A Practical Approach to the Error Estimation of Quasi-Monte Carlo Integrations

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Abstract

There have been few studies on practical error estimation methods of quasi-Monte Carlo integrations. Recently, some theoretical works were developed by Owen to analyze the quasi-Monte Carlo integration error. However his method given by those works is complicated to be implemented and needs huge computational efforts, so it would be of some interest to investigate into a simple error estimation method. In this paper, we will use a simple method, and give some theoretical considerations on the errors given by these two methods. Numerical experiments are also reported.

1 Introduction

We consider the error estimation problem of quasi-Monte Carlo(QMC) integrations of the integral

$$I = \int_{[0,1)^s} f(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{1}$$

For small s and smooth f there are many numerical integration methods and their error estimation methods. However, as s increases, the problem becomes more difficult. Monte Carlo(MC) method is frequently used for this problem. But it is well known that the statistical error, i.e. the standard deviation, of MC integration is $\mathrm{O}(1/\sqrt{N})$, where N is the number of the evaluations of the integrand, so that it is extremely time-consuming to obtain an accurate result. Recent years have seen successful applications of QMC method which uses low-discrepancy point set or sequence $\{\mathbf{x}_i\} \in [0,1)^s$ and computes the approximate value of (1) by

$$I_N = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i). \tag{2}$$

For the error of QMC method, we have the following Koksma-Hlawka inequality.

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) - \int_{[0,1)^s} f(\mathbf{x}) \mathrm{d}\mathbf{x} \right| \leq V(f) D_N^*,$$

where $V(f)$ is the total variation of f in the sense of Hardy and Klause, and D_N^* is the star discrepancy of the point set $\{\mathbf{x}_i\}$. Koksma-Hlawka inequality is a basis of the superiority of QMC method to MC, because if we use low-discrepancy point sequence, D_N^* (and also the absolute error of integral) goes to 0 with the rate $\mathrm{O}((\log N)^s/N)$ asymptotically as $N \to \infty$. However we cannot make an error estimation with Koksma-Hlawka inequality, because it is usually impossible to calculate the total variation $V(f)$. This is why we need a practical error estimation for QMC integrations.

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Here by the term "practical" we mean the method is accurate, fast, and easy to implement. Recently several works on error analysis of QMC integrations have been developed [2], [5], [8]. These approaches apply statistical error estimation methods to QMC integrations. Since low-discrepancy point sets are deterministic point sets, in order to do a statistical error estimation, point sets must be selected from some probability space. We need a probabilistic structure on the point sets.

In this paper we select two methods and compare their efficiencies. The first method uses scrambled nets, which were proposed by Owen [6]. The second one uses randomly shifted nets, which are based on the idea of Cranley and Patterson [1] for good lattice points methods. We use (t, m, s)-net as a low-discrepancy point set. Let us recall the definition of (t, m, s)-net [4]. A subset E of $[0,1)^s$ of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$$

with $a_i, d_i \in \mathrm{Z}$, $d_i \geq 0$, $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an *elementary interval in base b.*

Definition 1 *Let t and m be nonnegative integers and $t \leq m$. A (t, m, s)-net in base b is a set of b^m points in $[0,1)^s$ such that every elementary interval of volume b^{t-m} contains exactly b^t points of the point set.*

In the following, statistical error estimation methods are explained in Sect. 2. In Sect. 3 an analysis on the estimated error is given for one-dimensional case. An analysis for multidimensional case is developed in Sect. 4. Some numerical examples are presented in Sect. 5.

2 Statistical Error Estimation

We introduce two statistical error estimation methods for numerical integrations using (t, m, s)-net. The basic idea of these methods is a combination of MC and QMC. The general scheme of the methods is as follows. We select point sets $\{\mathbf{x}_i^{(j)}\}_{i=1}^N$, $j = 1, \ldots, M$, independently from a probability space, and compute the value of (2) for each point set.

$$S^{(j)} = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i^{(j)}), \quad j = 1, \ldots, M.$$

Then we calculate the estimate of I by

$$\hat{I} = \frac{1}{M} \sum_{j=1}^{M} S^{(j)}.$$

The error of the numerical integration is estimated using the variance of the evaluated values.

$$\hat{\sigma}^2 = \frac{1}{M(M-1)} \sum_{j=1}^{M} (S^{(j)} - \hat{I})^2.$$

In order to make a statistical statement about an error estimation we need a probabilistic structure on the point set. We will consider the following two structures in this paper.

Scrambled Net [6]. Let $\{\mathbf{z}_i\}$ be a (t, m, s)-net in base b. Suppose $\mathbf{z}_i = (z_i^1, \ldots, z_i^s)$ and $z_i^j = \sum_{k=1}^{\infty} z_{ijk} b^{-k}$ for integers $0 \leq z_{ijk} < b$. A scrambled net $\{\mathbf{x}_i\}$, $\mathbf{x}_i = (x_i^1, \ldots, x_i^s)$ is defined as $x_i^j = \sum_{k=1}^{\infty} x_{ijk} b^{-k}$, where x_{ijk} is a random permutation applied to z_{ijk}. Specifically x_{ijk} are determined as follows.

$$\begin{aligned} x_{ij1} &= \pi_j(z_{ij1}), \\ x_{ij2} &= \pi_{j z_{ij1}}(z_{ij2}), \\ &\vdots \\ x_{ijk} &= \pi_{j z_{ij1} z_{ij2} \ldots z_{ij,k-1}}(z_{ijk}). \end{aligned}$$

Here each π is a random permutation over $\{0, 1, \ldots, b-1\}$. In the second line the subscript z_{ij1} means that the permutation depends on the value of z_{ij1}. In the same way $\pi_{jz_{ij1}z_{ij2}\ldots z_{ij,k-1}}$ is a permutation depending on the values of $z_{ij1}, \ldots, z_{ij,k-1}$. All permutations are mutually independent. A scrambled net thus derived is also a (t, m, s)-net in base b.

This operation can be viewed as follows. For a given point set, divide equally each axis of the unit cube into b parts, and permute these parts in random order. Next, similarly, divide each part on each axis into b small parts equally, and permute these small parts in random order. The scrambled net is obtained by iterating this procedure. For the details of the scrambled net, the reader is referred to Owen [6], [7].

Randomly Shifted Net [1]. Let $\{\mathbf{z}_i\}$ be a (t, m, s)-net in base b and $\mathbf{u}$ be a random vector uniformly distributed over a unit cube. A randomly shifted net $\{\mathbf{x}_i\}$ is given by

$$\mathbf{x}_i = \mathbf{z}_i + \mathbf{u} \pmod{\mathbf{1}},$$

where (mod $\mathbf{1}$) means the componentwise (mod 1) operation.

