# A Metric for Positional Games

J. Mark Ettinger
Los Alamos National Laboratory
Mail Stop B-230
Los Alamos, New Mexico 87545 U.S.A.
ettinger@lanl.gov

**Abstract**

We define an extended real-valued metric, $\rho$, for positional games and prove that this class of games is a topological semigroup. We then show that two games are finitely separated iff they are path-connected and iff two closely related Conway games are equivalent. If two games are at a finite distance then this distance is bounded by the maximum difference of any two atoms found in the games. We may improve on this estimate when two games have the same form, as given by a form match. Finally, we show that if $\rho(G, H) = \infty$ then for all $X$ we have $G + X \not\equiv H + X$, a step towards proving cancellation for positional games.

Keywords: Combinatorial games, Milnor games, Conway games.

## 1 Introduction

The study of positional games is the study of two-player games of perfect information with no randomness and real-valued outcomes. Play is alternating with a nonterminated game having move options for both players. The origin of this field is in the paper [Milnor53] in which the key concept of the sum of games is first introduced. A move in a sum of games is a move in exactly one of the component games. It is then the other player's turn to move. Real world examples of positional games include Chess, Go and Tic-Tac-Toe. Conway later developed a closely related class of games called combinatorial games for which the classic reference is [Conway76]. Milnor restricted his attention to games of positive incentive, i.e. games in which both players are eager to move. It turns out that this restricted class of games is a group under the appropriate definition of inverse. In [Ettinger96-1] the full class of positional games, denoted $F_{\mathbf{R}}$, was studied and it was shown that this class has noninvertible games but satisfies the other axioms for semigroups. Generally speaking, the class of Milnor games is much more thoroughly understood than $F_{\mathbf{R}}$. One example is whether the cancellation property holds for all games in $F_{\mathbf{R}}$, i.e. does $G + X \equiv H + X$ imply $G \equiv H$? If $X$ is noninvertible this question is open.

The current investigations have their origin in this question of cancellation. Since $F_{\mathbf{R}}$ is a commutative semigroup a proof of cancellation would assure the possibility of extension to a group, though in addition to an abstract extension it is desired to obtain an extension with a natural game-theoretic interpretation for the new elements (see [Ettinger96-2] for details). We define an extended real-valued metric for positional games. We prove that addition is continuous under the topology generated by the metric. We then show that two games are finitely separated iff they are path-connected and iff two closely related Conway games are equivalent. If two games are at a finite distance then this distance is bounded by the maximum difference of any two atoms found in the games. We may improve on this estimate when two games have the same form, as given by a form match. Finally we relate these studies to the question of cancellation in $F_{\mathbf{R}}$ by showing that if $\rho(G, H) = \infty$ then for all $X$ we have $G + X \not\equiv H + X$. In this case $G$, $H$, and (all) $X$ are not candidates for counterexamples to cancellation so we need only search among $G$ and $H$ such that $\rho(G, H) < \infty$.

# 2 Positional Games and Conway Games

We consider two-player, non-random games of perfect information with real-valued outcomes. Omitted proofs may be found in [Ettinger96-1].

**Definition 1** *$F_{\mathbf{R}}$, the universe of hereditarily finite games on $\mathbf{R}$, is defined inductively as follows:*

*1. Every $r \in \mathbf{R}$ is a hereditarily finite game.*

*2. If $A$ and $B$ are non-empty, finite sets of hereditarily finite games then $\{A|B\}$ is a hereditarily finite game.*

*If* $G = \{A|B\}$ *then we write* $G^L = A$ *and* $G^R = B$. $G^L$ *is called the set of* left options *or* left moves *of* $G$ *and similarly for the right options of* $G$. *Also if* $H \in A$ *we write* $H \in_l G$ *and if* $H \in B$ *we write* $H \in_r G$. *So* $g_l \in_l G$ *iff* $g_l \in G^L$ *and* $g_r \in_r G$ *iff* $g_r \in G^R$. *Write* $g \in_s G$ *(*symmetric elementhood*) iff* $g \in_l G$ *or* $g \in_r G$.

To each game we assign an ordinal rank (called a birthday in [Conway76]) analogous to rank in set theory. Real numbers have rank = 0 and we call these games *atomic* games. Those games $G$ such that $G \in F_{\mathbf{R}} - \mathbf{R}$ are called *nonatomic games.* For nonatomic games we have

$$rank(G) = \max\{rank(g_l) + 1, rank(g_r) + 1\}$$

where the maximum is taken over all Left and Right options of $G$. For example $\{7| - 2\} \in F_{\mathbf{R}}$ has rank 1, $\{\{7| - 2\}|9\}$ has rank 2, etc.

