f^* empirically. One way to do that is by treating η and n_f^* as additional parameters of the low-energy effective theory, which, together with the rest of the low-energy constants, are to be determined by the usual procedure of comparing the correlation functions of the microscopic theory with the predictions of the low-energy theory.

In order to avoid cumbersome notation, we will henceforth assume that $\eta = 1$ in Eq. (3.14). Except for the trivial replacement of $n_f - n_f^*$ by $(n_f - n_f^*)^\eta$, nothing else would change in our analysis in case that the actual value of η is different.¹³

We conclude this subsection with several comments.

First, the reason why we consider only the fundamental representation is that, for any other representation, asymptotic freedom will be lost if we take the large- N_c limit while increasing the number of higher-representation flavors such that N_f/N_c is held fixed.

Second, the one-loop mass anomalous dimension is $\gamma_m = 6g^2C_2/(16\pi^2)$. Since $C_2 = (N_c^2 - 1)/(2N_c)$ for the fundamental representation, the mass anomalous dimension is $O(1)$ in large- N counting. Unlike the beta function, there is no reason that the mass anomalous dimension will become parametrically small near the bottom of the conformal window. Indeed, numerical simulations of, *e.g.*, the $N_f = 8$ theory, suggest that it is large [8] (see also Ref. [3]). A simplification that does take place is that, since the running comes to a halt, $\gamma_m(\alpha(\mu))$ is hardly changing over many energy decades. This has implications for the

¹¹ For a different approach to this problem, see Ref. [39].

¹² We would like to point out that the results of Ref. [40] are in disagreement with our main assumption (3.14).

However, the analysis of Ref. [40] involves a number of ingredients that do not constitute a systematic approximation, and, therefore, we believe that it does not disprove our assumption.

¹³ For the two-loop beta function, as well as for a crude model estimate of Eq. (3.14), see App. A.

effective low-energy theory, which we will discuss in Sec. III C below.

Finally, we recall that unlike in the usual large- N_c limit, in the Veneziano limit the axial anomaly does not vanish, because we now have $g^2 N_f \sim 1$ in Eq. (3.1). For this reason, the flavor-singlet pseudoscalar meson does not become light, and will not be considered in the rest of this paper.

B. Invariant potentials at tree level and the power counting

The fields of the effective low-energy theory include the non-linear field $\Sigma(x)$ introduced in Sec. II B, and the dilatonic meson field $\tau(x)$. The latter is inert under chiral transformations. Imposing for the time being only the usual chiral power counting, the leading-order lagrangian consists of terms of order $p^2 \sim m \sim \delta$, and is given by¹⁴

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_\pi + \tilde{\mathcal{L}}_\tau + \tilde{\mathcal{L}}_m + \tilde{\mathcal{L}}_d , \quad (3.15)$$

where

$$\tilde{\mathcal{L}}_\pi = \frac{f_\pi^2}{4} V_\pi(\tau - \sigma) e^{2\tau} \text{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) , \quad (3.16)$$

$$\tilde{\mathcal{L}}_\tau = \frac{f_\tau^2}{2} V_\tau(\tau - \sigma) e^{2\tau} (\partial_\mu \tau)^2 , \quad (3.17)$$

$$\tilde{\mathcal{L}}_m = -\frac{f_\pi^2 B_\pi}{2} V_M(\tau - \sigma) e^{y\tau} \text{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi) , \quad (3.18)$$

$$\tilde{\mathcal{L}}_d = f_\tau^2 B_\tau e^{4\tau} V_d(\tau - \sigma) . \quad (3.19)$$

$\tilde{\mathcal{L}}_\pi$ and $\tilde{\mathcal{L}}_\tau$ are the kinetic terms for pions and for the dilatonic meson, respectively. $\tilde{\mathcal{L}}_m$ is a generalized chiral mass term, whereas $\tilde{\mathcal{L}}_d$ accounts for the self-interactions of the dilatonic meson. Actually, $\tilde{\mathcal{L}}_d$ is $O(\delta^0)$, and not $O(\delta^1)$, in apparent violation of the rules of the effective theory. We will see later on how this is resolved. The relation of f_π and f_τ to the physical decay constants \hat{f}_π and \hat{f}_τ in the chiral limit will be discussed below. B_π and B_τ are additional low-energy constants of mass dimension one and two, respectively. As we will see below (see Sec. IV B), the dependence of the tree-level masses on the B parameters implies that they are $O(1)$ in large- N counting.

The effective action $\tilde{S} = \int d^4x \tilde{\mathcal{L}}(x)$ is invariant under dilatations. The transformation rules of the dynamical fields are

$$\Sigma(x) \rightarrow \Sigma(\lambda x) , \quad (3.20a)$$

$$\tau(x) \rightarrow \tau(\lambda x) + \log \lambda . \quad (3.20b)$$

The transformation rule of $\sigma(x)$ is given in Eq. (2.17d). Notice the similarity of the transformation rules of the dynamical field $\tau(x)$ and the source field $\sigma(x)$. The transformation rule of the chiral source $\chi(x)$ that follows from Eq. (3.18) is

$$\chi(x) \rightarrow \lambda^{4-y} \chi(\lambda x) . \quad (3.21)$$

¹⁴ We work in euclidean space.

The transformation rule of the bare chiral source, given in Eq. (2.17c), reflects its engineering dimension. Compatibility of Eqs. (3.21) and (2.17c) would require $y = 3$. In fact, the effective theory must be constructed using the renormalized chiral source, because it is the differentiation with respect to this renormalized source that produces finite correlation functions of the microscopic theory. The dilatation transformation rule of the renormalized chiral source is different from that of the bare source. We will derive it in Sec. III C below.

Because the shifts of $\sigma(x)$ and $\tau(x)$ cancel each other, the potentials V_I , $I = \pi, \tau, M, d$, undergo only a coordinates transformation

$$V_I(\tau(x) - \sigma(x)) \rightarrow V_I(\tau(\lambda x) - \sigma(\lambda x)) . \quad (3.22)$$

In other words, their scaling dimension is zero, which is why we will sometimes refer to them as invariant potentials. This transformation places no algebraic constraints on the coefficients of the power-series expansion,

$$V_I = \sum_{n=0}^{\infty} \frac{c_{I,n}}{n!} (\tau - \sigma)^n . \quad (3.23)$$

The coefficients $c_{I,n}$ therefore amount to infinitely many, linearly independent low-energy parameters.

In this subsection, we will establish a power-counting hierarchy

$$c_{I,n} \sim (n_f - n_f^*)^n \sim \delta^n . \quad (3.24)$$

This will allow us to set the potentials V_π , V_τ and V_M equal to one in the leading-order lagrangian, because they multiply terms that are already $O(\delta)$; and to truncate V_d to a linear function, as in Eq. (1.6).

When writing down the chiral lagrangian, it is customary to keep explicit the m dependence. Similarly, the $n_f - n_f^*$ dependence can be made explicit by performing the further expansion¹⁵

$$c_{I,n} = \sum_{k=0}^{\infty} \tilde{c}_{I,nk} (n_f - n_f^*)^k . \quad (3.25)$$

The new coefficients $\tilde{c}_{I,nk}$ are independent of all three expansion parameters m , $1/N$ and $n_f - n_f^*$. The power counting (3.24) can be equivalently stated as

$$\tilde{c}_{I,nk} = 0 , \quad k < n . \quad (3.26)$$

Below, we will mostly refer to the $c_{I,n}$ coefficients, but we will keep in mind that each of them is a function of $n_f - n_f^*$.

The hierarchy (3.24) is analogous to the one established in Refs. [18, 26] for the coefficients of the invariant potentials $V(\eta - \theta)$ discussed in Sec. II B, which occur in the low-energy effective theory that accommodates the $U(1)_A$ symmetry. The main difference is that the power-series coefficients of $V(\eta - \theta)$ are suppressed by powers of $1/N_c$ in the usual large- N_c limit. By contrast, here the power-series coefficients of the $V_I(\tau - \sigma)$ potentials are suppressed by powers of $n_f - n_f^*$, which is a new small parameter that is present in the Veneziano limit but not in the usual large- N_c limit.

¹⁵ Recall that we are assuming $\eta = 1$ in Eq. (3.14), see Sec. III A.

We will derive Eq. (3.24) by matching judiciously chosen correlation functions of the low-energy theory to the corresponding correlation functions of the microscopic theory, in the chiral limit. In order to understand the physical origin of Eq. (3.24), we begin by considering correlations functions of the microscopic theory obtained by differentiating n times with respect to the $\sigma(x)$ source field. As usual, the role of the external sources is to generate operator insertions, and thus in the microscopic theory we will assume that no two differentiations are done at coinciding points. We thus have

$$\Gamma_n^{\text{MIC}}(x_1, \dots, x_n) = (-1)^n \frac{\partial}{\partial \sigma(x_1)} \cdots \frac{\partial}{\partial \sigma(x_n)} W \quad (3.27a)$$

$$= \langle T_{an}(x_1) \cdots T_{an}(x_n) \rangle_{m=0} \quad (3.27b)$$

$$= \tilde{\beta}^n \langle F^2(x_1) \cdots F^2(x_n) \rangle . \quad (3.27c)$$

Here $W = \log Z$. It is understood that all source fields are set to zero after the differentiations are done (since we work in the chiral limit, this includes setting $\mathcal{S}(x) = \mathcal{P}(x) = 0$, cf. Eq. (2.11)). Allowing also for differentiations with respect to the scalar and pseudoscalar sources \mathcal{S}_{AB} and \mathcal{P}_{AB} gives us access to

$$\begin{aligned} \Gamma_{nkl}^{\text{MIC}}(x_1, \dots, x_n; y_1, \dots, y_k; z_1, \dots, z_\ell) &= \quad (3.28) \\ &= (-1)^{n+k+\ell} \frac{\partial}{\partial \sigma(x_1)} \cdots \frac{\partial}{\partial \sigma(x_n)} \frac{\partial}{\partial \mathcal{S}(y_1)} \cdots \frac{\partial}{\partial \mathcal{S}(y_k)} \frac{\partial}{\partial \mathcal{P}(z_1)} \cdots \frac{\partial}{\partial \mathcal{P}(z_\ell)} W \\ &= \tilde{\beta}^n \langle F^2(x_1) \cdots F^2(x_n) \bar{\psi}\psi(y_1) \cdots \bar{\psi}\psi(y_k) \bar{\psi}i\gamma_5\psi(z_1) \cdots \bar{\psi}i\gamma_5\psi(z_\ell) \rangle , \end{aligned}$$

where we have suppressed the flavor indices. We see that, by Eq. (3.14),

$$\Gamma_{nkl}^{\text{MIC}} = O(\delta^n) . \quad (3.29)$$

Here and in the rest of this subsection, $O(\delta^n)$ will always mean $O((n_f - n_f^*)^n)$. In Eq. (3.28), all the operators are renormalized ones.¹⁶ Correspondingly, the chiral sources used to generate $\Gamma_{nkl}^{\text{MIC}}$ are renormalized, too.

Turning to the low-energy effective theory, in order to probe the potentials $V_I(\tau - \sigma)$, we will carry out multiple differentiations at the *same* spacetime point. This immediately raises the issue of reconciling this procedure with the avoidance of the limit of coinciding points in the correlation functions of the microscopic theory. There is in fact no conflict. The point is that coinciding spacetime points do not have the same physical significance in the microscopic and the effective theories. While the pNGBs are represented in the effective theory as point particles, in reality they have a finite size, roughly of order $1/\Lambda_{IR}$. We will thus choose to consider operator insertions in the microscopic theory at non-coinciding points, but at distances much smaller than $1/\Lambda_{IR}$, such that they collapse to insertions at coinciding points in the effective low-energy theory. In addition, asymptotic states are obtained from operator insertions at large distances, in the same way in the microscopic and the effective theories. This will allow us to enforce the power counting of Eq. (3.29) on the effective theory's correlation functions that we will need in order to probe the $V_I(\tau - \sigma)$ potentials.

¹⁶ To avoid cumbersome notation, we have omitted the square brackets used in Sec. II to denote renormalized operators.

Among the correlation functions obtained in the effective theory by differentiating multiple times with respect to $\sigma(x)$ at the same spacetime point, there is a special subset, in which all the differentiations are applied to the same potential V_I . This gives rise to a diagram containing the n -th derivative of V_I , which, when evaluated at $\sigma = 0$, is simply (compare Eq. (3.23))

$$\left(-\frac{\partial}{\partial\sigma}\right)^n V_I \Big|_{\sigma=0} = \sum_{k=0}^{\infty} \frac{c_{I,n+k}}{k!} \tau^k = V_I^{(n)}(\tau). \quad (3.30)$$

The natural way for the correlation functions of the effective theory to satisfy Eq. (3.14), is that the power-series coefficients of V_I satisfy

$$c_{I,k} = O(\delta^n), \quad k \geq n, \quad (3.31)$$

because this implies that $V_I^{(n)}(\tau) = O(\delta^n)$ in its entirety. It is easy to see that conditions (3.24), (3.26) and (3.31) are mathematically equivalent. As it turns out, it is convenient to establish the power-counting hierarchy in the form of Eq. (3.31).

