Upper Bound on the Error Probability for Space-Time Codes in Fast Fading Channels

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Abstract—We present exact union bounds on the error probability of space-time codes in fast Rayleigh and Rician fading channels. Using Craig’s formula for the Gaussian Q-function, an exact pairwise error probability (PEP) of space-time codes is expressed in the form of a single finite-range integral whose integrand involves in the moment generating function (MGF) of a nonnegative random variable. Based on this expression, the exact union bounds for a geometrically uniform space-time trellis code (STTC) over fast fading channels can be obtained by using the transfer function bounding technique.

I. INTRODUCTION

Space-time coding scheme, proposed by Tarokh et al. [1], combines channel coding with antenna diversity and modulation to achieve reliable high-rate data transmission in wireless communications. Error performance of a geometrically uniform STTC over rapid fading channels can be evaluated using the transfer function, derived from its error state diagram, and the Chernoff bound applied to the PEP [2]. Although this union-Chernoff bound allows to considering all possible error events and gives a final expression for the average error probability in terms of the transfer function, it is too loose for most signal-to-noise ratio (SNR) ranges due to the use of Chernoff bounding. In [3], a tight upper bound on the PEP over Rayleigh fading channels was derived and was applied to the transfer function in order to evaluate the union bound on the bit error probability. However, this analysis is restricted to only Rayleigh fading and the PEP is still upper-bounded, though the bound is tighter than the Chernoff bound.

In [4], the exact PEP of space-time codes over rapid Rayleigh fading channels was derived using the method of residues and also in [5], a numerical technique with any desired degree of accuracy for calculating the PEP was presented using the Gauss-Chebyshev quadrature and the MGF approach. Since the final expressions for the PEP of [4] and [5] are not in a product form, it is impossible to combine the exact PEP with the transfer function of a STTC in order to evaluate the union bound. Therefore, instead of using transfer function approach, an estimate of the error probability was obtained by taking into account dominant error event paths of lengths up to a pre-determined specific value. In [6], an efficient distance spectrum computation method for STTCs was proposed using a state reduction technique and expurgated union bounds on the error probability in quasi-static fading channels were also derived.

In this paper, we derive an exact PEP of space-time codes over fast fading channels by using Craig’s expression for the Gaussian probability integral \( Q(x) \), which is an integral form with the finite limits independent of the argument \( x \) and the exponential integrand [8], [9]. The derived PEP is both exact and applicable to the transfer function bounding technique. Our results essentially are the generalization of the technique for accurate evaluation of the error performance of trellis-coded modulation (TCM) over fading channels in [9] to the case of multiple transmit and receive antennas.

II. SYSTEM MODEL

We consider a wireless communication system with \( n_t \) transmit antennas and \( n_r \) receive antennas. It is assumed that the channel coefficients between any pair of antennas are independent and identically distributed (i.i.d.) and perfectly known to the receiver. The signal received by the \( j \)th antenna at time \( t \) is given by

\[
\mathbf{r}_j(t) = \sum_{i=1}^{n_t} h_{ij}(t) \sqrt{E_s} c_i(t) + n_i(t)
\]

where \( h_{ij}(t) \) is a complex fading coefficient from the \( i \)th transmit antenna to the \( j \)th receive antenna, which is modeled as a sample of the complex Gaussian random variable with normalized mean square value, i.e., \( E[|h_{ij}(t)|^2] = 1 \), \( E_s \) is the energy per symbol, and \( n_i(t) \) is a zero-mean complex Gaussian noise with variance \( N_0/2 \) per dimension. The space-time symbol at time \( t \) is defined as \( \mathbf{c}_t = (c_i^{(0)}, c_i^{(1)}, \ldots, c_i^{(n_T)})^T \), where the superscript \( T \) denotes the transpose.