Here we give an interpretation on the methods for our investigation below. Both methods transform the original net into another point set by a bijection

$$\tau(\mathbf{z}_i) = \mathbf{x}_i.$$

We consider τ is randomly chosen from among all the bijections, instead of giving a probabilistic structure to $\{\mathbf{x}_i\}$. We denote by $\tau_{\mathrm{s}}^{(l)}$ the scrambling to the l-th digit in base b. The detailed definition of $\tau_{\mathrm{s}}^{(l)}$ is as follows.

$$\tau_{\mathrm{s}}^{(l)}(\mathbf{z}) = (\pi_1^{(l)}(z^1), \ldots, \pi_s^{(l)}(z^s)),$$

where $\mathbf{z} = (z^1, \ldots, z^s)$ and the permutation $\pi_j^{(l)}$, $1 \le j \le s$, for each element is defined as

$$\begin{aligned} \pi_j^{(l)} &= \pi_{jl} \circ \pi_{j,l-1} \circ \cdots \circ \pi_{j1}, \\ \pi_{jk} &= \pi_{jk1} \circ \pi_{jk2} \circ \cdots \circ \pi_{jk,b^{k-1}}. \end{aligned}$$

Here "$\circ$" means the composition of maps. π_{j1} is a random permutation on the equally divided b parts of unit cube along the j-th axis. π_{j2} is composed of b random permutations $\pi_{j21}, \ldots, \pi_{j2,b}$ on b small parts obtained by dividing each of b parts. See Fig. 1. The permutations π_{jk}, $k = 3, \ldots, l$, are defined similarly. For a randomly shifted net, the transformation τ_{t} is given by

$$\begin{aligned} \tau_{\mathrm{t}}(\mathbf{z}) &= \mathbf{z} + \mathbf{u} \pmod{\mathbf{1}} \\ &= (z^1 + u^1 (\mathrm{mod}\ 1), \ldots, z^s + u^s (\mathrm{mod}\ 1)), \end{aligned}$$

where $\mathbf{u} = (u^1, \ldots, u^s)$ is a random vector over a unit cube.

3 Error Analysis in One-Dimensional Case

The basic tools of our investigation are the b-adic Haar functions used by Owen [6]. Define the functions $\psi_c(x)$, $c = 0, 1, \ldots, b-1$, over $[0, 1)$ as

$$\psi_c(x) = \begin{cases} \sqrt{b} - \dfrac{1}{\sqrt{b}}, & \dfrac{c}{b} \le x < \dfrac{c+1}{b}, \\ -\dfrac{1}{\sqrt{b}}, & \text{otherwise.} \end{cases}$$

The b-adic Haar functions ψ_{ktc} are defined as

$$\psi_{ktc}(x) = b^{\frac{k-1}{2}} \psi_c(b^{k-1}x - t)$$

for integers $k > 0$, $0 \le t < b^{k-1}$ (This definition is slightly different from Owen's). Some useful properties of b-adic Haar functions are given below. We can show them by simple calculations.

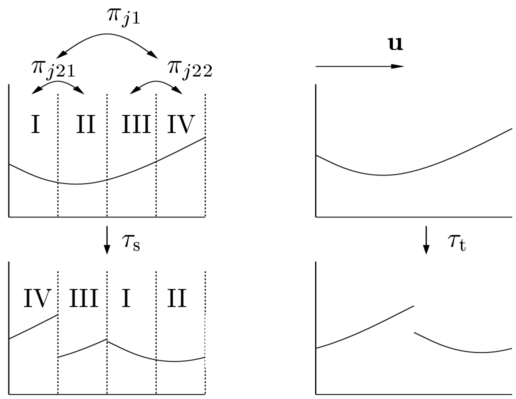

Figure 1: Scrambling and random shifting.

Proposition 1 *Let ψ_{ktc} be b-adic Haar functions.*

1. *For arbitrary $k > 0$ and $0 \le t < b^{k-1}$, the next relation holds.*

$$\sum_{c=0}^{b-1} \psi_{ktc}(x) = 0. \tag{3}$$

2. *For a $(t, m, 1)$-net $\{z_i\}$, if $0 < k \le m - t$ we have*

$$\sum_{i=1}^{b^m} \psi_{ktc}(z_i) = 0. \tag{4}$$

3.1 Mean Square Error

For the convenience of notation we set $\psi_0(x) = 1$, $0 \le x < 1$. The function set $\{\psi_{ktc}\}_{k=0}^{\infty}$ is a basis in $L^2[0,1)$, that is, any square integrable function f can be expanded by $\{\psi_{ktc}\}_{k=0}^{\infty}$.

$$f(x) = \sum_{k=0}^{\infty} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle \psi_{ktc}(x),$$

where $\langle , \rangle$ denotes the inner product defined as

$$\langle f, \psi_{ktc} \rangle = \int_0^1 f(x)\psi_{ktc}(x) \mathrm{d}x.$$

Note that $I = \langle f, \psi_0 \rangle$. Although the set $\{\psi_{ktc}\}$ is not an orthogonal system, the coefficients of expansion are given by inner products. For $k > 0$ we define

$$Q_k f(x) = \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle \psi_{ktc}(x).$$

Proposition 2 *For $\forall f(x) \in L^2[0,1)$ and each $k > 0$ we have*

$$\int_0^1 (Q_k f(x))^2 \mathrm{d}x = \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2.$$

Proof. We proceed by direct calculations.

$$\int_0^1 (Q_k f(x))^2 \mathrm{d}x = \sum_{t=0}^{b^{k-1}-1} \left(\sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2 \int_0^1 \psi_{ktc}^2(x) \mathrm{d}x + \sum_{c_1 \neq c_2} \langle f, \psi_{ktc_1} \rangle \langle f, \psi_{ktc_2} \rangle \int_0^1 \psi_{ktc_1}(x) \psi_{ktc_2}(x) \mathrm{d}x \right).$$

Using the fact that $\int_0^1 \psi_{ktc}^2(x)\mathrm{d}x = (b-1)/b$ and that $\int_0^1 \psi_{ktc_1}(x)\psi_{ktc_2}(x)\mathrm{d}x = -1/b$ for $c_1 \neq c_2$, together with (3) we have

$$\begin{aligned}
\int_0^1 (Q_k f(x))^2 \mathrm{d}x &= \sum_{t=0}^{b^{k-1}-1} \left(\frac{b-1}{b} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2 - \frac{1}{b} \sum_{c_1 \neq c_2} \langle f, \psi_{ktc_1} \rangle \langle f, \psi_{ktc_2} \rangle \right) \\
&= \sum_{t=0}^{b^{k-1}-1} \left(\frac{b-1}{b} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2 - \frac{1}{b} \left(- \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2 \right) \right) \\
&= \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2.
\end{aligned}$$

■

Under the interpretation given in Sect. 2 that a transformation is chosen randomly, the variance of the numerical integration by scrambled or randomly shifted net can be represented as follows.

$$V(I_N) = E\left[(I_N - I)^2\right] = E\left[\left(\frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{\infty} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f\tau, \psi_{ktc} \rangle \psi_{ktc}(z_i) \right)^2 \right]$$

$$= \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{t_1=0}^{b^{k_1-1}-1} \sum_{t_2=0}^{b^{k_2-1}-1} \sum_{c_1=0}^{b-1} \sum_{c_2=0}^{b-1} E[\langle f\tau, \psi_{k_1t_1c_1} \rangle \langle f\tau, \psi_{k_2t_2c_2} \rangle] \psi_{k_1t_1c_1}(z_i) \psi_{k_2t_2c_2}(z_j), \quad (5)$$

where we used an expression $f\tau$ for the composition of the functions f and τ. The difference between scrambling method and randomly shifting method is stated below.

Lemma 1 *If* $\mathrm{supp}\psi_{k_1t_1c_1} \cap \mathrm{supp}\psi_{k_2t_2c_2} = \emptyset$ *and* $l \geq \min(k_1, k_2)$, *or if* $\mathrm{supp}\psi_{k_1t_1c_1} \cap \mathrm{supp}\psi_{k_2t_2c_2} \neq \emptyset$ *and* $\mathrm{supp}\psi_{k_1t_1c_1} \neq \mathrm{supp}\psi_{k_2t_2c_2}$ *and* $l \geq \max(k_1, k_2)$, *then we have* $E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_1t_1c_1} \rangle \langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_2t_2c_2} \rangle\right] = 0.$

Proof. We set $k_1 \geq k_2$ without loss of generality. If the supports of $\psi_{k_1t_1c_1}$ and $\psi_{k_2t_2c_2}$ do not overlap, the permutations on the supports are mutually independent. Hence we have

$$E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_1t_1c_1} \rangle \langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_2t_2c_2} \rangle\right] = E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_1t_1c_1} \rangle\right] E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_2t_2c_2} \rangle\right].$$

If $l \geq k_2$, the last expectation is

$$E[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_2t_2c_2} \rangle] = \frac{1}{b} \sum_{c=0}^{b-1} \int_0^1 f(x) \psi_{k_2t_2c}(x) \mathrm{d}x = 0. \quad (6)$$

This yields the first part of the lemma.