There are a few more technical points we need to address before turning to the actual derivation. First, the zeroth-order term in the double expansion of each potential, $\tilde{c}_{I,00}$, is redundant, because each term in the lagrangian (3.15) is already multiplied by a low-energy constant. We will use this freedom to set $\tilde{c}_{M,00} = \tilde{c}_{\pi,00} = \tilde{c}_{\tau,00} = 1$. The role of $\tilde{c}_{d,00}$, the leading-order term in the expansion of V_d , is more subtle. The $\tilde{c}_{d,00}$ -dependent term is the only part in the entire lagrangian which is $O(1)$. All other terms start at $O(\delta)$, which is how the leading-order lagrangian normally scales according to the low-energy power counting. As we will show in Sec. IV B below, when we develop the diagrammatic expansion for each theory, there is a redefinition of the τ field that allows us to set $c_{d,0} = -c_{d,1}/4$. This is equivalent to setting $\tilde{c}_{00} = 0$ and $\tilde{c}_{01} = -\tilde{c}_{11}/4$, so that, after the redefinition, the entire leading-order lagrangian is $O(\delta)$.

For the diagrammatic expansion, we split the dilatonic meson field as

$$\tau(x) = v + \tilde{\tau}(x)/(e^v f_\tau) \equiv v + \tilde{\tau}(x)/\hat{f}_\tau, \quad (3.32)$$

where $v = \langle \tau \rangle$ is the classical solution, and the quantum field $\tilde{\tau}(x)$ has a canonically normalized kinetic term. Applying the τ field redefinition of Sec. IV B to Eq. (3.32) shifts the classical solution to $v = 0$ in the chiral limit, and the dependence on the original value of v gets absorbed into the decay constants,

$$\hat{f}_\pi = e^v f_\pi, \quad (3.33a)$$

$$\hat{f}_\tau = e^v f_\tau, \quad (3.33b)$$

and into the two other low-energy constants,

$$\hat{B}_\pi = e^{(y-2)v} B_\pi, \quad (3.34a)$$

$$\hat{B}_\tau = e^{2v} B_\tau. \quad (3.34b)$$

In these equations, v is the classical solution in the chiral limit before the field redefinition, given by Eq. (4.6) below. After the τ field redefinition, which sets $v = 0$ in the chiral limit, it can moreover be shown that the ground state is stable against the inclusion of higher-order terms in the classical potential.¹⁷

¹⁷ In fact, for what we need here it would be enough to set $\tilde{c}_{d,00} = 0$, leaving $\tilde{c}_{d,01}$ free; see Sec. V A.

As for the non-linear Σ field, in this paper we assume that $m > 0$ (or, if the chiral limit has been taken, that it was approached as $m \searrow 0$), hence its classical vacuum expectation value is $\langle \Sigma \rangle = 1$. The diagrammatic expansion is then generated as usual by expanding Σ in terms of the pion field, Eq. (3.5).

We will prove the power-counting hierarchy (3.31) using the diagrammatic expansion described above. The proof of Eq. (3.31) proceeds by induction. We assume that Eq. (3.31) has been proved up to $n - 1$, so that $c_{I,k} = O(\delta^k)$ for all $k \leq n - 1$, and $c_{I,k} = O(\delta^{n-1})$ for $k \geq n - 1$. In the induction step we must therefore improve the bound to $c_{I,k} = O(\delta^n)$ for $k \geq n$.

In the effective theory, the n differentiations with respect to the $\sigma(x)$ source will all be done at the same point. We will be interested in the special subset of diagrams mentioned above, in which all n differentiations have been applied to a single V_I , so that the diagram involves a single vertex

$$V_I^{(n)}(\tau(x)) . \quad (3.35)$$

We claim that the sum of diagrams with a single such vertex must be $O(\delta^n)$ by itself. When we apply the σ differentiations, we will also obtain diagrams with the factorizable form

$$V_{I_1}^{(k_1)}(\tau(x)) \cdots V_{I_\ell}^{(k_\ell)}(\tau(x)) , \quad k_1 + \cdots + k_\ell = n , \quad (3.36)$$

where at least two k_i 's are larger than zero, and therefore all $k_i < n$. Now, the factorizable-form diagrams may be obtained by first using ℓ non-coinciding points to obtain

$$V_{I_1}^{(k_1)}(\tau(x_1)) \cdots V_{I_\ell}^{(k_\ell)}(\tau(x_\ell)) , \quad (3.37)$$

and then taking the limit where all the ℓ spacetime points are brought together. Since $k_i < n$, the induction hypothesis implies $V_{I_i}^{(k_i)}(\tau(x_i)) = O(\delta^{k_i})$ for all i . The diagrams with the point-split structure (3.37) are thus $O(\delta^n)$, and the same is true in the limit of coinciding points. Therefore, the left-over part, which is the (sum of) diagrams with the non-factorizable vertex (3.35), must be $O(\delta^n)$ by itself.

The rest of the argument is more technical, and requires a case-by-case study. We will give here the proof of Eq. (3.31) for the power-series coefficients of V_d , which is the potential that is easiest to handle. The demonstration of Eq. (3.31) for the other three potentials V_M , V_π and V_τ makes use of the same ingredients, but is slightly more involved, and is relegated to App. B. The reader who is not interested in these details may skip to the next subsection.

Let us carry out no additional differentiations except for the n derivatives with respect to the $\sigma(x)$ field, and isolate the diagrams with a non-factorizable vertex. After dividing by $\hat{f}_\tau^2 \hat{B}_\tau$ we obtain the effective theory's correlation function

$$\Gamma_{d,n}^{\text{EFT}} = (\hat{f}_\tau^2 \hat{B}_\tau)^{-1} \langle \Omega_{d,n} + \Omega_{\pi,n} + \Omega_{\tau,n} \rangle , \quad (3.38)$$

where

$$\Omega_{d,n} = \hat{f}_\tau^2 \hat{B}_\tau V_d^{(n)}(\tau) e^{4\tau} , \quad (3.39)$$

$$\Omega_{\pi,n} = \frac{\hat{f}_\pi^2}{4} V_\pi^{(n)}(\tau) e^{2\tau} \text{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) , \quad (3.40)$$

$$\Omega_{\tau,n} = \frac{\hat{f}_\tau^2}{2} V_\tau^{(n)}(\tau) e^{2\tau} (\partial_\mu \tau)^2 . \quad (3.41)$$

The generalized mass term $\tilde{\mathcal{L}}_m$ does not contribute, because we did not differentiate with respect to the chiral source $\chi(x)$, but we did set $\chi(x) = m = 0$. The division by $\hat{f}_\tau^2 \hat{B}_\tau$ in Eq. (3.38) makes $\Gamma_{d,n}^{\text{EFT}}$ dimensionless, and, in addition, it removes the leading large- N dependence, so that $\Gamma_{d,n}^{\text{EFT}}$ is $O(1)$ in large- N counting.

Our first step is to establish that $c_{d,n} = O(\delta^n)$. Considering the expansion

$$\Omega_{d,n} = \sum_{k=0}^{\infty} \frac{c_{d,n+k}}{k!} \hat{f}_\tau^2 \hat{B}_\tau \tau^k e^{4\tau} , \quad (3.42)$$

we need to find a suitable limit where $\Gamma_{d,n}^{\text{EFT}}$ is dominated by the first term on the right-hand side of Eq. (3.42). If we can do so, we will have

$$\Gamma_{d,n}^{\text{EFT}} = c_{d,n} + O(\delta^{n+1}) , \quad (3.43)$$

and the desired result will follow from the matching to the microscopic theory.

We will derive Eq. (3.43) by working in the parameter region

$$m/\Lambda_{IR} \sim 1/N \sim p_i \cdot p_j \sim \epsilon_n , \quad (3.44a)$$

$$n_f - n_f^* \sim \delta , \quad (3.44b)$$

where $p_i \cdot p_j$ denotes the product of any two external momenta, and where

$$\epsilon_n \ll \delta^n . \quad (3.44c)$$

We begin with the observation that $\langle \Omega_{\pi,n} \rangle$ contains a pion loop, which, including the normalization factor from Eq. (3.38), is

$$\frac{\hat{f}_\pi^2 \langle \text{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) \rangle}{\hat{f}_\tau^2 \hat{B}_\tau} \sim \frac{\langle \text{tr}(\partial_\mu \pi \partial_\mu \pi) \rangle}{\hat{f}_\tau^2 \hat{B}_\tau} \sim \frac{N^2 m_\pi^4}{\hat{f}_\tau^2 \hat{B}_\tau} \sim \frac{m_\pi^4}{\Lambda_{IR}^4} \sim \epsilon_n^2 , \quad (3.45)$$

where we have used that the number of pions scales like $N_f^2 \sim N^2$, the large- N scaling of \hat{f}_τ and \hat{B}_τ , and Eq. (3.5). It follows that $\langle \Omega_{\pi,n} \rangle$ can be neglected. We next consider $\langle \Omega_{\tau,n} \rangle$. Since we have shifted the classical vacuum to $v = 0$, we may substitute $\tau = \tilde{\tau}/\hat{f}_\tau$. The contribution of $\langle \Omega_{\tau,n} \rangle$ thus involves the dilatonic meson tadpole¹⁸

$$\frac{\langle \hat{f}_\tau^2 (\partial_\mu \tau)^2 \rangle}{\hat{f}_\tau^2 \hat{B}_\tau} = \frac{\langle (\partial_\mu \tilde{\tau})^2 \rangle}{\hat{f}_\tau^2 \hat{B}_\tau} \sim \frac{m_\tau^4}{\hat{f}_\tau^2 \hat{B}_\tau} \sim \frac{m_\tau^4}{N^2 \Lambda_{IR}^4} . \quad (3.46)$$

As we will see in Sec. IV B below, $m_\tau^2 \sim \Lambda_{IR}^2 |n_f - n_f^*|$ in the chiral limit, and so, generically, the suppression of dilatonic meson loop comes from the smallness of both $|n_f - n_f^*|$ and $1/N$. Here we invoke the parameter region (3.44), because we must maintain the freedom to vary $n_f - n_f^*$. Therefore, we cannot neglect $\langle \Omega_{\tau,n} \rangle$ on the grounds of its m_τ dependence. However, Eq. (3.46) also involves a factor of $1/N^2 \sim \epsilon_n^2$, and this does justify the neglect of $\langle \Omega_{\tau,n} \rangle$.

¹⁸ Note that $\langle \tilde{\tau} \partial_\mu \tilde{\tau} \rangle = 0$.

At this stage we are left with $\langle \Omega_{d,n} \rangle$. The last step is to observe that, besides $c_{d,n}$, all other contributions to $\langle \Omega_{d,n} \rangle$ involve some number of dilatonic meson tadpoles $\langle \tau^2 \rangle$, each of which provides an additional suppression factor of $1/N^2$. This completes the proof that $c_{d,n} = O(\delta^n)$.

In the above argument, loop diagrams were always subleading. This is an example of a general result, first derived by Weinberg in the context of the chiral lagrangian [16], that the loop expansion of every correlation function in the effective theory obeys the standard power counting. Thus, every loop diagram is always suppressed relative to the tree diagrams that contribute to a given correlation function. Weinberg's result was generalized to the case where a dilatonic meson is present in Ref. [20]. While our setup is different from Ref. [20], both in terms of the expansion parameters (3.9) and the particular parameter range (3.44), the argument remains essentially the same. Henceforth, it will thus suffice to consider tree diagrams only.

In order to extend Eq. (3.31) to $c_{d,k}$ for $k > n$, we will need $\ell = k - n$ external τ legs. These external legs must be generated by differentiating with respect to one of the source fields, since Eq. (3.28) gives the complete set of correlation functions of the microscopic theory that can be matched to the effective theory. We will differentiate with respect to (the flavor trace of) the scalar source $\mathcal{S}(y)$. Including a suitable normalization factor and a trivial subtraction, we let

$$\Omega_{\mathcal{S}} = -\frac{1}{\hat{f}_{\pi}^2 \hat{B}_{\pi}} \frac{\partial \tilde{\mathcal{L}}}{\partial \mathcal{S}} \quad (3.47a)$$

$$\begin{aligned} &= \frac{1}{2} V_M(\tau) e^{y\tau} \text{tr}(\Sigma + \Sigma^{\dagger}) , \\ \tilde{\Omega}_{\mathcal{S}} &= \Omega_{\mathcal{S}} - N_f \quad (3.47b) \\ &= \frac{y\tilde{\tau}}{\hat{f}_{\tau}/N_f} - \frac{2}{\hat{f}_{\pi}^2} \text{tr}(\pi^2) + \dots , \end{aligned}$$

where we have used Eq. (3.18) and $\tilde{c}_{M,00} = 1$. This shows that $\tilde{\Omega}_{\mathcal{S}}$ can be used as an interpolating field for the dilatonic meson. As a generalization of Eq. (3.38), we will consider $\tilde{\Gamma}_{d,n,\ell}^{\text{EFT}}(p_1, \dots, p_{\ell})$, the Fourier transform of

$$\Gamma_{d,n,\ell}^{\text{EFT}} = (\hat{f}_{\tau}^2 \hat{B}_{\tau})^{-1} \left\langle (\Omega_{d,n}(x) + \Omega_{\pi,n}(x) + \Omega_{\tau,n}(x)) \tilde{\Omega}_{\mathcal{S}}(y_1) \cdots \tilde{\Omega}_{\mathcal{S}}(y_{\ell}) \right\rangle . \quad (3.48)$$

The demonstration that $c_{d,k} = O(\delta^n)$ for $k > n$ will proceed by an ‘‘inner’’ induction. Before turning to the general case, let us examine the two simplest cases.