We now define an addition operation on games. This operation first appeared in [Milnor53] and again in [Conway76]. The idea is that to make a play in $G + H$, one plays in exactly *one* of the games and leaves the other alone. It is then the other player's turn and he moves in exactly one component and so on until both components have ended.

**Definition 2** *We define game addition. If* $p$ *and* $q$ *are atomic games then they add as real numbers. Let*

$$p + \{g_{l_1}, ...|g_{r_1}, ...\} = \{g_{l_1} + p, ...|g_{r_1} + p, ...\}.$$

*Finally, if* $G$ *and* $H$ *are both non-atomic then let*

$$\{G^L|G^R\} + \{H^L|H^R\} = \{J^L|J^R\}$$

*where*

$$J^L = \{g_l + H, G + h_l : g_l \in G^L \text{ and } h_l \in H^L\}$$

*and*

$$J^R = \{g_r + H, G + h_r : g_r \in G^R \text{ and } h_r \in H^R\}.$$

For the rest of this paper unless otherwise stated, $p$,$q$,$r$ will designate atomic games, i.e. real numbers, $X$,$Y$,$Z$,$W$,$G$,$H$,$J$ will designate non-atomic games, $G^L$ will designate the set of left moves of $G$, $g_l$ will designate a left move of $G$ (i.e. $g_l \in_l G$), etc.

**Proposition 1** *Game addition is commutative and associative.*

We now define the operation of the Left and Right move operators, $\overline{L}$, $\overline{R}$, on a game. We wish to capture the notion of the determined outcome of a game if both players play completely optimally, i.e. in full knowledge of the entire game tree.

**Definition 3** *Let $G \in F_{\mathbf{R}}$. Then we define $\overline{L}(G)$ and $\overline{R}(G)$ as follows.*

1. $\overline{L}(p) = \overline{R}(p) = p$.
2. $\overline{L}(G) = \max\{\overline{R}(g_l) : g_l \in_l G\}$.
3. $\overline{R}(G) = \min\{\overline{L}(g_r) : g_r \in_r G\}$.

Notice the maximum and minimum are over finite sets. Thus $\overline{L}(G)$ and $\overline{R}(G)$ are the optimal outcomes which must occur if both players play optimally with Left starting and Right starting respectively.

We now are able to define our primary binary relations. The basic idea is that given two games, $G$ and $H$, $G$ may be *preferable* over $H$ from the perspective of the left player and $H$ may be *preferable* over $G$ from the perspective of the right player in sums with *test games.* First we operate on $G + X$ and $H + X$, where $X$ is an arbitrary test game, with $\overline{L}$ and $\overline{R}$ to produce the optimal outcomes. If the outcome of playing $G + X$ is *always* greater than the outcome of playing $H + X$ we say $G$ is "greater than" $H$.

**Definition 4** *Let $G, H \in F_{\mathbf{R}}$. Define*

$$G \geq_{\overline{L}} H \; iff \; \forall X \in F_{\mathbf{R}} \, \overline{L}(G + X) \geq \overline{L}(H + X).$$

*Define*

$$G \geq_{\overline{R}} H \; iff \; \forall X \in F_{\mathbf{R}} \, \overline{R}(G + X) \geq \overline{R}(H + X).$$

*Finally define*

$$G \geq H \; iff \; G \geq_{\overline{L}} H \; and \; G \geq_{\overline{R}} H.$$

If $G \geq H$ and $H \geq G$ then write $G \equiv H$. Write $G >_{\overline{L}} H$ if $G \geq_{\overline{L}} H$ and not $H \geq_{\overline{L}} G$. Similarly define $G >_{\overline{R}} H$ and $G > H$.