For one-dimensional case the latter condition of the lemma means that $\mathrm{supp}\psi_{k_1t_1c_1}$ is a proper subset of $\mathrm{supp}\psi_{k_2t_2c_2}$. The transformation $\tau_{\mathrm{s}}^{(l)}$ has the form $\pi_{1l} \cdots \pi_{1k_1} \cdots \pi_{1k_2} \cdots \pi_{11}$ when $l \geq k_1$. Conditioning on $\pi_{1k_2} \cdots \pi_{11}$ gives

$$E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_1t_1c_1} \rangle \langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_2t_2c_2} \rangle\right] = E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_2t_2c_2} \rangle E\left[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{k_1t_1c_1} \rangle | \pi_{1k_2} \cdots \pi_{11}\right]\right].$$

If $l \geq k_1$, the conditional expectation $E\left[\langle f\tau_{\rm s}^{(l)}, \psi_{k_1t_1c_1}\rangle | \pi_{1k_2}\cdots\pi_{11}\right]$ becomes zero the same as (6). ■

From the lemma, when we use a scrambled net to the l-th digit in base b, the coefficients in the expansion of the variance (5) with $\min(k_1, k_2) \leq l$ vanish if $k_1 \neq k_2$, or $k_1 = k_2$ and $t_1 \neq t_2$. Also the coefficients with $\max(k_1, k_2) \leq l$ vanish if $\mathrm{supp}\psi_{k_1t_1c_1} \subset \mathrm{supp}\psi_{k_2t_2c_2}$ and $\mathrm{supp}\psi_{k_1t_1c_1} \neq \mathrm{supp}\psi_{k_2t_2c_2}$, or if $\mathrm{supp}\psi_{k_2t_2c_2} \subset \mathrm{supp}\psi_{k_1t_1c_1}$ and $\mathrm{supp}\psi_{k_2t_2c_2} \neq \mathrm{supp}\psi_{k_1t_1c_1}$. The latter case corresponds to $\lfloor b^{k_2-k_1}t_1 \rfloor = t_2$ or $\lfloor b^{k_1-k_2}t_2 \rfloor = t_1$. The coefficients for $k_1, k_2 > l$ do not vanish.

On the other hand when we use a randomly shifted net, none of the coefficients vanish:

$$E[\langle f\tau_{\rm t}, \psi_{k_1t_1c_1}\rangle\langle f\tau_{\rm t}, \psi_{k_2t_2c_2}\rangle] \neq 0 \quad \text{for } \forall\psi_{k_1t_1c_1}, \psi_{k_2t_2c_2}.$$

Summarizing the above and using (4), we have the following results.

Theorem 1 *The variance of the numerical integration of $f \in L^2[0,1)$ by the scrambled (t,m,s)-net to the l-th digit in base b is given by*

$$\begin{aligned} V_{\rm s}(I_N) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\Bigg(&\sum_{k=m-t+1}^{l}\sum_{t=0}^{b^{k-1}-1}\sum_{c_1=0}^{b-1}\sum_{c_2=0}^{b-1} C^{\rm s}_{kktt c_1c_2}\psi_{ktc_1}(z_i)\psi_{ktc_2}(z_j) \\ +2&\sum_{k_1=l+1}^{\infty}\sum_{k_2=m-t+1}^{l}\sum_{\lfloor b^{k_2-k_1}t_1\rfloor=t_2}\sum_{c_1=0}^{b-1}\sum_{c_2=0}^{b-1} C^{\rm s}_{k_1k_2t_1t_2c_1c_2}\psi_{k_1t_1c_1}(z_i)\psi_{k_2t_2c_2}(z_j) \\ +&\sum_{k_1=l+1}^{\infty}\sum_{k_2=l+1}^{\infty}\sum_{t_1=0}^{b^{k_1-1}-1}\sum_{t_2=0}^{b^{k_2-1}-1}\sum_{c_1=0}^{b-1}\sum_{c_2=0}^{b-1} C^{\rm s}_{k_1k_2t_1t_2c_1c_2}\psi_{k_1t_1c_1}(z_i)\psi_{k_2t_2c_2}(z_j)\Bigg), \end{aligned} \tag{7}$$

where $C^{\rm s}_{k_1k_2t_1t_2c_1c_2} = E\left[\langle f\tau_{\rm s}^{(l)}, \psi_{k_1t_1c_1}\rangle\langle f\tau_{\rm s}^{(l)}, \psi_{k_2t_2c_2}\rangle\right]$. The variance of the integration of $f \in L^2[0,1)$ by the randomly shifted (t,m,s)-net is given by

$$\begin{aligned} V_{\rm t}(I_N) &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_1=m-t+1}^{\infty}\sum_{k_2=m-t+1}^{\infty}\sum_{t_1=0}^{b^{k_1-1}-1}\sum_{t_2=0}^{b^{k_2-1}-1} \\ &\quad\cdot\sum_{c_1=0}^{b-1}\sum_{c_2=0}^{b-1} E\left[\langle f\tau_{\rm t}, \psi_{k_1t_1c_1}\rangle\langle f\tau_{\rm t}, \psi_{k_2t_2c_2}\rangle\right]\psi_{k_1t_1c_1}(z_i)\psi_{k_2t_2c_2}(z_j). \end{aligned} \tag{8}$$

This result shows that the scrambling method eliminates the terms with $m - t < k \leq l$ in the expansion of the variance. On the other hand the random shifting method cannot eliminate them. See Fig. 2. If those terms are relatively small, the variance for shifted net is approximately equal to that for the scrambled net. We will show two methods give close values of variance by numerical experiments in the latter section.
Remark. Our result (7) agrees with Owen's result [6] when the number of permuted digits l becomes infinity. See Appendix A. ■

3.2 Mean Absolute Error

Another measure gives a different aspect of the integration error. Let us consider the mean absolute error.

$$R = E\left[\left|\int_0^1 f(x)\mathrm{d}x - \frac{1}{N}\sum_{i=1}^{N} f(x_i)\right|\right].$$

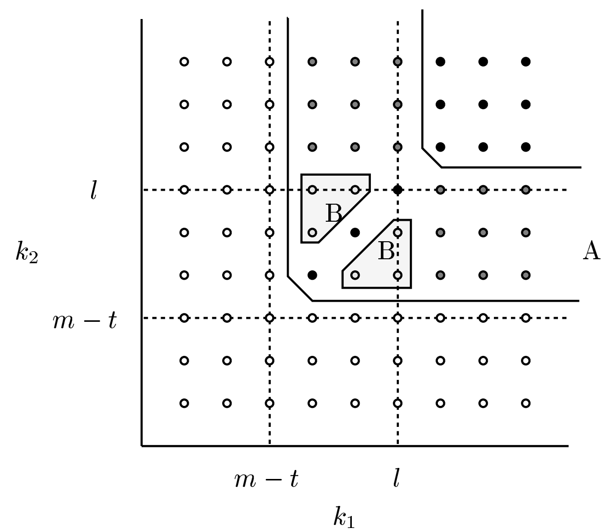

Figure 2: Vanishing terms. The terms corresponding to (k_1, k_2) with $\min(k_1, k_2) \leq m-t$, indicated by $\circ$, vanish due to the property of (t, m, s)-net. When we use the scrambled net, more terms vanish. If (k_1, k_2) is in the area A ($k_1, k_2 > m-t$ and $\min(k_1, k_2) \leq l$), the terms of not overlapping supports vanish. The overlapping supports terms vanish if (k_1, k_2) is in the area B ($k_1, k_2 > m-t$ and $\max(k_1, k_2) \leq l$).

