Detection of Critical Points for Shape Metamorphosis Animation

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Abstract

We apply topological analysis to functionally based shape metamorphosis. The time-dependent shape is defined using homotopy. The advantage of this method is the automatic generation of the intermediate shapes between the key shapes of different topology types. To complete the method, we have to find a way to automatically detect the critical points on the time axis while the shape undergoes topological changes. These critical points can be later used for generation of non-linear time steps distribution along the time axis, for example, for providing ease-in/ease-out effects in animation. We present a new method for analysis of shape metamorphosis based on the Morse theory, oriented to analysis of a height function. Although we analyze the shape in an N-dimensional space, the height function is defined in the N+2 dimensional space with N point coordinates and two additional coordinates of the defining function and time values. We can analyze how the critical points are changing in the given height level, which takes only zero value of the shape defining function. In this paper, we present this method in comparison with typical Morse theory analysis using simple objects in 2D and 3D spaces.

Keywords: Metamorphosis, critical points, homotopy, Morse theory.

1. Introduction

Recently, shape metamorphosis animation is used in movies, games, and TV commercials. Among the techniques of the shape metamorphosis, there are functionally based methods. The advantage of the functionally based shape metamorphosis is the automatic generation of intermediate shapes even between the key shapes of different topology types. Since the shape is approximated smoothly by the homotopy functions, we should only define the initial and target shapes. For creating animation, animators select time interval and key shapes for metamorphosis. Detecting the time moments when the topology changes are useful in defining the speed of transformation. The critical points on the time axis (moments of topology changes) can be used, for example, for ease-in/ease-out effects in animation with the transformation speed slowing down in these points.

There are many methods of defining metamorphosis between given shapes [5]. In this paper, we propose a new method based on the Morse theory for the analysis of the linear metamorphosis between functionally defined shapes. In addition, we proceed with comparison of the new proposed method with the typical Morse theory analysis. For the definition of shapes, we use the function representation (F-rep) [2, 6], where geometric solids are defined by inequalities \( F \geq 0 \) and \( F \) is an explicit real function of point coordinates. For modeling complex objects, we use R-functions, which provide analytical definitions of set-theoretic operations [2, 6].

The remainder of the paper is organized as follows. Section 2 reviews the previous works on the analysis of shape topology. We present a new method for analysis of shape metamorphosis based on the Morse theory in section 3. The analysis method based on the typical Morse theory is also explained in the same section. Next, several case studies in 2D and 3D spaces are presented in section 4 using both methods. In section 5, we conclude this paper and discuss future works. In this paper, all symbolic computations and illustrations are made using Mathematica®.

2. Previous Works

There are different techniques of the topology analysis. For example, the well-known method is to calculate the genus of an object. On the basis of triangulation of the object surface, one can detect the Euler characteristic using the number of vertices \( V \), edges \( E \), and faces \( F \). According to the Euler theorem, the Euler characteristic
\( e(X^3) \) in three-dimensional space is calculated as follows:
\[
e(X^3) = V - E + F.
\]

At the next step, the genus \( p \) that means the number of holes in the object is detected:
\[
p = \frac{2 - e(X^3)}{2}.
\]

Another topology analysis method is based on critical points [1]. Critical points are detected by using the condition that the gradient of the object’s defining function vanishes in such points. In addition, the points are classified as maximum points, minimum points, or saddle points on the basis of the signs of the eigenvalues of the Hessian. The topology analysis is performed based on the classification of the critical points. Our approach is close to this one, but is oriented towards the critical points detection on the time axis for shape metamorphosis.

There are also analysis methods using the Reeb graphs [7]. The Reeb graph represents the topological skeleton of an object. Contours of the object together with the Reeb graph represent the entire 3D object. By using this method, the metamorphosis can be presented by transitions between the different topology types. The authors of [3] classify possible topological transitions and show that for two given source and destination shapes there are several possible types of transitions. Each transition type of surface topology can be controlled explicitly by introducing additional key shape, which bridges the topological difference of the two input shapes. In this method, there is a need in the user intervention during the specification of the topological transition and the construction of the key shapes.

3. Methods of critical points detection

We discuss two methods for detection of critical points on the time axis, namely the moments of topology changes. Both methods are based on the Morse theory and use the homotopy function such as linear interpolation on the closed interval, \( I[0,1] \).