**Proposition 2** *The binary relations $\geq_{\overline{L}}$, $\geq_{\overline{R}}$, and $\geq$ are transitive.*

The next proposition shows that to check if $G \geq H$ it suffices to check one of $G \geq_{\overline{L}} H$ or $G \geq_{\overline{R}} H$. Define the *transitive closure* of a game $G$ to be all the games which are hereditarily elements of $G$. Formally, $trcls(p) = \{p\}$ and $trcls(G) = G^L \cup G^R \cup \bigcup_{g_s} trcls(g_s)$ where the union is taken over all options $g_s$ of $G$. Define the *support* or *alphabet* of a game $G$, $\alpha(G)$, to be all the set of atoms found in $G$, i.e. $\alpha(G) = trcls(G) \cap \mathbf{R}$. For $\alpha, \beta \in \mathbf{R}$ we write $\alpha \gg \beta$ to indicate that $\alpha$ is much larger than $\beta$. The exact meaning of "much larger" is dependent on the context in which it is being used. The proofs that utilize this notation could be formulated with precise bounds but this seems to obscure the main ideas so we incorporate this notation. In point of fact if $\beta > 0$ and we say let $\alpha \gg \beta$ then usually choosing $\alpha = 10\beta$ will perform the desired function. This is a useful heuristic. We write $\beta \ll \alpha(G)$ if for all $\gamma \in \alpha(G)$ we have $\beta \ll \gamma$ and $\alpha(G) \ll \alpha(H)$ if for all $\beta \in \alpha(G)$ we have $\beta \ll \alpha(H)$.

**Proposition 3** *$G \geq_{\overline{L}} H$ iff $G \geq_{\overline{R}} H$.*

**Proof.** By symmetry it suffices to prove one direction, say left to right. So we assume $G \not\geq_{\overline{R}} H$ and show $G \not\geq_{\overline{L}} H$. If $G \not\geq_{\overline{R}} H$ then there exists a game $X$ such that $\overline{R}(G + X) < \overline{R}(H + X)$. Now let $r \in \mathbf{R}$ such that

$$-r \ll \alpha(G) \cup \alpha(H) \cup \alpha(X) \ll r.$$

Now define $Y = \{X | -r\}$. Then

$$\overline{L}(G + Y) = \overline{R}(G + X)$$

and

$$\overline{L}(H + Y) = \overline{R}(H + X)$$

because Left has such a large incentive to prevent Right from choosing $-r$ in both games. Therefore

$$\overline{L}(G + Y) < \overline{L}(H + Y)$$

and we have shown $G \not\geq_{\overline{L}} H$.⊣

**Proposition 4** *(Comparison Theorem) If $G$ is nonatomic then $G \geq 0$ iff:*
*1a. $\overline{L}(G) \geq 0$, and*

*1b. For all $g_r$ there exists $g_{rl}$ such that $g_{rl} \geq 0$.*
*and $G \leq 0$ iff:*
*2a. $\overline{R}(G) \leq 0$, and*
*2b. For all $g_l$ there exists $g_{lr}$ such that $g_{lr} \leq 0$.*

**Proposition 5** *If $G \geq H$ then for all $J \in F_{\mathbf{R}}$ $G + J \geq H + J$.*

**Proposition 6** *$(F_{\mathbf{R}}/\equiv, \geq, +)$ is a commutative, partially-ordered, semigroup.*

**Proof** Follows from previous propositions. $\dashv$

$F_{\mathbf{R}}$ is not, in general, a group because many games do not have inverses. See [Ettinger96-1] for details. A game, $G$, is called *invertible* if there exists $H$ such that $G + H \equiv 0$. By the previous theorem we know that all invertible games are *cancellable*, i.e. if $H$ is invertible and $G + H \geq J + H$ then $G \geq J$. The questions of whether the cancellation property holds for $\geq$ and $\equiv$ *generally* are open.

**Definition 5** *We define $FM_{\mathbf{R}}$, the hereditarily finite Milnor games with real outcomes (henceforth called Milnor games), inductively as follows:*
*1. For all $r \in \mathbf{R}$, $r \in FM_{\mathbf{R}}$.*
*2. Let $G^L$ and $G^R$ be finite subsets of $FM_{\mathbf{R}}$ and $G = \{G^L | G^R\}$. If $\overline{L}(G) \geq \overline{R}(G)$ then $G \in FM_R$.*

Examination of [Milnor53] reveals that $FM_{\mathbf{R}}$ is the class of games under consideration in that paper and it is clear that $FM_{\mathbf{R}} \subseteq F_{\mathbf{R}}$. In Milnor's original terminology, these games have non-negative *incentive*.