Using Haar expansion of the integrand, the mean absolute error can be bounded as follows.

$$R \leq \frac{1}{N} \sum_{i=1}^{N} \sum_{k=m-t+1}^{\infty} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} E\left[|\langle f\tau, \psi_{ktc} \rangle| \right] |\psi_{ktc}(z_i)| . \tag{9}$$

The summation on k begins at $m-t+1$ due to the property (3). We need to estimate the values of expectation. For the scrambled net we note that b^k elementary intervals of $[tb^{-k}, (t+1)b^{-k})$ are randomly permuted, and that the permutation of the intervals of width smaller than t^{-k-1} has no effect on the expectation of the coefficients. Consequently we have a bound for the expectation of the coefficients for a fixed $k \geq l$.

$$\begin{aligned} E\left[|\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{ktc} \rangle| \right] &= \frac{1}{b^{k-1}} \sum_{t=0}^{b^{k-1}-1} \frac{1}{b} \sum_{c=0}^{b-1} \left| \int_0^1 f(x)\psi_{ktc}(x)\mathrm{d}x \right| \\ &\leq \sup_{t,c} \left| \int_0^1 f(x)\psi_{ktc}(x)\mathrm{d}x \right| . \end{aligned} \tag{10}$$

On the other hand, for the random shifting method the expectation is taken with respect to the uniform distribution t over $[0, 1)$:

$$\begin{aligned} E\left[|\langle f\tau_{\mathrm{t}}, \psi_{ktc} \rangle| \right] &= \frac{1}{b^{k-1}} \int_0^{b^{k-1}} \left| \int_0^1 f^*(x)\psi_{ktc}(x)\mathrm{d}x \right| \mathrm{d}t \\ &\leq \sup_{t,c} \left| \int_0^1 f^*(x)\psi_{ktc}(x)\mathrm{d}x \right| , \end{aligned}$$

where $f^*(x)$ is the periodic function defined as $f^*(x) = f(x \bmod 1)$. From (9) and (10) we have an upper bound for the mean absolute value error of the integration by the scrambled net.

$$R_{\mathrm{s}} \leq \frac{1}{N} \sum_{k=m-t+1}^{\infty} \sup_{t,c} \left| \int_0^1 f(x)\psi_{ktc}(x)\mathrm{d}x \right| \sum_{i=1}^{N} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} |\psi_{ktc}(z_i)|$$

$$\leq \sum_{k=m-t+1}^{\infty} b^{k/2} \sup_{t,c} \left| \int_0^1 f(x)\psi_{ktc}(x)\mathrm{d}x \right|.$$

Second inequality holds because only N terms are non-zero in the summation on i, t, and c for a fixed k, and $|\psi_{ktc}(z_i)|$ is bounded by $b^{k/2}$. Let us introduce the modulus of continuity of a function.

$$\omega_1(f;\delta) = \sup\{\int_0^1 |f(x+\xi) - f(x)|\mathrm{d}x :\ x+\xi \in [0,1),\ 0 < \xi < \delta\}.$$

Then we have

$$\sup_{t,c} \left| \int_0^1 f(x)\psi_{ktc}(x)\mathrm{d}x \right| \leq b^{k/2} \cdot b^{-k+1}\omega_1(f; b^{-k+1}).$$

Finally the mean absolute error of the scrambled net integration is bounded as follows.

$$R_{\mathrm{s}} \leq b \sum_{k=m-t+1}^{\infty} \omega_1(f; b^{-k+1}). \tag{11}$$

If we assume that $\omega_1(f; b^{-k+1})$ is bounded by $C(b^{-k})^\alpha$ with some constant C, which is the Hölder condition for the function f, then since $N = b^m$ we have

$$R_{\mathrm{s}} \leq Cb \sum_{k=m-t+1}^{\infty} (b^{-k})^\alpha = Cb(b^{-m+t-1})^\alpha \frac{1}{1-b^\alpha} = \frac{Cb^{\alpha(t-1)+1}}{1-b^\alpha}\frac{1}{N^\alpha}.$$

Thus we have obtained a relation between the integration error and the number of sample points.

We can also derive a bound for the error of the randomly shifted net.

$$R_{\mathrm{t}} \leq b \sum_{k=m-t+1}^{\infty} \omega_1^*(f^*; b^{-k+1}), \tag{12}$$

where $\omega_1^*(f;\delta) = \sup\{\int_0^1 |f(x+\xi) - f(x)|\mathrm{d}x :\ \ 0 < \xi < \delta\}$.

From (11) and (12), we know that the modulus of continuity of the integrand determines an upper bound of the integration error. If the integrand satisfies $\omega_1(f;\delta) \approx \omega_1^*(f^*;\delta)$, two upper bounds, R_{s} and R_{t} give an approximately same value.

4 Error Analysis in Multidimensional Case

We need multidimensional b-adic Haar functions to proceed with our analysis in multidimensional case. We use Owen's definition [6].

$$\psi_{\mathbf{ktc}}(\mathbf{x}) = \prod_{r=1}^{s} \psi_{k_r t_r c_r}(x_r),$$

where the subscripts of ψ mean $\mathbf{k} = (k_1, \ldots, k_s)$, $\mathbf{t} = (t_1, \ldots, t_s)$, and $\mathbf{c} = (c_1, \ldots, c_s)$, respectively. Let $\max \mathbf{k} = \max_{1 \leq i \leq s} k_i$, $\min \mathbf{k} = \min_{1 \leq i \leq s} k_i$, and $|\mathbf{k}| = \sum_{i=1}^{s} k_i$. We show some important properties, which can be proven easily.

Proposition 3 *Let $\psi_{\mathbf{ktc}}$ be s-dimensional b-adic Haar functions.*

1. *For each $\mathbf{k} \neq \mathbf{0}$ and $\mathbf{t} \geq \mathbf{0}$, the following equalities hold.*

$$\sum_{c_i=0}^{b-1} \psi_{\mathbf{ktc}}(\mathbf{x}) = 0, \quad i = 1, \ldots, s.$$

2. *Let $\{\mathbf{z}_i\}$ be a (t,m,s)-net in base b. For each $\mathbf{k}$ such that $0 < |\mathbf{k}| \leq m-t$ we have*

$$\sum_{i=1}^{b^m} \psi_{\mathbf{ktc}}(\mathbf{z}_i) = 0.$$

4.1 Mean Square Error

For $f \in L^2[0,1)^s$ we have an expression

$$f(\mathbf{x}) = \sum_{\mathbf{k}} \sum_{\mathbf{t}} \sum_{\mathbf{c}} \langle f, \psi_{\mathbf{ktc}} \rangle \psi_{\mathbf{ktc}}(\mathbf{x}),$$

where $\sum_{\mathbf{k}}$ means $\sum_{k_1=0}^{\infty} \cdots \sum_{k_s=0}^{\infty}$, $\sum_{\mathbf{t}}$ means $\sum_{t_1=0}^{b^{k_1-1}-1} \cdots \sum_{t_s=0}^{b^{k_s-1}-1}$, and $\sum_{\mathbf{c}}$ means $\sum_{c_1=0}^{b-1} \cdots \sum_{c_s=0}^{b-1}$, respectively. The variance of the integration is

$$V(I_N) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\mathbf{k}_1} \sum_{\mathbf{t}_1} \sum_{\mathbf{c}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{t}_2} \sum_{\mathbf{c}_2} E\left[\langle f\tau, \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \rangle \langle f\tau, \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \rangle\right] \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1}(\mathbf{z}_i) \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}(\mathbf{z}_j),$$

where we use τ to denote both scrambling and random shifting. We extend the Lemma 1 to obtain the following lemma. Let $\mathbf{k}_j = (k_1^{(j)}, \ldots, k_s^{(j)})$, $j = 1, 2$.