We start with $\ell = k - n = 1$. The leading contribution to $\Gamma_{d,n,1}^{\text{EFT}}$ is a tree diagram with a single dilatonic meson line connecting $\Omega_{d,n}(x)$ with $\tilde{\Omega}_{\mathcal{S}}(y_1)$. Expanding $V_d^{(n)}(\tau) e^{4\tau} = c_{d,n} + \tau(4c_{d,n} + c_{d,n+1}) + O(\tau^2)$ we find that, after amputation, the contribution to $\tilde{\Gamma}_{d,n,1}^{\text{EFT}}$ is equal to $4c_{d,n} + c_{d,n+1}$ up to an $O(1)$ factor. Since we already know that $c_{d,n} = O(\delta^n)$, the conclusion is that we must have $c_{d,n+1} = O(\delta^n)$ as well.

Proceeding to $\ell = k - n = 2$, now $\left\langle \Omega_{d,n}(x) \tilde{\Omega}_{\mathcal{S}}(y_1) \tilde{\Omega}_{\mathcal{S}}(y_2) \right\rangle$ receives contributions from three topologically distinct tree diagrams, each containing two dilatonic meson lines. The tree diagram we need is the one in which $\Omega_{d,n}(x)$ is connected by a line to each of $\tilde{\Omega}_{\mathcal{S}}(y_1)$ and $\tilde{\Omega}_{\mathcal{S}}(y_2)$. In addition, there two more tree-diagram topologies. In one case, the two

lines emanate from $\tilde{\Omega}_S(y_1)$, and connect it to $\Omega_{d,n}(x)$ and to $\tilde{\Omega}_S(y_2)$. In the other case, the roles of $\tilde{\Omega}_S(y_1)$ and $\tilde{\Omega}_S(y_2)$ are interchanged. At this point we use the external momenta as probes. Each tree-diagram topology has a unique dependence on the external momenta, and this allows us to isolate the tree diagram that we are after, the one in which the two lines emanate from $\Omega_{d,n}(x)$. After amputation, the result involves a linear combination of $c_{d,n}$, $c_{d,n+1}$ and $c_{d,n+2}$, and, since we already know that $c_{d,n}$ and $c_{d,n+1}$ are $O(\delta^n)$, it follows that the same is true for $c_{d,n+2}$.

The structure of the induction step is now clear. Having established that $c_{d,k} = O(\delta^n)$ for $n \leq k \leq n + \ell - 1$, we will in the next step consider $\tilde{\Gamma}_{d,n,\ell}^{\text{EFT}}(p_1, \dots, p_\ell)$. By Weinberg's theorem (and its generalizations), tree diagrams make the leading contribution. It is easily seen that $\Omega_{\pi,n}$ cannot contribute to any tree diagram, and can thus be neglected.¹⁹ The case of $\Omega_{\tau,n}$ is different. For $\ell \geq 2$, there are tree diagrams that receive contributions from both $\Omega_{d,n}$ and $\Omega_{\tau,n}$. However, due to the presence of two derivatives, in all cases the contribution of $\Omega_{\tau,n}$ will be suppressed by a relative factor $p_i \cdot p_j \sim \epsilon_n$. We are left with the tree diagrams constructed only from $\Omega_{d,n}$. Once again using the different momentum dependence we isolate the tree diagram in which the ℓ lines all emanate from $\Omega_{d,n}$, and, finally, using the induction hypothesis we conclude that $c_{d,n+\ell} = O(\delta^n)$.

For the generalization of Eq. (3.31) to the other leading-order potentials V_M , V_π and V_τ , as well as for a brief discussion of the potentials encountered at the next-to-leading order and beyond, the reader is referred to App. B.

C. The leading-order lagrangian

Having established the power counting for the coefficients of the invariant potentials, we are done with the $\sigma(x)$ source field, and we set it equal to zero. The power counting (3.31) allows us to replace V_M , V_π and V_τ by one in the leading-order lagrangian, and to truncate V_d to a linear function, as in Eq. (1.6). The result is the following leading-order effective lagrangian,

$$\mathcal{L} = \mathcal{L}_\pi + \mathcal{L}_\tau + \mathcal{L}_m + \mathcal{L}_d, \quad (3.49)$$

where²⁰

$$\mathcal{L}_\pi = \frac{f_\pi^2}{4} e^{2\tau} \text{tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma), \quad (3.50)$$

$$\mathcal{L}_\tau = \frac{f_\tau^2}{2} e^{2\tau} (\partial_\mu \tau)^2, \quad (3.51)$$

$$\mathcal{L}_m = -\frac{f_\pi^2 B_\pi}{2} e^{y\tau} \text{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi), \quad (3.52)$$

$$\mathcal{L}_d = f_\tau^2 B_\tau e^{4\tau} (c_0 + c_1 \tau). \quad (3.53)$$

In spite of the absence of the σ field, the effective action $S = \int d^4x \mathcal{L}(x)$ is invariant under a different scale transformation, where the dynamical effective fields transform in the same

¹⁹ Note that $\tilde{\Omega}_S$ produces at least two pions.

²⁰ In the rest of this paper we reserve the abbreviations c_n and \tilde{c}_{nk} for $c_{d,n}$ and $\tilde{c}_{d,nk}$, respectively.

way as before, and, in addition,

$$c_0 \rightarrow c_0 - c_1 \log \lambda , \quad (3.54a)$$

$$c_1 \rightarrow c_1 , \quad (3.54b)$$

where we have promoted the coefficients of $V_d(\tau)$ to global spurions. [As shown in App. C, this invariance can be generalized to the case that we set $\sigma(x) = 0$ in the lagrangian (3.15) while keeping the potentials $V_I(\tau)$ completely arbitrary.]

There is one more issue to address. As already mentioned, the effective theory must be constructed using the renormalized chiral source. We will now derive the dilatation, or scale, transformation of the renormalized chiral source field χ . As we will see, this fixes the value of y in Eq. (3.21) in terms of the mass anomalous dimension.

We start by reexamining the bare microscopic partition function in the presence of the σ field. This partition function is in fact invariant under a family of scale transformations that depend on a continuous parameter ζ . The transformation rules of the dynamical and source bare fields, Eqs. (2.17a) through (2.17c), remain the same as before, whereas Eqs. (2.17d) and (2.17e) are generalized to

$$\sigma(x) \rightarrow \sigma(\lambda x) + \zeta \log \lambda , \quad (3.55a)$$

$$\mu \rightarrow \lambda^{(1-\zeta)} \mu , \quad (3.55b)$$

$$\tilde{g}_0 \rightarrow \tilde{g}_0 . \quad (3.55c)$$

The new, dimensionless, parameter \tilde{g}_0 is defined in terms of the bare coupling g_0 and the renormalization scale μ , according to

$$\tilde{g}_0 = \mu^{d/2-2} g_0 . \quad (3.56)$$

The invariance of the microscopic partition function under this family of scale transformations follows from the fact that

$$\mu e^{\sigma(x)} \rightarrow \lambda \mu e^{\sigma(\lambda x)} , \quad (3.57)$$

independent of ζ . The transformation (2.17) is recovered for the special case $\zeta = 1$. Once we set $\sigma(x) = 0$, the only consistent choice in Eq. (3.55) is $\zeta = 0$. The entire scale transformation (3.57) is then “carried” by the renormalization scale μ , which transforms according to its engineering dimension, $\mu \rightarrow \lambda \mu$. This is the first ingredient we need.

Our $\zeta = 0$ scale transformation has something in common with a renormalization group transformation, because the renormalization scale μ changes under it in the same way. What distinguishes this scale transformation from a renormalization-group transformation, is that, in a renormalization-group transformation, the bare fields and parameters are kept invariant; whereas under our scale transformation all bare fields and parameters transform according to their engineering dimension.

In particular, the bare coupling g_0 transforms non-trivially under our scale transformation, in accordance with the fact that it is dimensionful for $d \neq 4$. The dimensionless \tilde{g}_0 is invariant. Because $Z(g)g = \mu^{d/2-2} g_0 = \tilde{g}_0$, it follows that the renormalized coupling g , too, is held fixed. Postponing momentarily which value of the renormalized coupling we should choose, the very fact that it is kept invariant allows us to obtain the transformation rule of the renormalized fermion mass m and chiral source $\chi(x)$, which is

$$\chi(x) \rightarrow \lambda^{1+\gamma_m} \chi(\lambda x) . \quad (3.58)$$

We see that the scale transformation now involves both the engineering and the anomalous dimension. The mass anomalous dimension is to be evaluated at the fixed value of the coupling, $\gamma_m = \gamma_m(g)$, and we thus arrive at the simple power in Eq. (3.58) above. Requiring compatibility of this new transformation rule with Eq. (3.21) we conclude that

$$y = 3 - \gamma_m . \quad (3.59)$$

The final remaining question is then what value of γ_m , or, equivalently, g , we should choose to fix y in the effective theory. By construction, the effective low-energy theory is an expansion in m and $n_f - n_f^*$.²¹ The starting point of the expansion is therefore $n_f = n_f^*$ (and $m = 0$). Let α_*^c be the location of the infrared fixed point, expressed in terms of the 't Hooft coupling (3.12), as the conformal window is entered. Assuming that the beta and gamma functions depend continuously on n_f , at $n_f = n_f^*$ the right choice for the argument of γ_m is $\alpha = \alpha_*^c$, and thus,

$$\gamma_m = \gamma_m(\alpha_*^c) \equiv \gamma_m^* . \quad (3.60)$$

The physical picture behind this choice is that, when n_f tends to n_f^* from below, chiral symmetry breaking will set in at a scale where, first, the beta function has become vanishingly small, and second, the value of the renormalized coupling tends to α_*^c . Corrections to Eq. (3.60) can be expanded in $n_f - n_f^*$, and thus they are subsumed in the V_I potentials that fully account for the dependence of the low-energy theory on $n_f - n_f^*$.

Numerical simulations find that the mass anomalous dimension is, within error, always in the range (see Ref. [3] and references therein)

$$0 \leq \gamma_m \lesssim 1 , \quad (3.61)$$

suggesting that the actual value of y should be somewhere in the range

$$2 \lesssim y \leq 3 . \quad (3.62)$$

For theoretical work supporting a value of γ_m near the upper bound in Eq. (3.61) for walking theories, see Refs. [33–35, 41, 42].

IV. THE EFFECTIVE THEORY AT TREE LEVEL

In this section we study the classical vacuum and the tree-level spectrum. When writing down the leading-order lagrangian (3.49) we have already set $\sigma(x) = 0$. The last preparatory step is to set the chiral source to its “expectation value” $\chi(x) = m$. We will assume that either $m > 0$, or else the chiral limit has been taken as $m \searrow 0$, so that, in all cases, the classical pion vacuum is $\Sigma = 1$ or, equivalently, $\text{tr} \langle \Sigma \rangle = N_f$.

The vacuum of the dilatonic meson is determined by minimizing the classical potential

$$V_d(\tau) = f_\tau^2 B_\tau U(\tau) , \quad (4.1a)$$

²¹ For this discussion, the expansion in $1/N$ only plays the role of allowing $n_f - n_f^*$ to tend to zero continuously.

where

$$U(\tau) = -\frac{f_\pi^2 B_\pi N_f m}{f_\tau^2 B_\tau} e^{y\tau} + V_d(\tau) e^{4\tau} \quad (4.1b)$$

$$= -\frac{m}{\mathcal{M}} e^{y\tau} + V_d(\tau) e^{4\tau} ,$$

$$\mathcal{M} \equiv \frac{f_\tau^2 B_\tau}{f_\pi^2 B_\pi N_f} . \quad (4.1c)$$

Here \mathcal{M} has the dimensions of mass. The rescaled potential $U(\tau)$ is dimensionless, and is $O(1)$ in large- N counting. This follows from the scaling laws of the decay constants (see Sec. III A) and the fact that B_π and B_τ are themselves $O(1)$ in large- N counting. The latter property follows from the dependence of the pion and dilatonic meson masses on these parameters (see Sec. IV B below), taking into account that meson masses are $O(1)$ in large- N counting. The pion mass is parametrically small in m , and dilatonic meson mass is parametrically small in $n_f - n_f^*$ and m combined, while neither mass is parametrically small in $1/N$.²²

The function $V_d(\tau)$ is linear. If we make use of Eq. (3.25) and the power-counting hierarchy (3.26) to keep only the $O(n_f - n_f^*)$ term in its expansion, we get

$$V_d(\tau) = c_0 + c_1 \tau = \tilde{c}_{00} + (n_f - n_f^*)(\tilde{c}_{01} + \tilde{c}_{11} \tau) , \quad (4.2)$$

where the low-energy constants \tilde{c}_{00} , \tilde{c}_{01} , and \tilde{c}_{11} are independent of all three expansion parameters in Eq. (3.9).