**Definition 6** *Let $\geq_M$ be the binary relation defined on $F_{\mathbf{R}}$ by taking test games in $FM_{\mathbf{R}}$. In other words, $G \geq_M H$ iff for all games $X \in FM_{\mathbf{R}}$ we have $\overline{L}(G+X) \geq \overline{L}(H+X)$ and $\overline{R}(G+X) \geq \overline{R}(H+X)$. We write $G \equiv_M H$ iff $G \geq_M H$ and $H \geq_M G$.*

**Definition 7** *Let $G$, $H \in F_{\mathbf{R}}$. Then the* Conway minus *of $G$, $cm(G)$, is defined by $cm(p) = -p$ if $p \in \mathbf{R}$. If $G = \{g_{l_1}, ... | g_{r_1}, ...\}$ is non-atomic then $cm(G) = \{cm(g_{r_1}), ... | cm(g_{l_1}), ...\}$.*

The game cm($G$) hereditarily reverses all move options of $G$, i.e. the game tree of cm($G$) is the "flipped" or "mirror image" version of the game tree of $G$.

We may again prove a proposition similar to proposition 3 which shows that we only need to consider one of the cases of a Left start or Right start and that one case implies the other.

**Proposition 7** *[Milnor53, p. 296] $(FM_{\mathbf{R}}/\equiv_M, +, cm)$ is an abelian group.*

We now define special operators that produce the best outcome subject to certain move constraints. For example, while playing in a sum $G + H$ instead of playing the game $G$ to obtain the best outcome $\overline{L}(G)$, in some situations Left will desire to play $G$ subject to the constraint that he move last in $G$, forcing Right to play in the other component $H$. The optimal result in this case is written as $(\overline{L}, L)(G)$ and represents the best possible outcome for Left given the top priority of a final Left move in $G$.

**Definition 8** *Let $\infty$ be greater than all elements of $\mathbf{R}$ and $-\infty$ be less than all elements of $\mathbf{R}$. We define the action of the* move-restricted operators, *$(\overline{L}$,L) and $(\overline{R}$, R) on games. Given a game $G$ the* move-modified games of *$G$, denoted $G_{LL}$ and $G_{RR}$, are obtained from $G$ by replacing all occurrences of atoms $p$ in $G$ by $\{-\infty|p\}$ and $\{p|\infty\}$ respectively. Formally, $p_{LL} = \{-\infty|p\}$, $\{g_{l_1}, ...|g_{r_1}, ...\}_{LL} = \{g_{l_1 LL}, ...|g_{r_1 LL}, ...\}$, and so on. Now define*

*1. $(\overline{L}$,L)($G$) = $\overline{L}(G_{LL})$.*

*2. $(\overline{R}$,R)($G$) = $\overline{R}(G_{RR})$.*

The move-modified games are utilized to assess appropriate penalties for taking or failing to take the final move in a game.

**Example 1** *Let $G = \{\{1|2\}|\{3|4\}\}$. $(\overline{L}$, L)($G$) = $-\infty$ and $(\overline{R}$, R)($G$) = $\infty$ because neither player has a move which insures a last move. Let $H = \{0, \{0|1\}| - 1\}$. Then $\overline{L}(H) = 1$ but $(\overline{L}, L)(H) = 0$.*

**Proposition 8** *1. $(\overline{L}$, L)($G$) $\leq \overline{L}(G)$.*

*2. $(\overline{R}$, R)($G$) $\geq \overline{R}(G)$.*

Intuitively, the above proposition is clear because in a play of a game with additional goals one can only do as well (and possibly worse) as play without any move restrictions. In the next section we will relate the move-restricted operators with winning conditions in Conway games, which are now reviewed.

The Conway universe of games, $UC$, is the class defined by the following inductive definition. The empty game, $\{|\} \in UC$. If $G^L, G^R$ are sets of games (possibly empty) then $\{G^L|G^R\} \in UC$. The class $UCF$, of hereditarily finite Conway games, is similar except that $\{G^L|G^R\} \in UCF$ iff $G^L, G^R$ are finite sets of hereditarily finite Conway games. The games in $UCF$ are called *short games* in [Conway76]. We use the same definitions for addition and comparison between games, denoted $\geq_C$, as is found in the above references and we maintain the same terminology of moves, options, etc. In [Conway76] the objects of interest were primarily the equivalence classes given by $\equiv_C$. Notice that $UC/\equiv_C$ is called **Ug** in [Conway76]. We write $G \equiv_C H$ to denote that $G$ and $H$ are equivalent in the sense of [Conway76] and write $0_C$ to denote the Conway equivalence class of the empty game, $\{|\}$.