Lemma 2 *If* $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \cap \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} = \emptyset$ *and* $l \geq \min(\min \mathbf{k}_1, \min \mathbf{k}_2)$, *or if* $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \cap \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \neq \emptyset$ *and* $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \neq \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}$ *and* $l \geq \min_i(\max(k_i^{(1)}, k_i^{(2)}))$, *then we have* $E[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \rangle \langle f\tau_{\mathrm{s}}^{(l)}, \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \rangle] = 0$.

Proof. Under the condition that the supports of two b-adic Haar functions have no intersection, we can show the lemma in a similar way to one-dimensional case.

If they have an intersection, we should note that even if $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \cap \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \neq \emptyset$ and $|\mathbf{k}_1| \geq |\mathbf{k}_2|$, $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1}$ is not necessarily a proper subset of $\operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}$ in multidimensional case. There exists, however, at least one index i such that $k_i^{(1)} > k_i^{(2)}$ for vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ if $|\mathbf{k}_1| \geq |\mathbf{k}_2|$ and $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \neq \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}$. For such i, $\operatorname{supp} \psi_{k_i^{(1)} t_i^{(1)} c_i^{(1)}}$ is a proper subset of $\operatorname{supp} \psi_{k_i^{(2)} t_i^{(2)} c_i^{(2)}}$. By Lemma 1 we have the latter part of the present lemma. ∎

From the lemma we immediately have the following variance estimation.

Theorem 2 *The variance of the integration of* $f \in L^2[0,1)^s$ *by the scrambled* (t,m,s)*-net to the* l*-th digit in base* b *is given by*

$$\begin{aligned} V_{\mathrm{s}}(I_N) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg(& \sum_{\substack{|\mathbf{k}| > m-t \\ \min \mathbf{k} < l}} \sum_{\mathbf{t}} \sum_{\mathbf{c}_1} \sum_{\mathbf{c}_2} C^{\mathrm{s}}_{\mathbf{k}_1\mathbf{k}_2\mathbf{t}_1\mathbf{t}_2\mathbf{c}_1\mathbf{c}_2} \psi_{\mathbf{ktc}_1}(\mathbf{z}_i) \psi_{\mathbf{ktc}_2}(\mathbf{z}_j) \\ & + 2 \sum_{\min \mathbf{k}_1 \geq l} \sum_{\substack{|\mathbf{k}_2| > m-t \\ \min \mathbf{k}_2 < l}} \sum_{\text{cond. A}} C^{\mathrm{s}}_{\mathbf{k}_1\mathbf{k}_2\mathbf{t}_1\mathbf{t}_2\mathbf{c}_1\mathbf{c}_2} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1}(\mathbf{z}_i) \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}(\mathbf{z}_j) \\ & + \sum_{\min \mathbf{k}_1 \geq l} \sum_{\min \mathbf{k}_2 \geq l} \sum_{\mathbf{t}_1, \mathbf{t}_2} \sum_{\mathbf{c}_1, \mathbf{c}_2} C^{\mathrm{s}}_{\mathbf{k}_1\mathbf{k}_2\mathbf{t}_1\mathbf{t}_2\mathbf{c}_1\mathbf{c}_2} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1}(\mathbf{z}_i) \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}(\mathbf{z}_j) \Bigg), \end{aligned}$$

where $C^{\mathrm{s}}_{\mathbf{k}_1\mathbf{k}_2\mathbf{t}_1\mathbf{t}_2\mathbf{c}_1\mathbf{c}_2} = E[\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \rangle \langle f\tau_{\mathrm{s}}^{(l)}, \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \rangle]$ *and* cond. A *in the second summation term means that* $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \cap \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \neq \emptyset$ *and* $\operatorname{supp} \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \neq \operatorname{supp} \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}$. *The variance of the integration of* $f \in L^2[0,1)^s$ *by the randomly shifted* (t,m,s)*-net is given by*

$$V_{\mathrm{t}}(I_N) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{|\mathbf{k}_1| > m-t} \sum_{|\mathbf{k}_2| > m-t} \sum_{\mathbf{t}_1} \sum_{\mathbf{t}_2} \sum_{\mathbf{c}_1} \sum_{\mathbf{c}_2} E_s\left[\langle f\tau_{\mathrm{t}}, \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1} \rangle \langle f\tau_{\mathrm{t}}, \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2} \rangle\right] \psi_{\mathbf{k}_1\mathbf{t}_1\mathbf{c}_1}(\mathbf{z}_i) \psi_{\mathbf{k}_2\mathbf{t}_2\mathbf{c}_2}(\mathbf{z}_j). \tag{13}$$

4.2 Mean Absolute Error

An upper bound for the mean absolute error in multidimensional case can be derived as in one-dimensional case.

$$\begin{aligned} R_{\mathrm{s}} &\leq \frac{1}{N}\sum_{i=1}^{N}\sum_{|\mathbf{k}|>m-t}\sum_{\mathbf{t}}\sum_{\mathbf{c}} E\left[|\langle f\tau_{\mathrm{s}}^{(l)}, \psi_{\mathbf{ktc}}\rangle|\right]|\psi_{\mathbf{ktc}}(\mathbf{z}_i)| \\ &\leq \frac{1}{N}\sum_{|\mathbf{k}|>m-t}\sup_{\mathbf{t},\mathbf{c}}\left|\int_{[0,1)^s} f(\mathbf{x})\psi_{\mathbf{ktc}}(\mathbf{x})\mathrm{d}\mathbf{x}\right|\sum_{i=1}^{N}\sum_{\mathbf{t}}\sum_{\mathbf{c}}|\psi_{\mathbf{ktc}}(\mathbf{z}_i)| \\ &\leq \sum_{|\mathbf{k}|>m-t} b^{|\mathbf{k}|/2}\sup_{\mathbf{t},\mathbf{c}}\left|\int_{[0,1)^s} f(\mathbf{x})\psi_{\mathbf{ktc}}(\mathbf{x})\mathrm{d}\mathbf{x}\right|. \end{aligned}$$

We note that the volume of supp$\psi_{\mathbf{ktc}}$ is $b^{-|\mathbf{k}|+s}$ and the maximum value of $\psi_{\mathbf{ktc}}(\mathbf{x})$ is bounded by $b^{|\mathbf{k}|/2}$, then an bound of the supremum can be obtained as follows.

$$\sup_{\mathbf{t},\mathbf{c}}\left|\int_{[0,1)^s} f(\mathbf{x})\psi_{\mathbf{ktc}}(\mathbf{x})\mathrm{d}\mathbf{x}\right| \leq b^{-|\mathbf{k}|/2+2s}\omega_1(f;\prod_{i=1}^{s}[0,b^{-k_i+1})).$$

where the modulus of continuity in multidimensional case is defined as follows.