We consider the fading coefficients may be modeled as
frequency-flat fast Rician fading, that is, \( h_{i,j}^{t} \sim N_{c}(\mu, \sigma^{2}) \),
and the \( n_x \times n_z \) channel matrix at time \( t \), denoted by \( H_i \), is defined as
\[
H_i = \begin{pmatrix}
h_{i,1}^{(1)} & h_{i,2}^{(1)} & \cdots & h_{i,n_z}^{(1)} \\
h_{i,1}^{(2)} & h_{i,2}^{(2)} & \cdots & h_{i,n_z}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
h_{i,1}^{(n_x)} & h_{i,2}^{(n_x)} & \cdots & h_{i,n_z}^{(n_x)}
\end{pmatrix}.
\]  
(2)

For fast fading, the fading coefficients \( h_{i,j}^{t} \) are assumed to remain constant during a symbol and vary independently from symbol to symbol. The Rician fading can be characterized by a Rician factor \( K = \mu^2/\sigma^2 \) which is the ratio of specular to diffuse energy. With normalized mean square value \( E[h_{i,j}^{t}]^2 = \mu^2 + \sigma^2 = 1 \), the mean and variance of \( h_{i,j}^{t} \) are \( \mu = \sqrt{K/(K+1)} \) and \( \sigma^2 = 1/(K+1) \), respectively.

### III. PAIRWISE ERROR PROBABILITY

Assuming that a space-time codeword \( C = (c_1, c_2, \ldots, c_L) \), where \( L \) is the codeword length, was transmitted, the conditional pairwise error probability, i.e., given the fading matrix sequence \( \{H_i\}_{i=1}^{L} \), the probability that the maximum likelihood receiver decides \( E = (e_1, e_2, \ldots, e_L) \), where \( e_t = (e_t^{(1)}, e_t^{(2)}, \ldots, e_t^{(n_z)})^T \), \( t = 1, 2, \ldots, L \), rather than \( C \) when only these two codewords are possible is given by
\[
P(C \rightarrow E | H_i, t = 1, \ldots, L) = Q\left(\frac{E_t}{\sqrt{2N_0} \sum_{i=1}^{L} \|H_i(c_i - e_i)\|^2}\right).
\]  
(3)

Define \( x_i = H_i(c_i - e_i) \) and \( z_i = x_i^H I_{n_z} x_i \) for \( t = 1, 2, \ldots, L \), where \( z_i \) is a noncentral quadratic form in complex Gaussian random variables and \( I_{n_z} \) and the superscript \( H \) denote an \( n_x \times n_z \) identity matrix and the transpose conjugate, respectively. We can rewrite (3) as
\[
P(C \rightarrow E | z_i, t = 1, \ldots, L) = Q\left(\frac{E_t}{\sqrt{2N_0} \sum_{i=1}^{L} z_i}\right)
\]  
(4)

and in order to evaluate the average pairwise error probability, we average (4) with respect to distributions of all \( z_i \). By using Craig’s formula for \( Q(x) \) [8], [9]
\[
Q(x) = \frac{1}{\pi} e^{x^2/2} \exp\left(-\frac{x^2}{2\sin^2 \theta}\right) d\theta
\]  
(5)

and the moment generating function (MGF) of \( z_i \), which is defined as a Laplace transform on its probability density function (pdf), the average pairwise error probability can be written as
\[
P(C \rightarrow E) = \frac{1}{\pi} e^{\eta^2/2} \prod_{i=0}^{\infty} \phi_{z_t} \left(\frac{E_t}{4N_0 \sin^2 \theta}\right) d\theta
\]  
(6)

where \( \eta \) is the set of all \( t \) for which \( e_t \neq e_i \) and the MGF of \( z_t \) is defined as
\[
\phi_{z_t}(s) = E\left[e^{-s z_t}\right] = \int_{0}^{\infty} \exp(-sz_t) p_{z_t}(z_t) dz_t.
\]  
(7)

The MGF (or the characteristic function) of a quadratic form in complex Gaussian variables is reported in [10] and (7) is easily written as
\[
\phi_{z_t}(s) = \left(\frac{K+1}{K+1+s\|e_t - e_i\|^2}\right)^{n_z}.
\]  
(8)

### IV. UNION BOUNDS AND NUMERICAL RESULTS

#### A. Rayleigh Fading

For a geometrically uniform STTC over Rayleigh fading channels \( (K = 0) \), the error probability does not depend on the transmitted codeword [1], [12]. Thus, we can use the classical transfer function bounding technique to evaluate the union bound. Since the integrand of (6) is in a product form, the Chernoff bound on the PEP required to apply the transfer function bound can be eliminated [9] and it is possible to combine the exact PEP of (6) with the transfer function in order to obtain the union bound for a STTC.