This section is organized as follows. In Sec. IV A we study the classical ground state approximating V_d by its leading constant term. We find that this truncation does not give rise to a consistent starting point for the diagrammatic expansion. In Sec. IV B we study the linear V_d of Eq. (4.2). We establish that a stable classical ground state exists, and derive expressions for the masses of the pion and of the dilatonic meson. In Sec. IV C we study the dilatation current and its anomalous divergence at the level of the leading-order effective theory. In Sec. IV D we discuss the limit $n_f \nearrow n_f^*$, where the “phase boundary” between chirally broken and infrared-conformal theories is approached from within the chirally broken phase. Finally, in Sec. IV E we return to the results of Sec. IV B, and give a more hands-on description of how these predictions can be tested within a given model.

A. Constant V_d

An unusual feature of the classical potential (4.1) is that it contains both an $O(1)$ term, and terms which are $O(\delta)$. The \tilde{c}_{00} term in V_d is $O(1)$, while the terms linear in $n_f - n_f^*$ or in m are $O(\delta)$ in the power counting of Sec. III A. If we consider the $O(1)$ part all by itself, the classical potential is $U(\tau) = \tilde{c}_{00} e^{4\tau}$. Boundedness from below then requires $\tilde{c}_{00} > 0$, for which the classical vacuum corresponds to $\langle \tau \rangle \rightarrow -\infty$, or, equivalently, $\langle e^\tau \rangle = 0$.

In order to better understand this behavior, let us take into account also the mass term, while still neglecting the linear term in V_d . In order to avoid unboundedness below we must

²² As discussed in Sec. III A, the Veneziano large- N limit is needed in order to enable us to treat $n_f - n_f^*$ as a continuous parameter that can be made arbitrarily small.

now also require $0 < y < 4$ (compare Eq. (3.62)). The classical potential then has a unique minimum $\langle \tau \rangle = v$ given by

$$e^{(4-y)v} = \frac{ym}{4\tilde{c}_0\mathcal{M}} . \quad (4.3)$$

Now v is finite for $m > 0$, while in the chiral limit $v \rightarrow -\infty$ is recovered.

The “runaway” solution (4.3) is not a consistent starting point for the low-energy expansion. In the case at hand, the physical decay constants are the rescaled decay constants in Eq. (3.33), except that v is the solution of Eq. (4.3). Since $e^v \propto m^{\frac{1}{4-y}}$, we see that the physical decay constants vanish in the chiral limit, which is unacceptable. The pion decay constant must tend to a finite value in the chiral limit. Likewise, we expect \hat{f}_τ to remain finite, as there is no reason that the dilatonic meson would cease to exist in the chiral limit.

Equation (4.2) would naively suggest that the term linear in τ can be neglected in the range $|n_f - n_f^*| \ll m/\Lambda_{IR}$. But, as we have seen, the outcome is inconsistent. In the next subsection, we turn our attention to the role of the linear term in V_d .

B. Linear V_d

In this subsection we show that the linear V_d of Eq. (4.2) gives rise to a well-behaved classical solution. Since we will be considering a fixed theory, the n_f dependence of c_0 and c_1 does not come into play. We will revisit the role of this n_f dependence in Sec. IV D below.

We begin by considering the classical potential for $m = 0$, where

$$U(\tau) = e^{4\tau}(c_0 + c_1\tau) . \quad (4.4)$$

This potential is bounded from below provided that

$$c_1 > 0 . \quad (4.5)$$

The saddle-point equation has a unique solution, the global minimum, given by

$$v = -1/4 - c_0/c_1 . \quad (4.6)$$

Although we are considering here a theory with fixed n_f , the question arises whether something might still go wrong if the selected theory corresponds to a very small $n_f - n_f^*$, because, according to Eq. (4.6), $|v|$ will diverge if we keep selecting theories with ever smaller $n_f - n_f^*$. We will now show that this is not the case.

Consider shifting the dilatonic meson field,

$$\tau \rightarrow \tau + \Delta . \quad (4.7)$$

A glance at Eq. (4.2) reveals that this field redefinition induces a corresponding redefinition of c_0 ,

$$c_0 \rightarrow c_0 + c_1\Delta . \quad (4.8)$$

We can use this “gauge freedom” to set c_0 to any value we like, and we will thus set

$$c_0 = -c_1/4 . \quad (4.9)$$

This corresponds to shifting the τ field by $\Delta = v$, where v is given by Eq. (4.6), so that, after the shift, the classical solution becomes

$$v = 0 . \quad (4.10)$$

The remaining dependence of the tree-level lagrangian (3.49) on Δ is absorbed into a redefinition of the decay constants f_π and f_τ , and the low-energy constants B_π and B_τ . Since Δ is by construction equal to the original value of v in Eq. (4.6), the values of the redefined f 's and the B 's are those given in Eqs. (3.33) and (3.34). Unlike in Sec. IV A, here we are working in the chiral limit, and the expansion around the chiral limit is therefore well behaved.

It is easy to check that the solution $v = 0$ is stable. Indeed, if we add higher-order terms, V_d generalizes to

$$V_d = c_1(\tau - 1/4) + \tau^2 f(\tau) , \quad (4.11)$$

where we have used Eq. (4.9). According to Sec. III B, $f(\tau) = O((n_f - n_f^*)^2)$, making it a small correction for any given theory for which $|n_f - n_f^*| \ll 1$. First, one can easily check that $v = 0$ remains a saddle point for any $f(\tau)$. Moreover, since $|f(\tau)| \ll c_1$, it follows that $v = 0$ always remains a minimum. The shifted classical solution $v = 0$ is therefore a good starting point for the diagrammatic low-energy expansion.

We next consider the tree-level masses. In the chiral limit, the pion is massless. The tree-level mass m_τ of the dilatonic meson is obtained from the second derivative of $U(\tau)$ evaluated at the saddle point,

$$m_\tau^2 = 4c_1 \hat{B}_\tau . \quad (4.12)$$

In large- N counting, all meson masses are $O(1)$. This implies that the low-energy constant \hat{B}_τ is $O(1)$ in large- N counting (a similar reasoning applies to \hat{B}_π). Equation (4.12) also implies that $m_\tau \sim |n_f - n_f^*|^{1/2} \sim \delta^{1/2}$, which resembles the familiar behavior of the pion mass in ordinary chiral perturbation theory, $m_\pi \sim m^{1/2}$.

Proceeding to the case $m > 0$, we cannot solve for v in closed form any more. The classical vacuum is now the solution of

$$\frac{ym}{\mathcal{M}c_1} = h(v) \equiv 4ve^{(4-y)v} . \quad (4.13)$$

We keep $c_0 = -c_1/4$ as in Eq. (4.9). \mathcal{M} is now defined in terms of the rescaled low-energy parameters of Eqs. (3.33) and (3.34). In order for the classical potential (4.1) to remain bounded below for $m > 0$, we need $0 < y < 4$. As follows from Eq. (3.62), the actual value of y appears to be comfortably within this range. Since both c_1 and m are generically $O(\delta)$, the left-hand side of Eq. (4.13) is $O(1)$, and so is the solution v . The function $h(v)$ is positive if and only if v is. Furthermore, for $v > 0$, $h(v)$ is a monotonically increasing function. Since we are assuming $m > 0$, it follows that v itself is a monotonically increasing function of m .

Taking into account that the kinetic terms now get rescaled by e^{2v} , the tree-level dilatonic meson mass is given by

$$m_\tau^2 = 4c_1 \hat{B}_\tau e^{2v} (1 + (4 - y)v) . \quad (4.14)$$

It is easy to see that m_τ increases monotonically with v , and thus with m . The tree-level pion mass is given by

$$m_\pi^2 = 2\hat{B}_\pi m e^{(y-2)v} = \frac{8c_1 \mathcal{M} \hat{B}_\pi}{y} v e^{2v} . \quad (4.15)$$

From the rightmost expression we learn that m_π , too, is monotonically increasing with v , and thus with m .

If we consider the limit where $m/(\mathcal{M}c_1) \ll 1$, we can solve Eq. (4.13) to first order in this ratio, obtaining

$$v = \frac{ym}{4\mathcal{M}c_1} . \quad (4.16)$$

The correction to the dilatonic meson mass is

$$\frac{m_\tau^2}{m_\tau^2(m=0)} = 1 + (6-y)v , \quad (4.17)$$

whereas the pion mass is given by

$$m_\pi^2 = 2\hat{B}_\pi m . \quad (4.18)$$

This is the familiar result from ordinary chiral perturbation theory. However, when $m/\mathcal{M} \sim c_1$, the leading-order dependence of m_π^2 on the quark mass, given by Eq. (4.15), is more complicated, and not linear.

C. Dilatation current in the leading-order effective theory

In this subsection we explore the basic properties of the dilatation current and its divergence to leading order in the effective theory. The actual derivation of the dilatation current is relegated to App. D. The result can be expressed as (see Eq. (D5))

$$S_\mu = x_\nu \Theta_{\mu\nu} = x_\nu (T_{\mu\nu} + K_{\mu\nu}/3) , \quad (4.19)$$

where $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are, respectively, the canonical and improved energy-momentum tensors of the effective theory at the leading order.²³

As expected, S_μ contains a term linear in the dilatonic meson field. Its origin is in the improvement term

$$\begin{aligned} K_{\mu\nu} &= \frac{f_\tau^2}{2} (\delta_{\mu\nu} \square - \partial_\mu \partial_\nu) e^{2\tau} \\ &= f_\tau e^v (\delta_{\mu\nu} \square - \partial_\mu \partial_\nu) \tilde{\tau} + \dots . \end{aligned} \quad (4.20)$$

where on the second line we have used Eq. (3.32). Plugging this into Eq. (4.19) gives

$$S_\mu = f_\tau e^v \partial_\mu \tilde{\tau} + \dots , \quad (4.21)$$

showing that the leading-order decay constant of the dilatonic meson is $f_\tau e^v$. [Note that, according to Eq. (D2a), the off-shell divergence $\partial_\mu S_\mu$ contains $\partial_\mu (x_\nu K_{\mu\nu}/3) = (\partial_\mu x_\nu) K_{\mu\nu}/3 = K_{\mu\mu}/3 = (f_\tau^2/2) \square e^{2\tau}$, which is equivalent to Eq. (4.21).]

The anomalous conservation equation may be read off from Eqs. (D2) and (D3). Using also Eqs. (3.59) and (3.60), it takes the form

$$\partial_\mu S_\mu = c_1 f_\tau^2 B_\tau e^{4\tau} + (1 + \gamma_m^*) \frac{f_\pi^2 B_\pi m}{2} e^{y\tau} \text{tr}(\Sigma + \Sigma^\dagger) . \quad (4.22)$$

²³ Since the fields of the effective theory are Lorentz scalars, the existence of an improvement term is to be expected [28].

If we now introduce

$$\bar{\psi}\psi(\text{EFT}) \equiv \frac{\partial \mathcal{L}}{\partial \mathcal{S}} = -\frac{f_\pi^2 B_\pi}{2} e^{y\tau} \text{tr}(\Sigma + \Sigma^\dagger), \quad (4.23)$$

with \mathcal{L} given by Eq. (3.49), we may rewrite Eq. (4.22) as

$$\partial_\mu S_\mu = c_1 f_\tau^2 B_\tau e^{4\tau} - (1 + \gamma_m^*) m \bar{\psi}\psi(\text{EFT}). \quad (4.24)$$

A comparison to the microscopic theory (Sec. II A) leads to the identification

$$\tilde{\beta} F^2(\text{EFT}) = -c_1 f_\tau^2 B_\tau e^{4\tau}. \quad (4.25)$$

If we take the expectation value of Eq. (4.25), and demand consistency with the microscopic theory, we obtain

$$\tilde{\beta} \langle F^2 \rangle = -c_1 f_\tau^2 B_\tau e^{4v}. \quad (4.26)$$

Since in the effective theory we were led to require that $c_1 > 0$, and we have that $\tilde{\beta} < 0$, it follows that $\langle F^2 \rangle > 0$, for $n_f \lesssim n_f^*$. We recall that F^2 can mix with the identity operator. Different prescriptions to subtract the identity part of $\langle F^2 \rangle$ can vary, and, as a result, even the sign of the subtracted $\langle F^2 \rangle$ is not uniquely defined. Interestingly, the effective theory appears to provide a natural framework to define the (subtracted) expectation value $\langle F^2 \rangle$, which, as we have seen, implies that it is positive.