$$\omega_1(f,\Delta) = \sup\{\int_{[0,1)^s}|f(\mathbf{x}) - f(\mathbf{x}+\mathbf{y})|\mathrm{d}\mathbf{x} : \mathbf{y}\in\Delta, \mathbf{x}+\mathbf{y}\in[0,1)^s\}.$$

As a result we have the error bound for the scrambled net integration.

$$R_{\mathrm{s}} \leq b^{2s}\sum_{|\mathbf{k}|>m-t}\omega_1(f;\prod_{i=1}^{s}[0,b^{-k_i+1})), \tag{14}$$

If we assume that f satisfies the condition

$$\omega_1(f;\prod_{i=1}^{s}[0,b^{-k_i+1})) \leq Cb^{-\alpha|\mathbf{k}|}$$

with some constant C, then we have the following bound.

$$R_{\mathrm{s}} \leq Cb^{2s}\sum_{|\mathbf{k}|>m-t} b^{-\alpha|\mathbf{k}|} = Cb^{2s}\sum_{k>m-t}\binom{s+k-1}{k}b^{-\alpha k}.$$

A similar result holds for random shifting method.

$$R_{\mathrm{t}} \leq b^{2s}\sum_{|\mathbf{k}|>m-t}\omega_1^*(f^*;\prod_{i=1}^{s}[0,b^{-k_i+1})), \tag{15}$$

where $f^*(\mathbf{x}) = f(\mathbf{x} \bmod \mathbf{1})$, and

$$\omega_1^*(f,\Delta) = \sup\{\int_{[0,1)^s}|f(\mathbf{x}) - f(\mathbf{x}+\mathbf{y})|\mathrm{d}\mathbf{x} :\ \mathbf{y}\in\Delta\}.$$

From (14) and (15), if $\omega_1(f;\Delta) \approx \omega_1^*(f^*;\Delta)$, two upper bounds give close values.

5 Numerical Experiments

We will show error estimations of multidimensional integrals for various functions by both the scrambling method and random shifting method.

Six kinds of functions proposed by Genz [3] are used for the experiments:

$$
\begin{aligned}
f_1(\mathbf{x}) &= \cos(2\pi u_1 + \sum_{j=1}^{s} a_j x_j) \quad \text{(Oscillatory)}, \\
f_2(\mathbf{x}) &= \prod_{j=1}^{s} (a_j^{-2} + (x_j - u_j)^2) \quad \text{(Product Peak)}, \\
f_3(\mathbf{x}) &= (1 + \sum_{j=1}^{s} a_j x_j)^{-s-1} \quad \text{(Corner Peak)}, \\
f_4(\mathbf{x}) &= \exp(-\sum_{j=1}^{s} a_j^2 (x_j - u_j)^2) \quad \text{(Gaussian)}, \\
f_5(\mathbf{x}) &= \exp(-\sum_{j=1}^{s} a_j |x_j - u_j|) \quad (C_0), \\
f_6(\mathbf{x}) &= \exp(-\sum_{j=1}^{s} a_j x_j) 1_{x_1 > u_1} 1_{x_2 > u_2} \quad \text{(Discontinuous)}.
\end{aligned}
$$

The parameters u_j are uniform random numbers over $[0,1)$. The parameters a_j are determined using $a_j = \gamma_j a'_j$, where a'_j are uniform random numbers over $[0,1)$, and γ_j is determined by $\gamma_j s^{e_j} \sum_{i=1}^{s} a'_i = h_j$. The numbers e_j and h_j are set as $(e_j) = (1.5, 2, 2, 1, 2, 2)$, and $(h_j) = (110, 600, 600, 100, 150, 100)$ respectively. The dimension s is set to 10.

We use Faure sequence and Sobol' sequence as (t, m, s)-net. In order to estimate $\hat{\sigma}$ scrambling and shifting are repeated 30 times respectively. Figs. 3 and 4 show the results with the error estimation for the functions with certain choice of parameters a_j and u_j. In each figure the abscissa is the number of scrambled or shifted sample points N, and estimated values are indicated by dots. The vertical bar on the dot shows the estimated error $3\hat{\sigma}$. The dashed line shows the true value of the integral. The net was scrambled to the $(m - t + 1)$-th digit in base b. The scrambling to lower digits had no effect on the error estimation. In these experiments both scrambled net and randomly shifted net gave almost the same accuracy of estimation.

6 Concluding Remarks

We presented a theoretical investigation for statistical error estimation methods of quasi-Monte Carlo integration and gave some numerical experiments. Our detailed analyses revealed the difference between the integration errors by a scrambled net and by a shifted net. On the other hand, our numerical experiments show that there exists no significant difference between estimated errors by two methods. Both of them gave accurate error estimates.

As for computational issue, the scrambling method requires complicated implementation and huge computational efforts, and is very time-consuming. It is difficult to apply the method to large scale problems. The random shifting method is simple and very fast. Based on these arguments and observations we consider that the random shifting method is a practical error estimation method for quasi-Monte Carlo integration.

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A Correspondence to Owen's Result

We compare our result and Owen's in one-dimensional case. Let the number of scrambled digit l be sufficiently large in (7) and rewrite $\tau_{\mathrm{s}}^{(l)}$ as τ_{s}, then we have

$$V_s(I_N) \simeq \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=m-t+1}^{\infty}\sum_{t=0}^{b^{k-1}-1}\sum_{c_1=0}^{b-1}\sum_{c_2=0}^{b-1} E\left[\langle f\tau_{\mathrm{s}}, \psi_{ktc_1}\rangle\langle f\tau_{\mathrm{s}}, \psi_{ktc_2}\rangle\right]\psi_{ktc_1}(z_i)\psi_{ktc_2}(z_j).$$

Let β_{k+1} be the sum for a fixed k in the $V_s(I_N)$.

$$\beta_{k+1} = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=0}^{b^{k-1}-1}\sum_{c_1=0}^{b-1}\sum_{c_2=0}^{b-1} E\left[\langle f\tau_{\mathrm{s}}, \psi_{ktc_1}\rangle\langle f\tau_{\mathrm{s}}, \psi_{ktc_2}\rangle\right]\psi_{ktc_1}(z_i)\psi_{ktc_2}(z_j).$$

In the summation $\psi_{ktc_1}(z_i)\psi_{ktc_2}(z_j)$ is nonzero only for such z_i, z_j that agree to $(k-1)$-th digit in base b. Divide β_{k+1} into two parts. One is the summation for such z_i, z_j that agree to k-th digit in base b, say S_1. Another is for such z_i, z_j that agree to $(k-1)$-th digit in base b but not in k-th digit, say S_2. Now we calculate the coefficients.

$$E[\langle f\tau_{\mathrm{s}}, \psi_{ktc}\rangle^2] = \frac{1}{b^k}\sum_{t=0}^{b^{k-1}-1}\sum_{c=0}^{b-1}\langle f, \psi_{ktc}\rangle^2, \tag{16}$$

$$E[\langle f\tau_{\mathrm{s}}, \psi_{ktc_1}\rangle\langle f\tau_{\mathrm{s}}, \psi_{ktc_2}\rangle] = \frac{1}{b^{k-1}}\frac{1}{b^2-b}\sum_{t=0}^{b^{k-1}-1}\sum_{c_1\neq c_2}\langle f, \psi_{ktc_1}\rangle\langle f, \psi_{ktc_2}\rangle$$

$$
\begin{aligned}
&= \frac{1}{b^k(b-1)} \sum_{t=0}^{b^{k-1}-1} \sum_{c_1=0}^{b-1} \langle f, \psi_{ktc_1} \rangle \langle f, \sum_{c_2 \neq c_1} \psi_{ktc_2} \rangle \\
&= \frac{1}{b^k(b-1)} \sum_{t=0}^{b^{k-1}-1} \sum_{c_1=0}^{b-1} \langle f, \psi_{ktc_1} \rangle \langle f, -\psi_{ktc_1} \rangle \\
&= -\frac{1}{b-1} E[\langle f\tau_{\mathrm{s}}, \psi_{ktc} \rangle^2]. \qquad (17)
\end{aligned}
$$