Let \( T(D(\theta), I) \) be the transfer function of the error state diagram of a STTC with \( 2^L \) states whose branch label associated with the transition from the state \( S_a \) to the state \( S_b \), where \( a, b \in \{0, 1, 2, \ldots, 2^{L-1}\} \), has the factor
\[
D_{ab}(\theta) = \begin{cases} 
I^{w_{a,b}} \cdot \phi_{z_t} \left(\frac{E_t}{4N_0 \sin^2 \theta}\right) & \text{if the transition } S_a \rightarrow S_b \text{ exists,} \\
0 & \text{otherwise,}
\end{cases}
\]  
(9)

where \( w_{a,b} \) is the Hamming weight of input bits associated
with this transition. The transfer function $T(D(\theta), I)$ can be determined by solving the usual nodal equations of the graph [7], given by (10).

Using the results of [9] and [11], the union bounds on the average bit and frame error probabilities are given by, respectively,

$$P_b \leq \frac{1}{\pi} \int_0^{\pi/2} \left. \frac{1}{m} \frac{\partial T(D(\theta), I)}{\partial I} \right|_{I=1} d\theta$$

(12)

and

$$P_f \leq 1 - \frac{1}{\pi} \int_0^{\pi/2} T(D(\theta), I) \left|_{I=1} \right. d\theta$$

(13)

where $m = \log_2 M$ is the number of bits per an $M$-PSK symbol. Note that the union-Chernoff bounds can be obtained by simply setting $\theta = \pi/2$.

Figs. 1 and 2 show the comparison of exact union bounds and union-Chernoff bounds on the bit and frame error probabilities with simulation results for 4-state and 16-state QPSK STTCs, presented by Tarokh et al. [1], with two transmit and two receive antennas in Rayleigh fading channels, respectively. In simulations, each frame consists of 130 symbols per transmit antenna. All figures are plotted against average symbol SNR per receive antenna, defined as $\text{SNR} = (E_s/N_0) \cdot \sum_{l=1}^{m} \{E[|h(l,i)|^2]\} = n_r E_s/N_0$. We observe that the exact union bounds are quite accurate, compared to the simulation results, at the error rate of interest, while the union-Chernoff bounds are rather loose for all range of SNRs.

**B. Rician Fading**

From (6) and (8), we see that the PEP of space-time codes over Rician fading channels depends on $\rho$, as well as the squared Euclidean distance between two space-time symbols, $\|e - e\|^2$. Therefore, the error probability of space-time codes over Rician fading channels depends on the transmitted codeword, even though the codes are geometrically uniform. In this case, the union bound on the error probability is evaluated as the linear combination of all possible PEPs, i.e.,

$$P_f \leq \sum_c \sum_{l=1}^{m} \frac{1}{i} \frac{\partial \rho^2}{\partial I} \prod_{n=1}^C \left( \left. \frac{E_s}{4N_0 \sin^2 \theta} \right|_{\theta=\pi} \right) d\theta$$

(14)

To show the results of our method in Rician fading channels, we evaluated the error probability of the binary (8,4,4) Reed-Muller code with BPSK, used in [5], over fast fading channels with two transmit and two receive antennas. In Fig. 3, the upper bounds on frame error probability are compared with the simulation results for $K = 0$ (Rayleigh), $K = 5$dB, and $K = 15$dB. Observing (6) and (8), we can see that the error probability of space-time codes over Rician fading channels does not always decrease as $K$ increases.
Fig. 2. Upper bounds on the error probability and simulation results for Tarokh’s 16-state QPSK STTC over fast Rayleigh fading channels. $n_r = 2$, $n_t = 2$, and $I = 130$.

Fig. 3 shows this case. In fact, an important parameter determining coding gain of space-time codes over fast Rician fading channels is the distribution of $\sum_{n=1}^{\infty} \rho_n \frac{\|\mathbf{e}_i - \mathbf{e}_j\|}{\|\mathbf{e}_i\|}$ rather than that of $\prod_{n=1}^{\infty} \|\mathbf{e}_i - \mathbf{e}_j\|$, unlike fast Rayleigh fading channels.

V. CONCLUSIONS

The exact PEP of space-time codes over fast Rayleigh and Rician fading channels are derived using Craig’s formula for the Gaussian Q-function. It has a form applicable to the transfer function bounding technique to evaluate the exact union bound on the error probability of a geometrically uniform STTC. Based on the observation of the exact PEP, it is found that the error probability of space-time codes over Rician fading channels depends on the transmitted codeword, even though the codes are geometrically uniform. Simulation results show that the exact union bounds are quite accurate at the error rate interesting for applications.

REFERENCES