Finally, another interesting consequence is that, in the chiral limit, we may use Eq. (4.12) (as well as (3.33) and (3.34)) to rewrite Eq. (4.26) as

$$-4\tilde{\beta} \langle F^2 \rangle = \hat{f}_\tau^2 m_\tau^2. \quad (4.27)$$

This result closely resembles the GMOR relation²⁴

$$-\frac{2m}{N_f} \langle \bar{\psi}\psi \rangle = \hat{f}_\pi^2 m_\pi^2, \quad (4.28)$$

which is reproduced using Eqs. (4.18) and (4.23). The existence of a GMOR-like relation for the dilatonic meson is not entirely obvious. The original GMOR relation is derived from the partial conservation of the axial current by making use of the fact that the pion pole is isolated from the 3-pion cut. By contrast, for any finite N_c and N_f , we expect that the mass of the dilatonic meson will remain non-zero in the chiral limit. Therefore, for $m = 0$ the dilatonic meson is unstable, and can decay into two pions. The $\tau \rightarrow \pi\pi$ vertex comes from expanding the pion kinetic term (3.50), and is given by $\sim f_\tau^{-1} \tau \text{tr}(\partial_\mu \pi \partial_\mu \pi)$, showing that f_τ^{-1} plays the role of the coupling constant. The resulting decay rate satisfies

$$\frac{\Gamma_\tau}{m_\tau} \sim \frac{m_\tau^2 N_f^2}{\hat{f}_\tau^2} \sim \frac{m_\tau^2}{\Lambda_{IR}^2} \sim |n_f - n_f^*|, \quad (4.29)$$

where in the first equality we have used that the number of pions is $\sim N_f^2$. We see that, thanks to the derivative interaction of the pions in the chiral limit, the ratio Γ_τ/m_τ is parametrically small, and so the dilatonic meson is a narrow resonance for n_f close to n_f^* .

²⁴ Note that $\bar{\psi}\psi$ is summed over flavors.

D. The limit $n_f \nearrow n_f^*$

In the previous subsections we have studied the tree-level structure of a single theory with specific values of N_c and N_f , and, thus, a fixed value of n_f . In this subsection, we explore the limit $n_f \nearrow n_f^*$, which necessarily involves the comparison of different theories.

Having in mind that we will now be varying n_f requires us to revisit two elements of the discussion of Sec. IV B. First, reinstating the n_f dependence (and remembering that we always consider $n_f < n_f^*$) the bound (4.5) reads

$$\tilde{c}_{11} < 0 . \quad (4.30)$$

Second, in Sec. IV B, the “gauge freedom” of shifting the τ field has allowed us to set its expectation value to zero in the chiral limit. In order to facilitate the comparison of different theories we may still shift the τ field, but the shift must now be uniform across these theories, that is, independent of n_f . Parametrically, this shift is $O(1)$, so that it does not mix different orders in the expansion in $n_f - n_f^*$. Comparing Eqs. (4.2) and (4.8), we see that the effect of an allowed shift is now

$$\tilde{c}_{00} \rightarrow \tilde{c}_{00} , \quad (4.31a)$$

$$\tilde{c}_{01} \rightarrow \tilde{c}_{01} + \tilde{c}_{11} \Delta . \quad (4.31b)$$

It follows that the value of the low-energy constant \tilde{c}_{01} is arbitrary. We will henceforth use the freedom to perform a uniform shift to set $\tilde{c}_{01} = 0$. By contrast, the leading-order coefficient \tilde{c}_{00} is invariant under the shift, and thus, it can have physical consequences.

The role of \tilde{c}_{00} is best illustrated by considering a concrete example. Starting from the rescalings in Eqs. (3.33) and (3.34), we consider the dimensionless ratio²⁵

$$\frac{m\hat{B}_\pi}{\hat{f}_\pi^2} = \frac{mB_\pi}{f_\pi^2} e^{-(1+\gamma_m^*)v} , \quad (4.32)$$

where we have used Eqs. (3.59) and (3.60) to trade y with the mass anomalous dimension γ_m^* at the sill of the conformal window, Eq. (3.60). The left-hand side of Eq. (4.32) can be determined in terms of the pion mass close to the chiral limit and the (physical) pion decay constant

$$\frac{m\hat{B}_\pi}{\hat{f}_\pi^2} = \frac{m_\pi^2}{2\hat{f}_\pi^2} , \quad (4.33)$$

where we have used Eq. (4.18). By plugging Eq. (4.2) into Eq. (4.6), we reexpress the classical solution as

$$v = -\frac{1}{4} - \frac{\tilde{c}_{00}}{\tilde{c}_{11}(n_f - n_f^*)} , \quad (4.34)$$

where we have used the new “gauge choice” $\tilde{c}_{01} = 0$. We see that the dimensionless ratio m_π^2/\hat{f}_π^2 depends (exponentially) on $\tilde{c}_{00}/\tilde{c}_{11}$. If we measure this ratio for sufficiently many

²⁵ Here we are, in effect, using the reference infrared scale $\Lambda_{IR} = 4\pi\hat{f}_\pi N_c^{-1/2}$ (compare Eq. (3.8)). The ratio (4.32) allows us to track the n_f dependence at fixed N . The N -dependence can easily be divided out for the purpose of comparing theories with different N .

walking theories (and close enough to the chiral limit), we will be able to extract from it the value of

$$\frac{\tilde{c}_{00}}{\tilde{c}_{11}} = \frac{B_\tau \tilde{c}_{00}}{B_\tau \tilde{c}_{11}} . \quad (4.35)$$

A linearly independent relation is obtained by measuring the mass of the dilatonic meson, which, in the present context, becomes (compare Eq. (4.12))

$$m_\tau^2 = 4\tilde{c}_{11}(n_f - n_f^*)B_\tau e^{2v} . \quad (4.36)$$

Using both measurements for a series of theories in the limit of small $n_f - n_f^*$ and m , we should thus be able to separately determine $\tilde{c}_{00}B_\tau$ and $\tilde{c}_{11}B_\tau$.

A technical comment is that, since the leading-order lagrangian depends on $B_\tau V_d$, we only have access to products such as $c_n B_\tau$ or $\tilde{c}_{nk} B_\tau$. This behavior is familiar from the standard chiral lagrangian, where only the product mB_π has an invariant meaning. In both cases, the separate determination of the B parameter requires an additional extraneous prescription.

Returning to Eq. (4.34) we find in the limit $n_f \nearrow n_f^*$ the asymptotic behavior

$$v \rightarrow \begin{cases} -\infty , & \tilde{c}_{00} > 0 , \\ +\infty , & \tilde{c}_{00} < 0 . \end{cases} \quad (4.37)$$

We conclude that $\hat{f}_\pi(n_f)$, $\hat{f}_\tau(n_f)$ and $\hat{B}_\tau(n_f)$ vanish in this limit when $\tilde{c}_{00} > 0$, or, alternatively, blow up when $\tilde{c}_{00} < 0$.²⁶ Intuitively, the correct behavior is that all dimensionful low-energy parameters will vanish as the conformal window is approached. This would imply that the “phase boundary” between chirally broken and infrared-conformal theories is smooth. We thus speculate that the correct sign is $\tilde{c}_{00} > 0$. However, this is merely a speculation. The reason is that when we look at, *e.g.*, the n_f dependence of \hat{f}_π , we are comparing the pion decay constants of different theories. Unless we have a common dimensionful reference scale, such a comparison is meaningless. Taking a low-energy parameter, such as f_π , as the common reference scale would only work if we have a way to determine f_π itself, and not just \hat{f}_π , from the microscopic theory. This requires insight into the microscopic theory not provided by the low-energy effective theory.

In order to avoid confusion, we note that in Sec. IV A we have considered a fixed theory, and the conclusion that the classical solution (4.3) is not a consistent starting point followed from the requirement that the chiral limit of that particular theory must not be singular. Once we take into account the linear term in V_d , the sign of \tilde{c}_{00} becomes unconstrained, and we need instead the constraint (4.5), or, equivalently, (4.30).

The description in terms of the effective low-energy theory developed in this paper necessarily breaks down inside the conformal window. The reason is that meson states no longer exist in the chiral limit, where, instead, all correlation functions obey power laws with non-trivial exponents. In view of Eq. (4.30), the product $c_1 = \tilde{c}_{11}(n_f - n_f^*)$ will be negative for $n_f > n_f^*$, and, as a result, the tree-level potential will become unbounded from below. It is reassuring to see that, through this classical behavior, the low-energy theory “knows” that it must cease to be valid as the conformal window is entered.

To conclude this section, we consider another dimensionless ratio, which, through the GMOR relation, directly influences the pion mass. This is the value of the flavor-singlet

²⁶ The same conclusion applies to $\hat{B}_\pi(n_f)$ provided that $\gamma_m^* < 1$.

fermion condensate, expressed in units of the pion decay constant,

$$\frac{\langle \bar{\psi}\psi \rangle}{\hat{f}_\pi^3} = -\frac{B_\pi}{f_\pi} e^{-\gamma_m^* v} . \quad (4.38)$$

If indeed \tilde{c}_{00} is positive (and since γ_m^* is expected to be positive, too), we have that $e^{-\gamma_m^* v} > 1$. This result then predicts a condensate enhancement that depends on the mass anomalous dimension, as well as, through Eq. (4.34), on the low-energy constants \tilde{c}_{00} and \tilde{c}_{11} . While condensate enhancement was predicted long ago within the gap-equation treatment [32–35, 39, 41], here we obtain it as a quantitative prediction within the framework of a systematic, low-energy expansion, provided that $\tilde{c}_{00} > 0$.

E. Testing the effective theory

In this section we have studied several tree-level predictions of the effective theory, first for a given model, and then as the limit $n_f \nearrow n_f^*$ is approached. In practice, the ability to test the effective theory by collecting data from many models is limited by its cost. Thus, in this subsection we return to a specific model with fixed N_c and N_f , and discuss in some more detail how the effective theory can be put to a test in this framework.

We will focus on the dependence of simple observables on the fermion mass m . Of course, virtually any physical observable of the low-energy sector can be used to test the predictions of the effective theory. The discussion below is only meant as an illustration, providing a more hands-on view on the results already presented in Sec. IV B.

We first discuss the determination of the exponent y , or, equivalently, the mass anomalous dimension γ_m^* at the sill of the conformal window (Eqs. (3.59) and (3.60)). The results of Sec. III, which are based on our main dynamical assumption (3.14), imply that for a given model, the mass anomalous dimension at the dynamical infrared scale $\gamma_m(\Lambda_{IR})$ can be expanded as

$$\gamma_m(\Lambda_{IR}) = \gamma_m^* + c_{\gamma,1}(n_f - n_f^*) + \dots . \quad (4.39)$$

In comparison with Eq. (3.14), the main difference is that at the scale Λ_{IR} , the trace anomaly vanishes with $n_f - n_f^*$, whereas the mass anomalous dimension stays finite. It then follows that we may reexpress $e^{y\tau}$ in Eq. (3.52) as

$$e^{y\tau} = e^{(3-\gamma_m^*)\tau} = e^{(3-\gamma_m(\Lambda_{IR}))\tau} (1 + c_{\gamma,1}(n_f - n_f^*)\tau + \dots) . \quad (4.40)$$

We conclude that the discrepancy arising from using $\gamma_m(\Lambda_{IR})$ of the model with fixed N_f and N_c , instead of γ_m^* , will be absorbed at the next-to-leading order by adjusting the coefficient $\tilde{c}_{M,11}$ occurring in the expansion of V_M (see Eq. (3.18) and Sec. V A below).

With an estimate for y at hand, we may now use the numerical results for the pion mass m_π , measured at several different values of the fermion mass m , and substitute them into Eq. (4.15) in order to determine the functional form of $v(m)$ together with the low-energy constant \hat{B}_π . The latter may be found by considering the small- m behavior, because, by construction, the shifted dilaton expectation value of Sec. IV B vanishes in the chiral limit.

At this point we may carry out two tests of the effective theory. First, the function $h(v)$, defined in Eq. (4.13), must depend linearly on m once we substitute $v = v(m)$. Another test is that the m -dependence of the dilatonic meson mass should agree with Eq. (4.14), which, in turn, allows us to determine the product $c_1 \hat{B}_\tau$.

A word of caution is that, since we are using the predictions of the leading-order lagrangian, the results are only expected to provide a good description of the low-energy sector when all the expansion parameters, m/Λ_{IR} , $1/N_c$ and $n_f - n_f^*$, are sufficiently small.