According to Owen, introduce for $k \geq 0$

$$
M_k = \sum_{i=1}^{N} \sum_{j=1}^{N} 1_{\lfloor b^k z_i \rfloor = \lfloor b^k z_j \rfloor},
$$

where $1_{(\cdot)}$ is the indicator function that returns 1 if $(\cdot)$ is true, 0 otherwise.

$$
\begin{aligned}
\sum_{i,j=1}^{N} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \sum_{\lfloor b^k z_i \rfloor = \lfloor b^k z_j \rfloor} \psi_{ktc}(z_i)\psi_{ktc}(z_j) &= M_k \left(b^{k-1} \left(\sqrt{b} - \frac{1}{\sqrt{b}} \right)^2 + b^{k-1} \left(\frac{-1}{\sqrt{b}} \right)^2 (b-1) \right) \\
&= M_k b^{k-1}(b-1). \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i,j=1}^{N} \sum_{t=0}^{b^{k-1}-1} \sum_{c_1 \neq c_2} \sum_{\lfloor b^k z_i \rfloor = \lfloor b^k z_j \rfloor} \psi_{ktc_1}(z_i)\psi_{ktc_2}(z_j) &= M_k \left(2b^{k-1} \left(\sqrt{b} - \frac{1}{\sqrt{b}} \right) \left(\frac{-1}{\sqrt{b}} \right) (b-1) + \right. \\
&\qquad \left. b^{k-1} \left(\frac{-1}{\sqrt{b}} \right)^2 (b-1)(b-2) \right) \\
&= -M_k b^{k-1}(b-1). \qquad (19)
\end{aligned}
$$

From (16), (17), (18), and (19) we have

$$
\begin{aligned}
S_1 &= \sum_{i,j=1}^{N} \sum_{\lfloor b^k z_i \rfloor = \lfloor b^k z_j \rfloor} \sum_{t=0}^{b^{k-1}-1} \left(\sum_{c=0}^{b-1} E[\langle f\tau_{\mathrm{s}}, \psi_{ktc} \rangle^2] \psi_{ktc}(z_i)\psi_{ktc}(z_j) \right. \\
&\qquad \left. + \sum_{c_1 \neq c_2} E[\langle f\tau_{\mathrm{s}}, \psi_{ktc_1} \rangle \langle f\tau_{\mathrm{s}}, \psi_{ktc_2} \rangle] \psi_{ktc_1}(z_i)\psi_{ktc_2}(z_j) \right) \\
&= M_k \left(1 + \frac{1}{b-1} \right) (b-1) b^{k-1} E_s[\langle f, \psi_{ktc} \rangle] \\
&= M_k \sum_{t=0}^{b^k-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc} \rangle^2
\end{aligned}
$$

Next we need two following sums in order to calculate S_2.

$$
\begin{aligned}
&\sum_{i,j=1}^{N} \sum_{\substack{\lfloor b^{k-1} z_i \rfloor = \lfloor b^{k-1} z_j \rfloor \\ \lfloor b^k z_i \rfloor \neq \lfloor b^k z_j \rfloor}} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \psi_{ktc}(z_i)\psi_{ktc}(z_j) \\
&= \left(2b^{k-1} \left(\sqrt{b} - \frac{1}{\sqrt{b}} \right) \left(\frac{-1}{\sqrt{b}} \right) + b^{k-1} \left(\frac{-1}{\sqrt{b}} \right)^2 (b-2) \right) (M_{k-1} - M_k) \\
&= -b^{k-1}(M_{k-1} - M_k). \qquad (20)
\end{aligned}
$$

$$\sum_{i,j=1}^{N} \sum_{\substack{\lfloor b^{k-1} z_i \rfloor = \lfloor b^{k-1} z_j \rfloor \\ \lfloor b^k z_i \rfloor \neq \lfloor b^k z_j \rfloor}} \sum_{t=0}^{b^{k-1}-1} \sum_{c_1 \neq c_2} \psi_{ktc_1}(z_i) \psi_{ktc_2}(z_j)$$

$$= \sum_{\substack{\lfloor b^{k-1} z_i \rfloor = \lfloor b^{k-1} z_j \rfloor \\ \lfloor b^k z_i \rfloor \neq \lfloor b^k z_j \rfloor}} \sum_{t=0}^{b^{k-1}-1} \sum_{c_1=0}^{b-1} \psi_{ktc_1}(z_i)(-\psi_{ktc_1}(z_j))$$

$$= b^{k-1}(M_{k-1} - M_k).$$

Then we have

$$\begin{aligned} S_2 &= E[\langle f\tau_{\mathrm{s}}, \psi_{ktc}\rangle^2](-b^{k-1})(M_{k-1} - M_k) + \left(-\frac{1}{b-1}\right) E[\langle f\tau_{\mathrm{s}}, \psi_{ktc}\rangle^2] b^{k-1}(M_{k-1} - M_k) \\ &= -\frac{b^k}{b-1} E[\langle f\tau_{\mathrm{s}}, \psi_{ktc}\rangle^2](M_{k-1} - M_k) \\ &= -\frac{M_{k-1} - M_k}{b-1} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc}\rangle^2. \end{aligned}$$

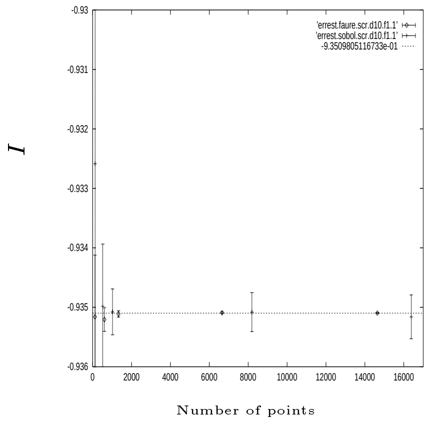
It follows that

$$\begin{aligned} \beta_{k+1} &= S_1 + S_2 = \left(M_k - \frac{M_{k-1} - M_k}{b-1}\right) \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc}\rangle^2 \\ &= \frac{bM_k - M_{k-1}}{b-1} \sum_{t=0}^{b^{k-1}-1} \sum_{c=0}^{b-1} \langle f, \psi_{ktc}\rangle^2 \\ &= \frac{bM_k - M_{k-1}}{b-1} \int_0^1 (Q_k f(x))^2 \mathrm{d}x. \end{aligned}$$

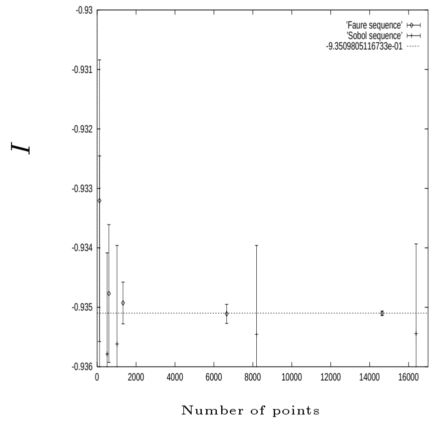
Finally,

$$V_s(I_N) = \frac{1}{N^2} \sum_{k=m-t+1}^{\infty} \frac{bM_{k+1} - M_k}{b-1} \int_0^1 (Q_k f(x))^2 \mathrm{d}x.$$

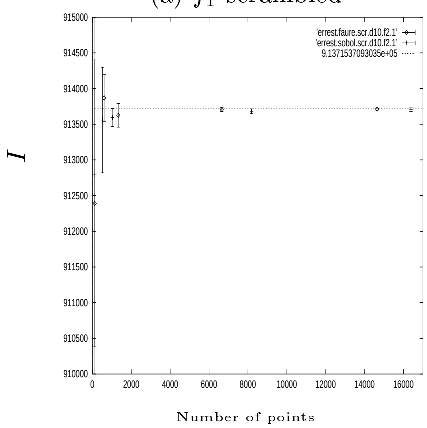
This result agrees to Owen's result in one-dimensional case.