3.1. Method using typical Morse theory analysis

The first method uses typical Morse theory analysis. The steps of this method are described below:

1. Create F-rep models for the initial shape and the target shape in an N dimensional space.

2. Create an F-rep model for the N+1 dimensional object using a homotopy function for metamorphosis.

3. Create the height function from the created F-rep models.

4. Detect the critical points of the height function.

5. Detect the moments when the topology changes.

At the first step, the initial and target shapes for the shape metamorphosis are defined using F-rep. These shapes are key shapes of the shape metamorphosis and can have different topology types.

At the second step, the N+1 dimensional (time dependent) object is generated by the homotopy function. The homotopy function is used for the generation of the N+1 dimensional metamorphosis between the N dimensional initial and a target shapes. The intermediate shapes, cross-sections of the N+1 dimensional shape metamorphosis, can be generated automatically along the time axis by using the homotopy function.

At the third step, we create the height function by expressing the time variable in terms of point coordinates as
\[
t = h[x_1, x_2, \ldots, x_n].
\]

At the next step, we detect the critical points of the height function \( h \). The critical points are special points, where the gradient of the height function vanishes:
\[
\nabla h = \left( \frac{\partial}{\partial x_1} h, \frac{\partial}{\partial x_2} h, \ldots, \frac{\partial}{\partial x_n} h \right) = 0.
\]

The F-rep model and the critical points are defined in the Euclidean space. Therefore, the irrelevant solutions \( (x_1, x_2, \ldots, x_n) \) (e.g., including complex numbers) of the above system are eliminated from the set of critical points.

After the detection of the critical point \( p_i \), the next step is only to assign the coordinate values of the critical points as the argument of the height function as \( h(p_i) \), and thus to calculate the time values for the topology changes in the shape metamorphosis. Since the time variable is defined on a closed interval, \( I[0,1] \), the irrelevant time values can be also excluded.

3.2. Analysis of the F-rep function level

In the second method, the F-rep function is considered a height function and we detect the topology change moments in the particular function level \( F=0 \). The steps
of this method are described below.

1. Create F-rep models for the initial shape and target shape in N dimensional space
2. Create the height function in an N+2 dimensional space using a homotopy function and the F-rep models.
3. Detect the moments of topology changes in the particular height level.

At the first step the initial and target shapes in the N dimensional space are defined using F-rep. This step is equivalent to the first step of the previous method.

At the second step, the height function \( h \) which is defined in an N+2 dimensional space is generated by using homotopy function and the F-rep models of the initial and target shapes. Two additional coordinates are function values and time values. Because the time coordinate and the function coordinate are added to the defined N dimensional object, the height function is defined in an N+2 dimensional space. The F-rep function defining the entire metamorphosis is considered a time-dependent height function:

\[
h = f(x_1, x_2, ..., x_n, t).
\]

We are interested only in those critical points, which are located on the surface of some intermediate shapes, so only analysis of the particular level \( h=0 \) is performed.

At the final step, we detect the critical points of the topology changes. The detection of critical points is performed in the zero level of the height function. This is the main difference from the typical Morse theory analysis. We keep the zero level and analyze how the critical points are changing in this level in time. The critical points have to satisfy the following conditions:

\[
\frac{\partial}{\partial x_1} h = \frac{\partial}{\partial x_2} h = ..., \frac{\partial}{\partial x_n} h = 0 \quad \text{and} \quad h(x_1, x_2, ..., x_n, t) = 0
\]

, where \( t \) is time variable and \( x_i \) are geometric point coordinates.

The condition \( h=0 \) is added for analysis of only the particular level. The irrelevant solutions \((x_1, x_2, ..., x_n, t)\) are excluded from the set of critical points. The time value of the critical points defines the moments of the topology changes.

4. Case studies in 2D and 3D

In this section, we describe our case studies in 2D and 3D spaces. Two examples of shape metamorphosis in 2D are included in 4.1. One is a quite simple case of a disk to a ring metamorphosis and another is a more complex case. In 4.2, we give two 3D examples.

4.1. Examples in 2D

Here, we consider 2D shape metamorphosis from a disk to a ring. In this metamorphosis, the initial shape and the target shape are defined as \( f_{ini}(x, y) \) and \( f_{tar}(x, y) \) respectively (see Figure 1):

\[
f_{ini}[x, y] = s^2 - x^2 - y^2 \\
f_{tar}[x, y] = -x^4 - y^4 - 2x^2 y^2 + 2(R^2 + r^2)x^2 + 2(R^2 + r^2)y^2 - (R^2 - r^2)^2
\]

, where \( s = 2, R = 2, r = 1 \).