As we have explained, the effective theory is restricted to theories with fermions in the fundamental representation because only in this case can we turn the ratio N_f/N_c into an effectively continuous parameter by invoking the Veneziano limit. While this would necessarily be of a speculative nature, let us give a thought to models with higher-representation fermions (such as the $SU(3)$ -sextet model studied in Refs. [11, 13]). Consider the euclidean partition function defined by varying *continuously* the number of flavors N_f . While the theory is non-local whenever N_f is not an integer, it is a well-defined statistical mechanics system. That system might become infrared conformal at some (non-integer!) value of N_f , which we denote as N_f^* . In the fortuitous event that there exist an integer N_f such that, first, $N_f < N_f^*$ so that the N_f -flavor theory undergoes spontaneous chiral symmetry breaking, and, second, $N_f^* - N_f \ll 1$, the low-energy sector of the N_f -flavor higher-representation theory might admit an expansion in m/Λ_{IR} and $N_f - N_f^*$.

Testing this scenario can be done along similar lines. For the actual effective field theory framework (with fermions in the fundamental representation), the success of the low-energy description at a given order depends on the smallness of the expansion parameters $1/N_c$ and $n_f - n_f^*$ for the model under consideration; the fermion mass can always be dialed such that m/Λ_{IR} is small enough. Similar tests of the low-energy sector can be carried out in any model with higher-representation fermions and a light scalar meson, and the hope would be that that model is well-described by an expansion in $N_f - N_f^*$.

V. EFFECTIVE FIELD THEORY AT ONE LOOP

In this section we study the effective theory at the next-to-leading order (NLO). In Sec. V A we write down the NLO lagrangian. Then, as an example, we work out in Sec. V B the one-loop effective potential from a dilatonic meson loop, and confirm that the NLO lagrangian contains the necessary counterterms to renormalize it.

A. Next to leading order lagrangian

Like the standard chiral lagrangian, our low-energy expansion involves a derivative expansion, as well as an expansion in the fermion mass m , or, more generally, in the chiral source χ . As we will shortly see, the systematic expansion in ∂_μ and χ provides the main organizing principle for the successively higher orders in the effective theory.

Our setup contains two additional small parameters: $1/N$, and $n_f - n_f^*$. The systematic expansion in $n_f - n_f^*$ is realized in a very simple way: each operator that we encounter while expanding in ∂_μ and χ will be multiplied by its own invariant potential $V_I(\tau - \sigma)$, which, after setting $\sigma = 0$, reduces to $V_I(\tau)$. The invariance property of these potentials under the dilatation (3.20b), namely, the transformation rule $V_I(\tau(x)) \rightarrow V_I(\tau(\lambda x))$, is maintained by promoting the power-series coefficients of the potentials to (global) spurions, whose transformation properties are derived in App. C. At each order in the expansion defined by the power counting of Sec. III A the potentials $V_I(\tau)$ can be truncated consistent with that order, and the effective theory thus contains a finite number of low-energy constants.

As for $1/N$, the reason why we invoke the Veneziano limit is to justify the treatment of

$n_f - n_f^*$ as a continuous parameter. The only explicit N -dependence is that the Σ field takes values in $SU(N_f)$. In addition, the decay constants scale with N , as discussed in Sec. III A. The parameter $1/N$ does not play any direct role in organizing the effective lagrangian.

Before we turn to the construction of the NLO lagrangian, let us reconsider the freedom to shift the τ field and its effect on the LO lagrangian. In order to develop the low-energy expansion of a given theory, in this section we will choose to shift the τ field by $\Delta = -\tilde{c}_{00}/(\tilde{c}_{11}(n_f - n_f^*))$. This shift sets $\tilde{c}_{00} = 0$, thereby eliminating the only $O(1)$ term present in the LO lagrangian, and the entire LO lagrangian becomes $O(\delta)$. This new ‘‘gauge choice’’ leaves \tilde{c}_{01} intact. Recall that, in Sec. IV D, we made use of an n_f -independent shift to argue that the value of \tilde{c}_{01} is unphysical. Nevertheless, here it is advantageous to keep it free, because this manifestly maintains dilatation invariance of the $O(n_f - n_f^*)$ part of V_d (see Eq. (4.2) and App. C).

The classical vacuum is now given by $v = -1/4 - \tilde{c}_{01}/\tilde{c}_{11}$, which is $O(1)$, and hence any factors of e^v which arise from the expansion of the τ field, Eq. (3.32), are $O(1)$ too.²⁷ The classical solution will now be modified by higher-order terms of the kind studied in Eq. (4.11). But the corresponding change in v will be $O(n_f - n_f^*)$, so that the systematic nature of the low-energy expansion is respected.

The NLO lagrangian consists of four kinds of operators. The first kind comes from picking the $O(\delta^2)$ terms from the LO lagrangian of Eq. (3.15) while setting $\sigma = 0$. This amounts to picking the terms $(n_f - n_f^*)(\tilde{c}_{I,01} + \tilde{c}_{I,11}\tau)$ from the expansions of V_M , V_π and V_τ , because these potentials multiply operators that are $O(\delta)$ by themselves. The role of V_d is again special, because $\tilde{\mathcal{L}}_d$ is the only part of the effective theory which involves no small parameters other than $n_f - n_f^*$. The $O(\delta^2)$ terms coming from V_d give rise to the NLO operators

$$Q_1^d = \tilde{c}_{02} (n_f - n_f^*)^2 f_\tau^2 B_\tau e^{4\tau} , \quad (5.1a)$$

$$Q_2^d = \tilde{c}_{12} \tau (n_f - n_f^*)^2 f_\tau^2 B_\tau e^{4\tau} , \quad (5.1b)$$

$$Q_3^d = \tilde{c}_{22} (\tau^2/2) (n_f - n_f^*)^2 f_\tau^2 B_\tau e^{4\tau} . \quad (5.1c)$$

The dilatation transformation properties of the low-energy constants \tilde{c}_{nk} , given in App. C, ensure that the spacetime integral of $Q_1^d + Q_2^d + Q_3^d$ is invariant.

The remaining terms in the NLO lagrangian arise from the expansion in ∂_μ and χ , which we will refer to as the ‘‘chiral expansion.’’ Since these terms are already $O(\delta^2)$ on account of their dependence on derivatives and on the chiral source χ , it follows that the invariant potentials multiplying all these terms may be set equal to 1. At the next-to-next to leading order (NNLO), we will need the $O((n_f - n_f^*)^3)$ part of V_d , the $O((n_f - n_f^*)^2)$ parts of V_M , V_π and V_τ , and the $O(n_f - n_f^*)$ parts of invariant potentials that multiply NLO operators arising from the chiral expansion.

The first kind of chiral-expansion NLO operators comes in one-to-one correspondence with the NLO operators of standard chiral perturbation theory [18]. Each NLO operator

$$\tilde{Q}_i = \tilde{Q}_i(\Sigma, \chi, \partial_\mu) , \quad (5.2)$$

appears in the lagrangian in the form

$$Q_i^\pi = e^{4\tau} \tilde{Q}_i(\Sigma, e^{(y-4)\tau} \chi, e^{-\tau} \partial_\mu) . \quad (5.3)$$

²⁷ This reduces to the solution $v = 0$ of Eq. (4.10) in the special case that $\tilde{c}_{01} = -\tilde{c}_{11}/4$.

The NLO action constructed from the operators Q_i^π is dilatation invariant. A second set of chiral-expansion NLO operators consists of operators that depend on the dilatonic meson field only. There are three of them,²⁸

$$Q_1^\tau = [(\partial_\mu \tau)^2]^2, \quad (5.4a)$$

$$Q_2^\tau = (\square \tau)^2, \quad (5.4b)$$

$$Q_3^\tau = \square \tau (\partial_\mu \tau)^2. \quad (5.4c)$$

Finally, there are operators that depend on both Σ and derivatives of τ ,

$$Q_1^{\text{mix}} = \partial_\mu \tau \partial_\nu \tau \text{tr}(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma), \quad (5.5a)$$

$$Q_2^{\text{mix}} = (\partial_\mu \tau)^2 \text{tr}(\partial_\nu \Sigma^\dagger \partial_\nu \Sigma), \quad (5.5b)$$

$$Q_3^{\text{mix}} = \square \tau \text{tr}(\partial_\nu \Sigma^\dagger \partial_\nu \Sigma), \quad (5.5c)$$

$$Q_4^{\text{mix}} = e^{(y-2)\tau} (\partial_\mu \tau)^2 \text{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi), \quad (5.5d)$$

$$Q_5^{\text{mix}} = e^{(y-2)\tau} \square \tau \text{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi). \quad (5.5e)$$

B. Effective potential at NLO

In this subsection we calculate, as an example, the one-loop effective potential coming from a dilatonic meson field in the loop.²⁹ As in the previous subsection, we use the τ shift to set $\tilde{c}_{00} = 0$ while leaving \tilde{c}_{01} free. The easiest way to obtain the effective potential is via the background field method. Instead of a constant classical vacuum v , we introduce a background field $u(x)$, assumed to be slowly varying, and replace Eq. (3.32) by

$$\tau(x) = u(x) + \tilde{\tau}(x)/f_\tau, \quad (5.6)$$

where, as before, $\tilde{\tau}(x)$ is the quantum part.³⁰ We will be applying an adiabatic approximation, where derivatives of the background field are neglected. The result of the calculation is a functional of the background field $u(x)$, in which we are eventually to replace $u(x)$ by the full field $\tau(x)$.

We start by expanding S to quadratic order in $\tilde{\tau}$,

$$S^{(2)} = \frac{\mu^{d-4} f_\tau^2}{2} \int d^d x \left(e^{2u} (\partial_\mu \tilde{\tau})^2 + \mathcal{V}''(u) \tilde{\tau}^2 \right), \quad (5.7)$$

where now (compare Eq. (4.1))

$$\mathcal{V}(u) = B_\tau V_d(u) e^{4u} - \frac{f_\pi^2 B_\pi}{2 f_\tau^2} e^{yu} \text{tr}(\chi^\dagger \Sigma + \Sigma^\dagger \chi), \quad (5.8)$$

²⁸ We omit the low-energy constants that multiply each operator in Eqs. (5.4) and (5.5).

²⁹ We did not calculate the contribution to the effective potential coming from a pion loop, for which the power counting should work in the usual way.

³⁰ Because the classical background is not constant, we normalize the quantum part only by the original f_τ .

and, integrating over $\tilde{\tau}$, we find

$$V_1 = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \log(e^{2u} p^2 + \mathcal{V}''(u)) \quad (5.9a)$$

$$= -\frac{1}{2} \frac{\partial}{\partial \alpha} \int \frac{d^d p}{(2\pi)^d} (e^{2u} p^2 + \mathcal{V}''(u))^{-\alpha} \Big|_{\alpha=0} \quad (5.9b)$$

$$= -\frac{1}{2} \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}} (e^{-2u} \mathcal{V}''(u))^{\frac{d}{2}} \quad (5.9c)$$

$$= -\frac{1}{64\pi^2} (e^{-2u} \mathcal{V}''(u))^2 \times \left(\frac{2}{4-d} - \gamma + \frac{3}{2} - \log\left(\frac{e^{-2u} \mathcal{V}''(u)}{4\pi\mu^2}\right) + O(d-4) \right). \quad (5.9d)$$

We confirmed this result by a diagrammatic calculation, see App. E. The effective potential is thus given by Eq. (5.9d) with the replacement $u(x) \rightarrow \tau(x)$.

The divergence of the effective potential is proportional to $(e^{-2\tau} \mathcal{V}''(\tau))^2$. It is straightforward to check that the counterterms needed to remove it are a linear combination of the NLO operators of Sec. V A. As an example, if we set $\chi = 0$ then the required counterterms are the three NLO operators in Eq. (5.1). This calculation provides a non-trivial check of the constraints that follow from the power counting and from the symmetries of the effective theory.

VI. DISCUSSION

We begin this concluding section with a summary of the main technical ingredients of our work. We have proposed a low-energy effective action that accommodates dilatation symmetry into the framework of the chiral lagrangian. Previously, the chiral lagrangian was extended to accommodate the anomalous $U(1)_A$ symmetry. What these two extensions have in common is that the new symmetry, be it dilatations or $U(1)$ axial transformations, is abelian. In both cases, the dependence of the low-energy theory on the singlet dynamical field associated with the abelian symmetry is virtually unconstrained at the algebraic level. For the low-energy theory to have any predictive power, we must be able to establish a power-counting hierarchy that will suppress interactions with increasingly large number of singlet particles.

For the $U(1)_A$ symmetry, the needed hierarchy was established in the usual large- N_c limit, in which the axial anomaly vanishes [18, 26]. The low-energy expansion is then a double expansion: in the fermion mass m , and in $1/N_c$. In the case of dilatations, we are able to propose a suitable starting point only for fermions in the fundamental representation. The low-energy expansion involves a triple limit: the chiral limit $m \rightarrow 0$; the Veneziano large- N limit in which the ratio $n_f = N_f/N_c$ effectively becomes a continuous parameter; and the limit in which n_f approaches the sill of the conformal window n_f^* from below.