# 3 The Metric

We now introduce the main object of the present study, a measure of distance between games.

**Definition 9** *We work in the extended real number system where $-\infty < r < \infty$ for all real numbers $r$. Define*

$$\rho_L(G,H) = \sup_X |\overline{L}(G+X) - \overline{L}(H+X)|$$

*and*

$$\rho_R(G,H) = \sup_X |\overline{R}(G+X) - \overline{R}(H+X)|.$$

*Let $\rho(G,H) = \max\{\rho_L(G,H), \rho_R(G,H)\}$.*

The proof of Proposition 3 shows that this multiplicity is redundant and leads to the following result.

**Proposition 9** *For all $G$ and $H$, $\rho_L(G,H) = \rho_R(G,H)$.*

In light of this proposition we will only have need to discuss $\rho(G,H) = \sup_X |\overline{L}(G+X) - \overline{L}(H+X)|$.

**Proposition 10** *$\rho : F_{\mathbf{R}} \times F_{\mathbf{R}} \to \mathbf{R} \cup \{\infty\}$ is an extended real-valued metric.*

**Proof.** We need only prove the triangle inequality.

$$\begin{aligned}
\rho(G,H) &= \sup_X |\overline{L}(G+X) - \overline{L}(H+X)| \\
&= \sup_X |\overline{L}(G+X) - \overline{L}(J+X) + \overline{L}(J+X) - \overline{L}(H+X)| \\
&\leq \sup_X |\overline{L}(G+X) - \overline{L}(J+X)| + |\overline{L}(J+X) - \overline{L}(H+X)| \\
&\leq \sup_X |\overline{L}(G+X) - \overline{L}(J+X)| + \sup_Y |\overline{L}(J+X) - \overline{L}(H+X)| \\
&= \rho(G,J) + \rho(J,H) \dashv
\end{aligned}$$

Notice that $G \equiv H$ iff $\rho(G,H) = 0$. The next definition formalizes the separation of the atoms belonging to two games. The *alphabetic distance* is not itself a metric but will provide a bound on $\rho(G,H)$.

**Definition 10** *The* alphabetic distance *between two games $G$ and $H$ is $d_\alpha(G,H) = \max_{\beta \in \alpha(G), \gamma \in \alpha(H)} |\beta - \gamma|$.*

Before further investigation of $\rho$ we first study a similar metric for Milnor games, $FM_{\mathbf{R}}$.

**Definition 11** *For $G, H \in FM_{\mathbf{R}}$ define*

$$\rho'(G,H) = \sup_{X \in FM_{\mathbf{R}}} |\overline{L}(G+X) - \overline{L}(H+X)|.$$

In other words, $\rho'$ is the natural equivalent of $\rho$ for the restricted class, $FM_{\mathbf{R}}$. Notice that for all Milnor games $G$ and $H$ we have $\rho(G,H) \geq \rho'(G,H)$. However $\rho'$ is not the restriction of $\rho$ to $FM_{\mathbf{R}}$. This is because the extra test games in $F_{\mathbf{R}}$ allow for cases like $G = 0$ and $H = \{0|0\}$ where $\rho'(G,H) = 0$ (a consequence of results in [Milnor53]) but $\rho(G,H) = \infty$ (shown later in this paper). However unlike $\rho$, $\rho'$ is real-valued, i.e. all Milnor games are a finite $\rho'$ distance apart.

**Proposition 11** *$\rho' : FM_{\mathbf{R}} \times FM_{\mathbf{R}} \to \mathbf{R}$ is a metric. In fact $\rho'(G,H) \le d_\alpha(G,H)$.*

**Proof.** Fix $G$, $H$, $X \in FM_{\mathbf{R}}$ and suppose $\overline{L}(G+X) \ge \overline{L}(H+X)$. By [Milnor53, p.294] we have $\overline{L}(G+X) \le \overline{L}(G) + \overline{L}(X)$ and $\overline{L}(H+X) \ge \overline{R}(H) + \overline{L}(X)$. Therefore

$$\begin{aligned} \overline{L}(G+X) - \overline{L}(H+X) &\le (\overline{L}(G) + \overline{L}(X)) - (\overline{R}(H) + \overline{L}(X)) \\ &= \overline{L}(G) - \overline{R}(H) \\ &\le d_\alpha(G,H) \dashv \end{aligned}$$

**Theorem 1** *Game addition is uniformly continuous under the topology generated by $\rho$. Therefore $F_{\mathbf{R}}$ is a topological semigroup. $\overline{L}$ and $\overline{R}$ are also continuous functions.*

**Proof.** If $\rho(H,J) \le \epsilon$ then $\rho(G+H, G+J) \le \epsilon$ since for any $X$ we have $|\overline{L}((G+X)+H) - \overline{L}((G+X)+J)| \le \epsilon. \dashv$

We now pursue the classification of the components of finitely separated games and bounds on these distances. We will require the notion of substituting one atom for another in a game $G$.