![Figure 1: Initial shape (left) and target shape (right) of different topology types](image)

By adding the time coordinate, the 3D object is generated from these shapes using linear homotopy function \( f[x, y, t] \):

\[
f[x, y, t] = f_{ini}[x, y] \cdot (1-t) + f_{tar}[x, y] \cdot t
\]

\[
= 4 - x^2 - y^2 - t \cdot (13 + 11y^2 + y^4 + x^2 \cdot (2y^2 - 11))
\]

Then, the height function \( h_1 \) is created by solving the equation \( f[x, y, t] = 0 \) for \( t \) variable, which is a height function in the first approach presented above:
Figure 2 shows the surface defined by the height function $h_1$ in the closed time interval $I[0,1]$. According to the figure, we can find out intuitively that the topology can change at some time moment (height level). Therefore, the moment that the topology changes is single in this case.

At the next step, to analyze the time moment when the topology changes, we should detect the critical points of the height function. The critical points are special points satisfying the condition of the vanishing gradient:

$$\frac{\partial}{\partial x} h_1 = \frac{\partial}{\partial y} h_1 = 0. $$

The only one critical point is detected as a solution of the above system in this case: $(x, y) = (0, 0)$. After the detection of the critical point, the next step is only to assign the critical point coordinates to the height function $h_1(x, y)$. Then, the height value is a moment of the topology change. The detected result is $t = h_1(0,0) = 0.307692$. Figure 3 shows the intermediate shape of the metamorphosis at this time moment. The correctly detected moment of the topology change is the one when a hole is poked in the disk.

Applying another method proposed above, we use the same functions $f_{in}[x, y]$, $f_{inr}[x, y]$ and $f[x, y, t]$. At the next step, the height function $h_2$ is created from the homotopy function $f[x, y, t]$ (defining a hypersurface in 4D space). The height function is described below.

$$h_2 = f[x, y, t]$$

$$= -4 - x^2 - y^2$$

$$- t \cdot (13 + x^4 - 11y^2 + y^4 + x^2 \cdot (2y^2 - 11))$$

At the final step, we detect the critical points satisfying the following conditions:

$$\frac{\partial}{\partial x} h_2 = \frac{\partial}{\partial y} h_2 = 0 \quad \text{and} \quad f[x, y, t] = 0.$$

The second condition $f[x, y, t] = 0$ means that the height $h_2$ is fixed to zero value and we want to detect only critical points on the surface of an intermediate shape. After the detection, the exclusion of irrelevant points is performed. The result by using this method is $(x, y, t) = (0, 0, 0.307692)$. The time value of this point is the same moment when the topology changes, which was detected using the first method. The shape at this moment is also shown in Figure 3. Using the second method, the critical point is detected without the complex transformations of the metamorphosis expression.

Next, we consider a more complex example, shape metamorphosis from two disks in general positions.
(Figure 4a) to a ring (Figure 4h). In this case, there should be more than one time moment of the topology changes. In addition, in the both methods, the initial shape is defined by using R-function for union. One disk is defined as $\text{disk}1[x, y]$, and another as $\text{disk}2[x, y]$.

$$\text{disk}1[x, y] = s1^2 - (x - 2)^2 - (y - 4)^2$$
$$\text{disk}2[x, y] = s2^2 - x^2 - (y + 2)^2$$

, where $s1 = s2 = 2$.

Then, the initial shape of two disks union is exactly defined by the R-function as below.

$$f_{\text{ini}}[x, y] = \text{disk}1[x, y] + \text{disk}2[x, y]$$
$$+ \sqrt{(\text{disk}1[x, y])^2 + (\text{disk}2[x, y])^2}$$
$$= -2(8 + x \cdot (x - 2) + y \cdot (y - 2))$$
$$+ \sqrt{(x \cdot (x - 4) + (y - 4)^2)^2 + (x^2 + y \cdot (y + 4))^2}$$

Similarly to the previous example, we apply both methods of critical points detection and we detect the same results. In the first method, after excluding irrelevant solutions the result is as follows:

$$(x, y) = [(0.016682, -5.246709),$$
$$(4.272867, 8.472384),$$
$$(2.004681, 0.527721),$$
$$(0.010182, -2.303490),$$
$$(-1.476238, 1.375145)]$$

Assigning the critical points coordinates to the height function results in two valid time moments $t_1=0.211949$ and $t_2=0.309162$.