The main hypothesis underlying our derivation is that, when we increase both N_c and N_f such that n_f tends to n_f^* from within the chirally broken phase, the trace of the energy-momentum tensor in the chiral limit tends to zero when probed at the scale at which spontaneous chiral symmetry breaking sets in. This hypothesis is compatible with the notion of a smooth ‘‘conformal phase transition,’’ and with the existence of an infrared fixed-point

as the conformal window is entered. While the low-energy theory is not directly sensitive to the fixed-point value of the coupling, it depends explicitly on the corresponding value of mass anomalous dimension.

For $n_f \nearrow n_f^*$, conservation of the dilatation current in the chiral limit is in effect recovered. However, instead of the dilatonic meson becoming a true Nambu-Goldstone boson, this limit is at the edge of the region of validity of the effective theory. The effective theory breaks down through a diverging vacuum expectation value $v = \langle \tau \rangle$ at $n_f = n_f^*$, leading to unphysical values of the decay constants f_π and f_τ . For $n_f > n_f^*$, the classical potential of the effective theory becomes unbounded from below. This singular behavior suggests that pions cease to exist as the conformal window is entered, consistent with the absence of chiral symmetry breaking inside the conformal window. It also suggests the absence of a true Nambu-Goldstone boson for dilatation symmetry inside the conformal window. However, we emphasize that as the effective theory breaks down at the conformal sill, it does not tell us anything about the physics on the other side of the sill.

An external dilaton source $\sigma(x)$ which, in the microscopic theory, is coupled to the trace anomaly, *i.e.*, to the quantum part of the operatorial trace of the energy-momentum tensor, communicates information about the operatorial trace to the effective theory. By matching σ -dependent correlation functions between the microscopic and the effective theory we were able to establish the desired power-counting hierarchy. Successive powers of $\tau - \sigma$ in the expansion of the invariant potentials $V_I(\tau - \sigma)$ are suppressed by equal powers of $n_f - n_f^*$.

In comparison to the low-energy framework that includes the $U(1)_A$ symmetry, a simplification that occurs in our case is that all relevant currents are renormalization-group invariant. This includes the energy-momentum tensor, the dilatation current, and the non-singlet axial currents. A related choice we have made is that we opted to avoid the introduction of external gauge fields of any kind. In particular, we derived the dilatation current in the effective theory directly using the Noether procedure.

In the microscopic theory, we have avoided multiple operator insertions at coinciding points. As a result, we did not probe any c -number trace anomalies. In flat space, these c -number anomalies reduce to four-derivative terms. Whether the four-derivative terms of the next-to-leading order lagrangian may be constrained via matching conditions similar to those of Refs. [43, 44] we leave as an open question. Studying this question requires the promotion of the global dilatation symmetry to local Weyl invariance, which involves the introduction of the metric tensor [45], and perhaps also Weyl gauging [46].

The anomalous conservation law of the dilatation current in the effective theory suggests that, up to a suitable proportionality constant, the field strength operator of the microscopic theory, F^2 , is represented in the low-energy theory by $e^{4\tau}$, where τ is the dilatonic meson field. Now, the classical potential of the effective theory is $e^{4\tau}(c_0 + c_1\tau)$ in the chiral limit. If we make use of the above identification, it takes the form $F^2(c_0 + (c_1/4)\log F^2)$. This effective lagrangian was proposed long ago as a phenomenological lagrangian for Yang-Mills theory [47, 48]. The difference is that, in our case, the classical potential arises in an effective theory that provides a systematic low-energy expansion of gauge theories near (but below) the bottom of the conformal window.

Among the notable predictions of the tree-level effective theory is the result that the masses of both the dilatonic meson and the pion are monotonically increasing with the fermion mass m . In the region where $m_\pi \sim m_\tau$, the dependence of the pion mass on m is more complicated than the standard GMOR relation. We have also derived a GMOR-like relation for the dilatonic meson in the chiral limit. The reason why such a relation exists,

in spite of the fact that the dilatonic meson can decay into two pions, is that the pions have a derivative coupling in the chiral limit, and, as a result, the ratio of the dilatonic meson's decay rate to its mass tends to zero when n_f tends to n_f^* . Finally, we obtained a quantitative prediction for the enhancement of the fermion condensate,³¹ which, at the leading order, depends on the mass anomalous dimension at the sill of the conformal window, as well as on two other low-energy constants.

The GMOR-like relation for the dilatonic meson gives rise to a definition of the expectation value $\langle F^2 \rangle$ within the low-energy theory. The definition of this expectation value in the microscopic theory is ambiguous, because of the power-divergent mixing of F^2 with the identity operator; we leave the implications of a definition through the low-energy theory as an open question. For related discussions in the context of a lattice definition of the energy-momentum tensor, see, *e.g.*, Refs. [49, 50]. The behavior of $\langle F^2 \rangle$ also plays a role in the finite-temperature phase transition, see for example Refs. [39, 51].

Our work was motivated by lattice studies of asymptotically free models whose spectrum contains a light flavor-singlet scalar meson in certain parts of the bare-parameter phase diagram. These models naturally provide a test bed for the effective field theory framework we have developed. The framework must be put to a test because, unlike the standard chiral lagrangian, we cannot derive the low-energy expansion from first principles; it is a consequence of our main dynamical assumption, Eq. (3.14).

At the same time, the new low-energy framework provides a useful analytic tool. It is notoriously difficult to firmly establish by numerical studies whether a particular model with a slowly running coupling is chirally broken or infrared conformal.³² In a theory with an IRFP, the infrared physics satisfies hyperscaling relations. By contrast, the effective low-energy theory developed here predicts a distinctively different dependence of the light spectrum on the fermion mass m , already at leading order. Moreover, in an infrared-conformal theory, the inverse linear size of the system is always a relevant operator, whereas in the effective low-energy theory finite volume corrections occur only at the next-to-leading order. These qualitative differences may help diagnose whether a specific model is infrared conformal or chirally broken.

The low-energy framework would be best tested by exploring many theories with fermions in the fundamental representation with varying numbers of colors and flavors. Since cost considerations may make such an extensive study difficult, we have briefly commented on tests that can be carried out within a given model with fixed values of N_c and N_f .

Finally, while the low-energy framework is not applicable to theories with higher-representation fermions, we noted that in this case one may attempt a similar expansion, in which, in place of the two small parameters $1/N$ and $n_f - n_f^*$, there is a single small parameter $N_f - N_f^*$. The meaning of N_f^* is that, by varying the number of flavors N_f continuously, the resulting (in general non-local) statistical mechanics system is assumed to become infrared conformal at some non-integer value $N_f = N_f^*$. The physical model where N_f is the largest integer smaller than N_f^* is chirally broken, and might be amenable to an expansion in m/Λ_{IR} and $N_f - N_f^*$ if the latter happens to be numerically small.

³¹ See, however, the discussion of the sign of \tilde{c}_{00} in Sec. IV D.

³² As an example, a recent study using domain-wall fermions suggests that the $SU(3)$ theory with $N_f = 10$ fundamental fermions is infrared conformal [52], whereas another recent study using staggered fermions suggests that the $N_f = 12$ theory could be chirally broken [53]. The results of these two studies are in conflict.

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Appendix A: Two-loop beta function in the Veneziano limit

In terms of an 't Hooft coupling with a slightly different normalization (compare Eq. (3.12))

$$\tilde{\alpha} = g^2 N_c / (16\pi^2) , \quad (\text{A1})$$

the two-loop beta function for N_f Dirac fermions in the fundamental representation takes the form

$$\beta(\tilde{\alpha}) \equiv \frac{\partial \tilde{\alpha}}{\partial \log \mu} = -b_1 \tilde{\alpha}^2 - b_2 \tilde{\alpha}^3 , \quad (\text{A2a})$$

$$b_1 = 11/3 - 2n_f/3 , \quad (\text{A2b})$$

$$b_2 = 34/3 - n_f(13/3 - 1/N_c^2) , \quad (\text{A2c})$$

where n_f was defined in Eq. (1.1). In the Veneziano limit we may drop the last term on the right-hand side of Eq. (A2c). In the two-loop approximation, an IRFP exists for n_f values in the following range

$$2.61 \simeq 34/13 < n_f < 5.5 . \quad (\text{A3})$$

For $n_f > 5.5$, asymptotic freedom is lost. The two-loop prediction for the value of the IRFP is

$$\tilde{\alpha}_*(n_f) = (11 - 2n_f)/(13n_f - 34) . \quad (\text{A4})$$

Of course, Eq. (A4) is not reliable when n_f is close to 34/13. Moreover, the two-loop prediction is oblivious to the possible onset of spontaneous chiral symmetry breaking. As discussed in Sec. III A, for walking theories we may combine the gap equation and the two-loop beta function to postulate that chiral symmetry breaking sets in when the running coupling reaches α_c , provided that $\alpha_c < \alpha_*$. The critical value of the rescaled coupling is $\tilde{\alpha}_c = \alpha_c/(4\pi) = 1/6$. Substituting this value into Eq. (A2) provides a model estimate for Eq. (3.14) (dividing out $[F^2]$), given by

$$\beta(\tilde{\alpha}_c) = \beta(1/6) = \frac{25}{648} (n_f - 4) , \quad (\text{A5})$$

which corresponds to $n_f^* = 4$ and $\eta = 1$. In other words, the estimated range of the conformal window in the Veneziano limit is $n_f^* = 4 < n_f < 5.5$.

Appendix B: Details of the inductive proof

In this appendix we complete the proof of Eq. (3.31) by considering the remaining leading-order potentials V_M , V_π and V_τ . We will also briefly discuss the potentials encountered at the next-to-leading order and beyond.

We start with V_M . In order to probe this potential we add to the n differentiations with respect to σ one more differentiation with respect to the scalar source \mathcal{S} , all at the same spacetime point. We obtain

$$\Gamma_{M,n}^{\text{EFT}} = \langle \Omega_{M,n}(x) + \Omega_{M,0}(x) (\Omega_{d,n}(x) + \Omega_{\pi,n}(x) + \Omega_{\tau,n}(x)) \rangle , \quad (\text{B1})$$

where

$$\Omega_{M,n} = \frac{1}{2N_f} V_M^{(n)}(\tau) e^{y\tau} \text{tr}(\Sigma + \Sigma^\dagger) . \quad (\text{B2})$$

Note that $\Omega_{M,0} = \Omega_{\mathcal{S}}/N_f$. The additional normalization factor of $1/N_f$ relative to Eq. (3.47) is introduced in order that $\Gamma_{M,n}^{\text{EFT}}$ will be $O(1)$ in large- N counting. The $\Omega_{M,n}$ term in Eq. (B1) is obtained by applying all σ -derivatives to V_M . The other term is obtained when only the \mathcal{S} differentiation is applied to $\tilde{\mathcal{L}}_m$, while the σ -derivatives are applied to some other term in the lagrangian.

For the parameter range (3.44) we may neglect all contributions of $\Omega_{\pi,n}$ and $\Omega_{\tau,n}$ relative to the leading contribution of $\Omega_{d,n}$, for the same reasons as in Sec. III B. As for $\Omega_{d,n}$ itself, at this point we have already established that V_d satisfies the power-counting hierarchy (3.31). Therefore, the contribution coming from $\Omega_{M,n}$ must also be $O(\delta^n)$. It is easily seen that this requires $c_{M,n} = O(\delta^n)$. The proof that $c_{M,k} = O(\delta^n)$ for $k > n$ works in the same way as in Sec. III B, by adding $k - n$ external τ legs.

The next potential to consider is V_π . Probing it requires an operator that can serve as an interpolating field for a pion state, which may be obtained by differentiating with respect to the pseudoscalar source \mathcal{P} . Keeping the (suppressed) flavor indices open, and using the same normalization factor as in Eq. (3.47), we have

$$\begin{aligned} \Omega_{\mathcal{P}} &= -\frac{1}{2\hat{f}_\pi^2 \hat{B}_\pi} \frac{\partial \tilde{\mathcal{L}}}{\partial \mathcal{P}} \\ &= \frac{i}{4} V_M(\tau) e^{y\tau} (\Sigma^\dagger - \Sigma) \\ &= V_M(\tau) e^{y\tau} (\pi/\hat{f}_\pi + \dots) . \end{aligned} \quad (\text{B3})$$

The basic correlation function we now consider involves the n differentiations with respect to $\sigma(x)$, plus two differentiations with respect to \mathcal{P} at spacetime points which are far apart. We thus start by considering $\tilde{\Gamma}_{\pi,n}^{\text{EFT}}(q_1, q_2)$ which is the Fourier transform of

$$\Gamma_{\pi,n}^{\text{EFT}}(x, z_1, z_2) = \left\langle \left(\Omega_{d,n}(x) + \Omega_{\pi,n}(x) + \Omega_{\tau,n}(x) \right) \Omega_{\mathcal{P}}(z_1) \Omega_{\mathcal{P}}(z_2) \right\rangle . \quad (\text{B4})$$

Once again we may ignore $\Omega_{d,n}$, for which we have already proved Eq. (3.31). The only way to construct a tree diagram out of the remaining terms is to connect $\Omega_{\pi,n}$ by two pion lines to $\Omega_{\mathcal{P}}(z_1)$ and $\Omega_{\mathcal{P}}(z_2)$ (with a spacetime derivative acting on each line, see Eq. (3.40)). Since Weinberg's theorem ensures that loop diagrams are subleading, we conclude that the leading contribution to $\tilde{\Gamma}_{\pi,n}^{\text{EFT}}(q_1, q_2)$ is proportional to $c_{\pi,n}$, implying that $c_{\pi,n}$ satisfies Eq. (3.31). As usual, in order to probe $c_{\pi,k}$ for $k > n$ we add $k - n$ additional τ legs generated by insertions of $\tilde{\Omega}_{\mathcal{S}}$.