**Definition 12** *A* specified atom *of a game $G$ is a sequence of games, $\overline{p} = (g_0, g_1, g_2, ..., g_n)$ where $g_0 = G$, $g_n = p$ is an atom, and $g_{i+1} \in_s g_i$. By $\overline{p}(i) = (g_{i+1}, ..., g_n)$ we denote the* specified atom of $g_{i+1}$ derived from $\overline{p}$. *If $\overline{p}$ is a specified atom of $G$ and $r$ is an atom, then the* game obtained by replacing $\overline{p}$ in $G$ with $r$, *denoted $G(\overline{p}, r)$ is defined inductively by $G(\overline{p}, r) = r$ if $G = p$ (i.e. $G$ is atomic), $G(\overline{p}, r) = \{g_1(\overline{p}(0), r), g_{l_2}, ...|g_{r_1}, ...\}$ if $G = \{g_1, g_{l_2}, ...|g_{r_1}, ...\}$ and $G(\overline{p}, r) = \{g_{l_1}, g_{l_2}, ...|g_1(\overline{p}(0)r), g_{r_2}, ...\}$ if $G = \{g_{l_1}, g_{l_2}, ...|g_1, g_{r_2}, ...\}$. Finally we write $G(\overline{p}_1, r_1; \overline{p}_2, r_2; ...; \overline{p}_n, r_n)$ for $G(\overline{p}_1, r_1)(\overline{p}_2, r_2)...(\overline{p}_n, r_n)$ and $G_r$ for $G(\overline{p}_1, r; \overline{p}_2, r; ...; \overline{p}_n, r)$ where $\{\overline{p}_i\}$ is the set of* all *specified atoms in $G$.*

**Example 2** *If $G = \{1|\{2|\{1|3\}\}\}$ and the specified atom $\overline{p}$ is $\overline{p} = (G, 1)$ then $G(\overline{p}, 5) = \{5|\{2|\{1|3\}\}\}$ and $G_0 = \{0|\{0|\{0|0\}\}\}$.*

In some circumstances we may be able to estimate the distance between two games. Given a game $G$, specified atoms $\overline{p}_1$,...,$\overline{p}_n$, and atoms $r_1$,...,$r_n$, $G(\overline{p}_1, r_1; ...; \overline{p}_n, r_n)$ has the "same form" as $G$ but with certain atoms replaced. The next definition captures this notion of two games having the same form and the proposition verifies the intuition that two games with the same form are separated by a distance bounded by the maximum difference of corresponding atoms.

**Definition 13** *A* form match *from the atomic game $p$ to the atomic game $q$ is the unique function $f : \{p\} \to \{q\}$ (i.e. $f(p) = q$). A* form match *from the nonatomic game $G$ to the nonatomic game $H$ is a quadruple of functions, $f =< f_1, f_2, f_3, f_4 >$, where:*

*1. $f_1 : G^L \to H^L$ and is bijective.*

*2. $f_2 : G^R \to H^R$ and is bijective.*

*3. The domain of $f_3$ is $G^L$ and $f_3(g_l) = f'$ where $f'$ is a form match from $g_l$ to $f_1(g_l)$.*

*4. The domain of $f_4$ is $G^R$ and $f_4(g_r) = f''$ where $f''$ is a form match from $g_r$ to $f_2(g_r)$.*

*If there is a form match from $G$ to $H$ we say that $G$ and $H$* have the same form. *The* distance *of a form match $f$, denoted $|f|$, is $|p - q|$ if $f$ is an atomic form match and $max_{g_l,g_r}\{|f_3(g_l)|, |f_4(g_r)|\}$ for a nonatomic form match.*

**Proposition 12** *$G$ and $H$ have the same form iff there exists a set of specified atoms of $G$, $\{\overline{p}_1, ..., \overline{p}_n\}$, and a set of atoms $\{r_1, ...r_n\}$ such that $H = G(\overline{p}_1, r_1; ...; \overline{p}_n, r_n)$.*

**Proof.** A simple induction.⊣

Notice that if $f$ is a form match from $G$ to $H$ then $|f| \leq d_\alpha(G, H)$ and the previous proposition shows that rank$(G) =$ rank$(H)$.