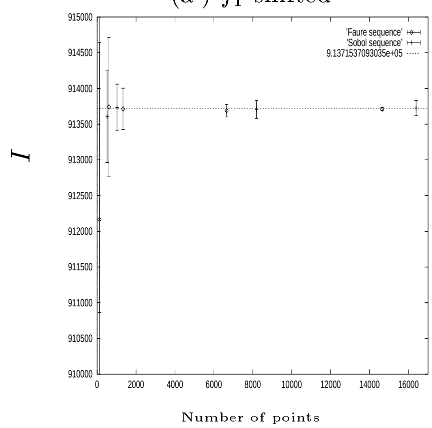
(a) f_1 scrambled



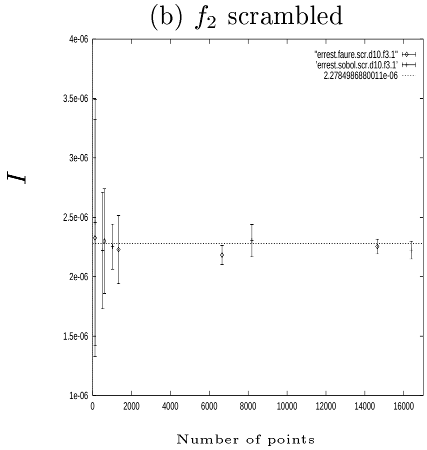
(a′) f_1 shifted



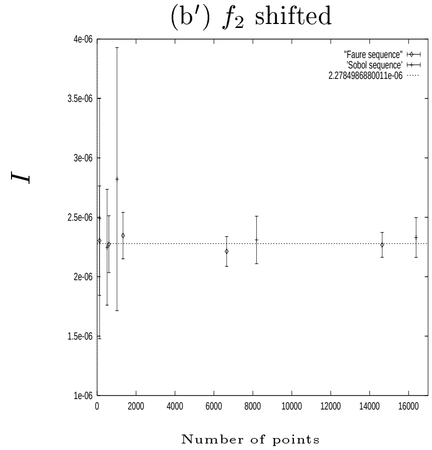
(b) f_2 scrambled



(b′) f_2 shifted

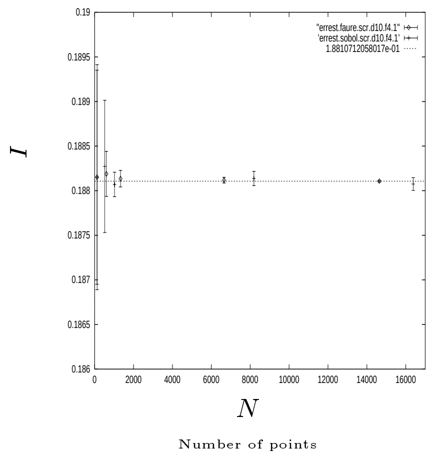


(c) f_3 scrambled

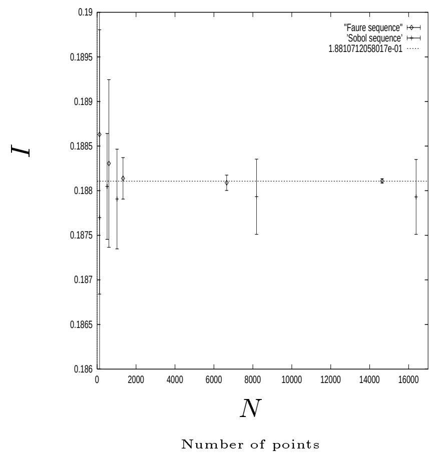


(c′) f_3 shifted

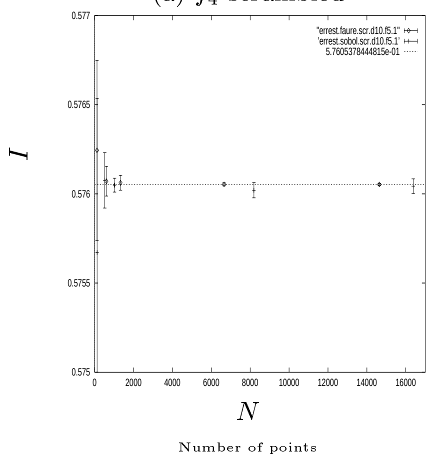
Figure 3: Error estimations for Genz test functions (1)



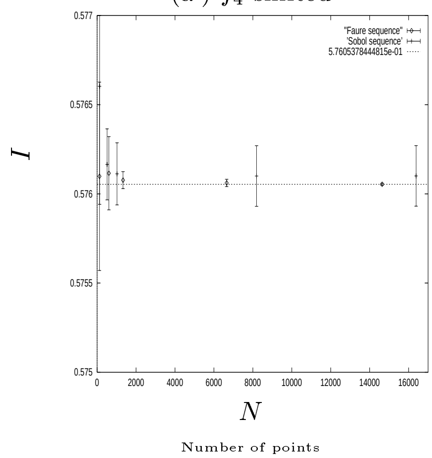
(a) f_4 scrambled



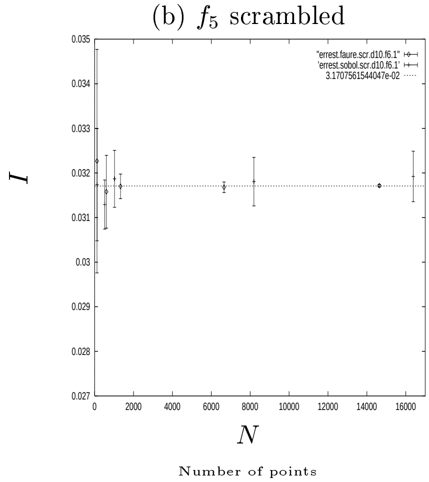
(a′) f_4 shifted



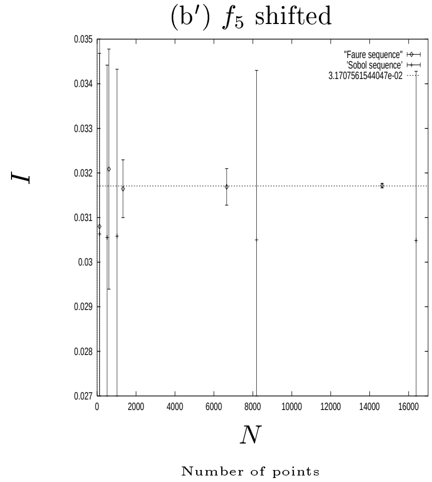
(b) f_5 scrambled



(b′) f_5 shifted



(c) f_6 scrambled



(c′) f_6 shifted

Figure 4: Error estimations for Genz test functions (2)