Finally we consider V_τ . The first correlation function we consider is $\tilde{\Gamma}_{d,n,2}^{\text{EFT}}(p_1, p_2)$ (see Eq. (3.48)). The contribution of $\Omega_{\pi,n}$ can be dropped since it will necessarily involve loops. Based on its momentum dependence, we isolate the tree diagram in which $\Omega_{d,n} + \Omega_{\tau,n}$ is

connected by a τ line to each of the two insertions of $\tilde{\Omega}_S$. After amputation, the leading-order result is proportional to

$$8\hat{B}_\tau c_{d,n} + p_1 \cdot p_2 c_{\tau,n} , \quad (\text{B5})$$

(where, in the case of $\Omega_{d,n}$, the two powers of τ are obtained by expanding $e^{4\tau}$ to second order). We may now use the momentum dependence once more to isolate the $c_{\tau,n}$ term, which must therefore satisfy Eq. (3.31). The generalization to $c_{\tau,k}$ for $k > n$ is similar.

Invariant potentials $V_I(\tau - \sigma)$ occur not only at the leading order, but also at all higher orders. Every higher-order term that arises from what we have referred to in Sec. V A as the chiral expansion will be multiplied by its own invariant potential. The proof of Eq. (3.31) generalizes to all these potentials, too.

A concrete example illustrates how this works. Consider once again $\tilde{\Gamma}_{d,n,2}^{\text{EFT}}(p_1, p_2)$, which we have used in order to study the n -th order term in the potential V_τ occurring in the leading-order lagrangian. We have isolated the relevant (amputated) tree diagram, and calculated it up to second order in the external momenta, see Eq. (B5). If we extend the calculation of this correlation function to $O(p^4)$, there will be two new kinds of contributions. First, there will be one-loop diagrams involving the LO vertices, and thus the LO potentials. An additional $O(p^4)$ contribution will come from the NLO operator Q_2^τ of Eq. (5.4b), which serves as a counterterm. Since we already know that the LO potentials satisfy Eq. (3.31), we can now isolate the contribution of Q_2^τ , and thus, prove that the invariant potential by which Q_2^τ is multiplied must satisfy Eq. (3.31) too. When we consider more external τ legs, the other two NLO operators in Eq. (5.4) will in general contribute as well. But, the fact that these operators are linearly independent means that we must always be able to isolate the contribution of each one of them, and thus, enforce the power-counting hierarchy on the corresponding invariant potential.

Appendix C: Transformation properties of the potentials $V_I(\tau)$

Here we show that the invariance of a potential $V(\tau - \sigma)$ under the dilatation transformation of Eqs. (2.17d) and (3.20b) can be maintained when we set $\sigma = 0$. The invariance of $V(\tau)$, or, more precisely, the transformation rule $V(\tau(x)) \rightarrow V(\tau(\lambda x))$, is achieved by promoting the power-series coefficients to global spurions, whose transformation rules will be derived below. Dropping the coordinates dependence, we start by rearranging the expansion of $V(\tau - \sigma)$ in powers of τ ,

$$\begin{aligned} V &= \sum_{n=0}^{\infty} \frac{c_n}{n!} (\tau - \sigma)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{c_n}{k!(n-k)!} \tau^k (-\sigma)^{n-k} \\ &= \sum_{k=0}^{\infty} \frac{\hat{c}_k(\sigma)}{k!} \tau^k , \end{aligned} \quad (\text{C1})$$

where we have introduced

$$\hat{c}_k(\sigma) = \sum_{m=0}^{\infty} \frac{c_{m+k}}{m!} (-\sigma)^m . \quad (\text{C2})$$

Note that $c_n = \hat{c}_n(0)$. Under the transformation $\sigma \rightarrow \sigma + \Delta$ (where $\Delta = \log \lambda$) we have, trivially, $\hat{c}_n(\sigma) \rightarrow \hat{c}_n(\sigma + \Delta)$. The spurionic transformation of c_n is derived by interpreting it as $\hat{c}_n(0)$,

$$c_n = \hat{c}_n(0) \rightarrow \sum_{m=0}^{\infty} \frac{c_{m+n}}{m!} (-\Delta)^m . \quad (\text{C3})$$

The shift by $\Delta = \log \lambda$ is parametrically $O(1)$, *i.e.*, it does not mix different orders in the expansion of the c_n 's in powers of $n_f - n_f^*$, see Eq. (3.25). Noting that the double-expansion coefficients \tilde{c}_{nk} vanish for $k < n$, we find that their spurionic transformation rule is

$$\tilde{c}_{nk} \rightarrow \sum_{m=0}^{k-n} \frac{\tilde{c}_{m+n,k}}{m!} (-\Delta)^m , \quad k \geq n , \quad (\text{C4})$$

which now involves only a finite number of terms on the right-hand side.

Appendix D: Derivation of the leading-order dilatation current

In this appendix we derive the expression for the dilatation current to leading order in the effective theory. For more generality, we start from Eq. (3.15), where we set $\sigma(x) = 0$ and $\chi(x) = m$. The infinitesimal transformations are (compare Eqs. (2.17) and (3.20))

$$\delta\tau = \theta(x_\nu \partial_\nu \tau + 1) , \quad (\text{D1a})$$

$$\delta\Sigma = \theta x_\nu \partial_\nu \Sigma . \quad (\text{D1b})$$

Promoting θ to a local parameter we find (allowing for integration by parts since we may choose $\theta(x)$ to vanish at infinity)

$$\delta_\theta \tilde{\mathcal{L}}_\tau = -\theta \partial_\mu \left[x_\nu \left(T_{\mu\nu}^\tau + \frac{1}{3} K_{\mu\nu} \right) \right] + \theta \frac{f_\tau^2}{2} e^{2\tau} (\partial_\mu \tau)^2 V'_\tau , \quad (\text{D2a})$$

$$\delta_\theta \tilde{\mathcal{L}}_\pi = -\theta \partial_\mu (x_\nu T_{\mu\nu}^\pi) + \theta \frac{f_\pi^2}{4} e^{2\tau} \text{tr}[(\partial_\mu \Sigma^\dagger)(\partial_\mu \Sigma)] V'_\pi , \quad (\text{D2b})$$

$$\delta_\theta \tilde{\mathcal{L}}_d = \theta f_\tau^2 B_\tau [\partial_\nu (x_\nu e^{4\tau} V_d) + e^{4\tau} V'_d] , \quad (\text{D2c})$$

$$\begin{aligned} \delta_\theta \tilde{\mathcal{L}}_m = & -\frac{\theta f_\pi^2 B_\pi m}{2} \left[\partial_\nu \left(x_\nu e^{y\tau} V_M \text{tr}(\Sigma + \Sigma^\dagger) \right) \right. \\ & \left. + e^{y\tau} ((y-4)V_M + V'_M) \text{tr}(\Sigma + \Sigma^\dagger) \right] , \end{aligned} \quad (\text{D2d})$$

where

$$T_{\mu\nu}^\tau = \frac{f_\tau^2 e^{2\tau} V_\tau}{2} (2(\partial_\mu \tau)(\partial_\nu \tau) - \delta_{\mu\nu} (\partial_\rho \tau)^2) , \quad (\text{D3a})$$

$$T_{\mu\nu}^\pi = \frac{f_\pi^2 e^{2\tau} V_\pi}{4} \text{tr}[(\partial_\mu \Sigma^\dagger)(\partial_\nu \Sigma) + (\partial_\nu \Sigma^\dagger)(\partial_\mu \Sigma) - \delta_{\mu\nu} (\partial_\rho \Sigma^\dagger)(\partial_\rho \Sigma)] , \quad (\text{D3b})$$

$$K_{\mu\nu} = f_\tau^2 (\delta_{\mu\nu} \square - \partial_\mu \partial_\nu) \Upsilon , \quad (\text{D3c})$$

and $\Upsilon(\tau)$ is defined by

$$\frac{\partial \Upsilon}{\partial \tau} = e^{2\tau} V_\tau . \quad (\text{D4})$$

Note that $K_{\mu\nu}$ is transversal by construction. From these results we read off the expression for the dilatation current,

$$S_\mu = x_\nu \Theta_{\mu\nu} , \quad (\text{D5})$$

$$\begin{aligned} \Theta_{\mu\nu} &= T_{\mu\nu}^\tau + T_{\mu\nu}^\pi - \delta_{\mu\nu}(\tilde{\mathcal{L}}_d + \tilde{\mathcal{L}}_m) + \frac{1}{3} K_{\mu\nu} \\ &= T_{\mu\nu} + \frac{1}{3} K_{\mu\nu} , \end{aligned} \quad (\text{D6})$$

where $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are the canonical and improved energy-momentum tensors, respectively.

In order to obtain the leading-order expression for S_μ , we substitute in the above results $V_M = V_\pi = V_\tau = 1$, and use Eq. (4.2) for V_d . The solution of Eq. (D4) is then $\Upsilon = e^{2\tau}/2$, and the final expression for $K_{\mu\nu}$ may be found in Eq. (4.20).

Appendix E: Resummation of one-loop diagrams

Here we calculate the one-loop effective potential by resumming diagrams. As in Sec. V we set $\tilde{c}_{00} = 0$, but keep \tilde{c}_{01} free. The quantum field $\tilde{\tau}$ is defined by Eq. (3.32). We encounter two types of vertices. The first possibility is that the vertex comes from the kinetic term (3.51), in which case both derivatives must act on the internal lines. The rest of this vertex is given by

$$\mathcal{A} = e^{2\tilde{\tau}/\hat{f}_\tau} - 1 . \quad (\text{E1})$$

The other type of vertex comes from expanding \mathcal{L}_d or \mathcal{L}_m , and, apart from the two internal lines, it is given by

$$\mathcal{B} = e^{-2v}(\mathcal{V}''(v + \tilde{\tau}/\hat{f}_\tau) - \mathcal{V}''(v)) , \quad (\text{E2})$$

where \mathcal{V} was defined in Eq. (5.8). The one-loop diagram with n insertions of \mathcal{A} and ℓ insertions of \mathcal{B} can be obtained as

$$\Gamma_{n,\ell} = \frac{1}{2} \left(\frac{\mathcal{A}^n}{n!} \frac{\partial^n}{\partial \lambda^n} \right) \left(\frac{\mathcal{B}^\ell}{\ell!} \frac{\partial^\ell}{\partial (m_\tau^2)^\ell} \right) I_0 \Big|_{\lambda=1} , \quad (\text{E3})$$

where in $d \neq 4$ dimensions,

$$I_0 = \Gamma(-d/2) \left(\frac{m_\tau^2}{4\pi\lambda} \right)^{d/2} . \quad (\text{E4})$$

Summing over n and ℓ , and using $m_\tau^2 = e^{-2v}\mathcal{V}''(v)$, we find

$$\begin{aligned} V_1 &= - \sum_{n,\ell=0}^{\infty} \Gamma_{n,\ell} \\ &= - \frac{1}{2} \exp \left(\mathcal{B} \frac{\partial}{\partial (m_\tau^2)} \right) \exp \left(\mathcal{A} \frac{\partial}{\partial \lambda} \right) I_0 \Big|_{\lambda=1} \\ &= - \frac{\Gamma(-d/2)}{2} \left(\frac{e^{-2(v+\tilde{\tau}/\hat{f}_\tau)} \mathcal{V}''(v + \tilde{\tau}/\hat{f}_\tau)}{4\pi} \right)^{d/2} \\ &= - \frac{\Gamma(-d/2)}{2} \left(\frac{e^{-2\tau} \mathcal{V}''(\tau)}{4\pi} \right)^{d/2} , \end{aligned} \quad (\text{E5})$$

in agreement with Eq. (5.9). The role of the first exponential on the second line is to replace m_τ^2 in Eq. (E4) by $\mathcal{B} + m_\tau^2 = e^{-2v}\mathcal{V}''(\tau)$, while the role of the second is to replace λ by $e^{-2v}e^{2\tau}$. The factors of e^{2v} then cancel out, leading to the final result which depends only on the full $\tau(x)$ field. The (minimally subtracted) renormalized effective potential may be obtained from Eq. (5.9d) by discarding the pole part, and substituting $u(x) \rightarrow \tau(x)$.

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