**Proposition 13** *If $f$ is a form match from $G$ to $H$ then $\rho(G, H) \leq |f|$.*

**Proof.** If $f$ is an atomic form match then the result is clear. Now assume the proposition for all games of smaller rank than $G$. First suppose

$$\overline{L}(G + X) = \overline{R}(G + x_l)$$

(i.e. $x_l$ is an optimal Left move). In $H + X$ Left has the option to move to $H + x_l$. By induction on rank$(X)$ we have

$$|\overline{R}(G + x_l) - \overline{R}(H + x_l)| \leq |f|.$$

This implies

$$\overline{L}(H+X) \geq \overline{L}(G+X) - |f|.$$

Now assume

$$\overline{L}(G+X) = \overline{R}(g_l + X)$$

(i.e. $g_l$ is an optimal Left move). Note $f_3(g_l)$ is a form match from $g_l$ to some $h_l$ and $|f_3(g_l)| \leq |f|$. So by induction

$$|\overline{R}(g_l+X) - \overline{R}(h_l+X)| \leq |f_3(g_l)| \leq |f|.$$

This again implies

$$\overline{L}(H+X) \geq \overline{L}(G+X) - |f|.$$

A symmetric argument shows

$$\overline{L}(G+X) \geq \overline{L}(H+X) - |f|$$

so we have

$$|\overline{L}(G+X) - \overline{L}(H+X)| \leq |f|$$

and the proof is complete.⊣

This result gives us an estimate on $\rho(G,H)$ in the case where there exists a form match from $G$ to $H$, namely $\min\{|f| : f$ is a form match from $G$ to $H\}$.

The following lemmas are used in the proof of the main theorem which characterizes the games which are a finite distance apart.

**Lemma 1** *Suppose $\{\overline{p}_1, ..., \overline{p}_n\}$ is the complete set of all specified atoms in $G$. Then $G$ and $G(\overline{p}_1, r_1; ...; \overline{p}_n, r_n)$ are path-connected for any set of real numbers $\{r_1, ..., r_n\}$.*

**Proof.** Let $G(t) = G(\overline{p}_1, tr_1 + (1-t)p_1; ...; \overline{p}_n, tr_n + (1-t)p_n)$. Then by the previous proposition $t : [0,1] \to G(t)$ is a path from $G = G(0)$ to $G(\overline{p}_1, r_1; ...; \overline{p}_n, r_n) = G(1)$.⊣

**Lemma 2** *$G_0 \equiv_C H_0$ implies $G_0 \equiv H_0$.*

**Proof.** By the comparison theorem it is easy to check that $H_0 + cm(H_0) \equiv 0$. It is also easy to check that $G_0 + cm(H_0) \equiv 0$ if $G_0 \equiv_C H_0$.⊣

**Lemma 3** *$G_0 \equiv_C 0_C$ iff $(\overline{L}, L)(G) = -\infty$ and $(\overline{R}, R)(G) = \infty$.*

**Proof.** The move-restricted operators capture the ability to take the final move in a game. Both conditions hold iff neither player can move first with the goal of taking the final move in the game.$\dashv$

**Theorem 2** *The following are equivalent:*

1. $G_0 \equiv_C H_0$.
2. *$G$ and $H$ are path connected.*
3. $\rho(G, H) < \infty$.

**Proof.** We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). $G$ is path-connected to $G_0$ and $H$ is path-connected to $H_0$ by Lemma 1. So (1) implies $G_0 \equiv H_0$ by Lemma 2 and this implies (2). (2) implies (3) is a straightforward topological argument. Any path between $G$ and $H$ is the continuous image of a compact space and is therefore compact. Cover the path by balls of radius $\epsilon$ and take a finite subcovering. The triangle inequality then implies (3). To prove (3) $\Rightarrow$ (1), assume (1) does not hold. Then $G_0 + cm(H_0) \not\equiv_C 0_C$ so at least one player, say Right, has an opening move which results in a Conway win, i.e. insures he may make the final move. The proof of Lemma 3 implies $(\overline{R}, R)(G+cm(H_0)) < \infty$ whereas $(\overline{R}, R)(H+cm(H_0)) = \infty$. Let $X = \{-s|s\}$ where $-s \ll \alpha(G) \cup \alpha(H) \ll s$. Then