Applying the second method, we detect the critical points in the particular level of the height function $h_2$. After the exclusion of irrelevant points, the result of the second method is as follows:

$$(x, y, t) = [(2.004681, 0.527721, 0.309162),$$
$$(-1.476238, 1.375145, 0.211949)]$$

In the second method, the detected results are also $t_1=0.211949$ and $t_2=0.309162$. Note that exclusion of irrelevant item is performed only one time in the second method. On the other hand, in the first method, the exclusion is performed two times.

Figure 4: The figures from (a) to (h) are the sequence of shape metamorphosis from two disks to a ring, (d) and (f) correspond to two detected critical points.
Figure 4 shows the key frames of the metamorphosis, which have been created using the detected critical time moments of the topology changes. The shape in (a) is the initial shape. The next shapes (b) and (c) are the shapes at the time moments 0.05 and 0.16 respectively. The shapes from (a) to (c) have the same topology type. The shape of (d) is the special shape at the moment \( t_1 = 0.211949 \) when the topology changes for the first time. Next, the shape (e) is shown at the time 0.26 with the same topology type with (d). The shape (f) is the special shape at the moment \( t_2 = 0.309162 \) when the topology changes for the second time. The shape (g) is shown at the time 0.35 and the shape (h) is the target shape. The shapes (f), (g), and (h) have the same topology types.

4.2. Example in 3D

In this section, we consider the shape metamorphosis from a sphere to a torus in 3D space. In the shape metamorphosis, the initial shape and the target shape are defined by \( \text{ini} \), \( \text{tar} \) as follows (see Figure 5):

\[
\begin{align*}
 f_{\text{ini}}[x, y, z] &= s^2 - x^2 - y^2 - z^2, \\
 f_{\text{tar}}[x, y, z] &= x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 \\
 &+ 2y^2z^2 - 2(R^2 + r^2)x^2 + 2(R^2 - r^2)y^2 \\
 &- 2(R^2 + r^2)z^2 + (R^2 - r^2)^2,
\end{align*}
\]

where \( s = 2, R=2, r = 1 \).

The metamorphosis is defined by the homotopy function:

\[
\begin{align*}
 f[x, y, z, t] &= f_{\text{ini}}[x, y, z] \cdot (1-t) + f_{\text{tar}}[x, y, z] \cdot t \\
 &= (1-t) \cdot (4-x^2-y^2-z^2) \\
 &+ t \cdot (9-10x^2+x^4+6y^2+2x^2y^2+y^4-10z^2 \\
 &+ 2x^2z^2+2y^2z^2+z^4),
\end{align*}
\]

Next, the height function \( h_3 \) is created to detect the critical point of the height function by the first method:

\[
\begin{align*}
 t = h_3 &= \frac{f_{\text{ini}}[x, y, z]}{f_{\text{ini}}[x, y, z] - f_{\text{tar}}[x, y, z]} \\
 &= (4-x^2-y^2-z^2)/(5-9x^2+x^4+7y^2 \\
 &+ 2x^2y^2+y^4-9z^2+2x^2z^2+2y^2z^2+z^4)
\end{align*}
\]

The critical points should satisfy the vanishing gradient conditions as in the 2D examples above:

\[
\begin{align*}
 \frac{\partial}{\partial x} h_3 &= \frac{\partial}{\partial y} h_3 &= \frac{\partial}{\partial z} h_3 = 0.
\end{align*}
\]

After the all critical points are detected, the exclusion of irrelevant points is performed. Next, the assignment of the suitable point coordinates to the height function is made and thus the corresponding time moments are detected.

In the second method, since the height coordinate and the time coordinate are added to the function, the height function defines a hypersurface in 5D space as shown below:

\[
\begin{align*}
 h_4 &= f[x, y, z, t] \\
 &= (1-t) \cdot (4-x^2-y^2-z^2) \\
 &+ t \cdot (9-10x^2+x^4+6y^2+2x^2y^2+y^4-10z^2 \\
 &+ 2x^2z^2+2y^2z^2+z^4)
\end{align*}
\]

Then, the moments of the topology change are detected at the critical points in the particular level \( h_4 \). The critical points satisfy the following conditions:

\[
\begin{align*}
 \frac{\partial}{\partial x} h_4 &= \frac{\partial}{\partial y} h_4 &= \frac{\partial}{\partial z} h_4 = 0 \quad \text{and} \\
 f[x, y, z, t] &= 0.
\end{align*}
\]

In this example, we detect the same result using both methods: \( t=0.034482 \). The corresponding intermediate shape of the metamorphosis is shown in Figure 6.
Figure 6: Intermediate shape of 3D metamorphosis at the detected critical time moment of the hole being poked in the deformed sphere.