$$\overline{R}(G + cm(H_0) + X) = \gamma + 0 + -s \ll \beta + 0 + s = \overline{R}(H + cm(H_0) + X)$$

where $\gamma, \beta \in \alpha(G)$. We may make $s$ as large as desired and this proves $\rho(G, H) = \infty$ if (1) is false. Therefore we have shown (3) $\Rightarrow$ (1).$\dashv$

In fact we may improve on this result by deriving a bound on $\rho(G, H)$ if we know the distance is finite. This bound will be a consequence of the next proposition which is a finer result linking the Conway inequality with the metric.

**Proposition 14** *If $G_0 \geq_C H_0$ then*

1. $\overline{L}(G + X) \geq \overline{L}(H + X) - d_\alpha(G, H)$.
2. $\overline{R}(H + X) \leq \overline{R}(G + X) + d_\alpha(G, H)$.

**Proof.** If $G$ and $H$ are atomic the result clearly holds. Now assume the result holds by induction for all games $G$, $H$, and $X$ of smaller rank and that (1) fails to hold. If

$$\overline{L}(H + X) = \overline{R}(H + x_l)$$

for some $x_l$ (i.e. the Left move to $x_l$ is optimal) then we have

$$\overline{R}(H+x_l) \leq \overline{R}(G+x_l) + d_\alpha(G,H)$$

by induction. But

$$\overline{R}(G+x_l) \leq \overline{L}(G+X)$$

so we have

$$\overline{L}(H+X) \leq \overline{L}(G+X) + d_\alpha(G,H)$$

which is a contradiction. Therefore the optimal Left move in $H+X$ must be to some $h_l + X$. Now since $G_0 \geq_C H_0$ either there exists $h_{lr}$ such that $G_0 \geq_C h_{lr0}$ or there exists $g_l$ such that $g_{l0} \geq_C h_{l0}$. If the latter holds then

$$\overline{L}(H+X) = \overline{R}(h_l+X) \leq \overline{R}(g_l+X) + d_\alpha(G,H) \leq \overline{L}(G+X) + d_\alpha(G,H)$$

where the first inequality uses the fact that $d_\alpha(g_l, h_l) \leq d_\alpha(G,H)$ and we again have a contradiction. If the former holds then

$$\overline{L}(H+X) = \overline{R}(h_l+X) \leq \overline{L}(h_{lr}+X)$$

and

$$\overline{L}(G+X) \geq \overline{L}(h_{lr}+X) - d_\alpha(G,h_{lr})$$

by induction. Note that

$$d_\alpha(G,h_{lr}) \leq d_\alpha(G,H)$$

and combining these inequalities we obtain

$$\overline{L}(G+X) \geq \overline{L}(H+X) - d_\alpha(G,H)$$

which is again a contradiction. So (1) must hold and the proof of (2) is similar.⊣

**Corollary 1** *If $\rho(G,H) < \infty$ then $\rho(G,H) \leq d_\alpha(G,H)$.*

**Proof.** If $\rho(G,H) < \infty$ then $G_0 \equiv_C H_0$ by the above theorem. The proposition then immediately yields the result.⊣

One may recall from the comments in the introduction that these investigations had their origin in the search for a proof of the cancellation property for $F_{\mathbf{R}}$. The following results shows that if there exists $G$, $H$, and $X$ such that $G \not\equiv H$ but $G+X \equiv H+X$ then $\rho(G,H) < \infty$ and thus, in particular, $G_0 \equiv_C H_0$.

**Proposition 15** *If* $\rho(G,H)=\infty$ *then for all* $X$*,* $\rho(G+X,H+X)=\infty$.

**Proof.** If $\rho(G,H)=\infty$ then $G_0 \not\equiv H_0$ by Theorem 2. Therefore

$$(G+X)_0 = G_0 + X_0 \not\equiv_C H_0 + X_0 = (H+X)_0.$$

So $\rho(G+X,H+X)=\infty$.$\dashv$

**Corollary 2** *If* $G_0 \not\equiv_C H_0$ *then for all* $X$*,* $G+X \not\equiv H+X$.

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