Now, we consider a more complex example, the shape metamorphosis from a torus to union of two spheres in general positions. The initial shape and the target shape are defined by $f_{ini}[x, y, z]$ and $f_{tar}[x, y, z]$ as follows:

$$f_{ini}[x, y, z] = 9 - 10x^2 + x^4 + 6y^2 + 2x^3y^2 + y^4 - 10z^2 + 2x^2z^2 + 2y^2z^2 + z^4$$

$$f_{tar}[x, y, z] = 8 - (x - 3)^2 - x^2 - 2y^2 - (z - 3)^2 - (z + 3)^2 + \sqrt{(4 - x^2 - y^2 - (z - 3)^2)^2 + (4 - (x - 3)^2 - y^2 - (z + 3)^2)^2}$$

Similarly to the previous example, we apply both methods of critical points detection and we detect different results. In the first method, assigning the critical points coordinates to the height function results in three valid time moments $t_1=0.828221$ and $t_2=0.994141$, $t_3=0.997014$. However, using the second method, we detect the same results and two additional points. The results of the second method are as follows:

$$(x, y, t) = \{(0, 0, 1.000000, 0.799999), (0, 0, 1.000000, 0.799999), (0.007179, 0.1.825826, 0.828221), (0.050425, 0, 6.977086, 0.994141), (6.483631, 0, -6.392539, 0.997014)\}$$

This means the two events of topology evolution happen at a short moment $t_1=0.799999$. At this time moment detected by the second method, the topology changes twice at the short period of time. The changes are described from (b) to (d) in Figure 7.

Figure 7: The figures from (a) to (h) are the sequence of shape metamorphosis from a torus to two spheres, (c) and (e), (f), (g) correspond to the detected critical points.

Figure 7 shows the key frames of the metamorphosis, which have been created using the detected critical time moments of the topology changes. The shape in (a) is the initial shape. The shapes (b) and (d) are the shapes at the time moments 0.60 and 0.82 respectively. The shape of (c) is the special shape at the moment $t_1=0.799999$ when the topology changes for the first and second time. The
shape (e) is the shape at the moment \( t_2 = 0.828221 \) when the topology changes for the third time. Next, the shape (f) and (g) is shown at the time \( t_3 = 0.994141 \) and \( t_4 = 0.997014 \). The shape (h) is the target shape. According to Figure 7, the topology duly changes at the moment \( t_1 \). In the first method, the critical points depend on the height. Therefore, it is difficult for the first method to detect the moments when the topology changes during short time periods. For solving this problem, higher-precision analysis based on typical Morse theory is needed.

Note that the first method takes too much time for calculation. For the last example, the first method takes 3490.29 seconds for detecting the critical points of height function and the second method takes 9.94 seconds for detecting the moments that the topology changes in the particular height level. The proposed second method can perform several times faster than the first method. For the reference, in this testing, the spec of CPU and memory are as follows:

- **CPU**: AMD Athlon(tm) XP 1800+, 1.5 GHz
- **Memory**: 512 MB RAM

### 5. Discussions and Conclusions

Critical point detection is useful in metamorphosis animation for the selection of key frames. For example, this can provide ease-in/ease-out effects. In this work, we proposed a method of detection of the moments of the topology changes in the particular level of the defining F-rep function. This method is simpler than the method based on typical Morse theory analysis. The reason is that the complicated transformations of the expressions for representing the time as a height function are not necessary in this method. Our method simply uses the F-rep of the metamorphosis as a time-dependent height function and we analyze the moments of the topology changes of its zero level along the time axis. The latter method can be performed in much faster than the method based on typical Morse theory analysis. We detected the valid results by both methods in our 2D and 3D case studies.

In this paper, we used the linear homotopy function for metamorphosis. The proposed method can also be used for the analysis of non-linear shape metamorphosis. The difference is only to use some non-linear function for homotopy description for the shape metamorphosis. This will be one of the subjects of our future work.

Future work will also include more detailed comparison of applicability of discussed methods for different metamorphosis models, development of numerical methods for analysis of complex metamorphosis, and comparison with other topology analysis methods like one presented in [8] where the Reeb graphs are height function independent.

### References