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OBSTRUCTIONS TO THE EXISTENCE AND SQUEEZING OF LAGRANGIAN COBORDISMS

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Abstract. Capacities that provide both qualitative and quantitative obstructions to the existence of a Lagrangian cobordism between two $(n-1)$-dimensional submanifolds in parallel hyperplanes of $\mathbb{R}^{2n}$ are defined using the theory of generating families. Qualitatively, these capacities show that, for example, in $\mathbb{R}^4$ there is no Lagrangian cobordism between two ∞-shaped curves with a negative crossing when the lower end is "smaller". Quantitatively, when the boundary of a Lagrangian ball lies in a hyperplane of $\mathbb{R}^{2n}$, the capacity of the boundary gives a restriction on the size of a rectangular cylinder into which the Lagrangian ball can be squeezed.

1. Introduction

A fundamental problem in symplectic topology is to understand the boundary between flexibility (when symplectic objects behave like topological objects) and rigidity (when symplectic behavior is more restrictive). In this paper, we investigate flexibility and rigidity in the setting of existence questions for Lagrangian cobordisms in the standard symplectic $\mathbb{R}^{2n}$.

1.1. Questions and Results.

Consider $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ with coordinates $(x_1, \dots x_n, y_1, \dots y_n)$ and the standard symplectic form $\omega = \sum dx_i \wedge dy_i$. Consider an $(n-1)$-dimensional submanifold L_a in $\{y_n = a\}$ and an $(n-1)$-dimensional submanifold L_b in $\{y_n = b\}$. A natural and important question to ask is whether or not there is a Lagrangian cobordism between these submanifolds; that is, does there exist an n-dimensional submanifold $L_{[a,b]}$ of $\mathbb{R}^{2n}$ with $\omega|_{TL_{[a,b]}} = 0$ that intersects $\{y_n = a\}$ transversally to form L_a and $\{y_n = b\}$ transversally to form L_b?

Topological data provides restictions on such a cobordism. We will first focus on $\mathbb{R}^4$, though the restrictions and our results will have higher dimensional analogues. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the projection to the (x_1, y_1) coordinates. It is easy to see that a cobordism between two connected curves L_a and L_b can exist only if $\pi(L_a)$ and $\pi(L_b)$ bound the same signed area and have the same winding number. Further, if $\mathrm{wr}(L_a)$ denotes the writhe of the diagram $\pi(L_a)$ with respect to the blackboad framing, the canonical isomorphism between the tangent and normal bundles of a Lagrangian submanifold then leads to the following restriction on the Euler characteristic

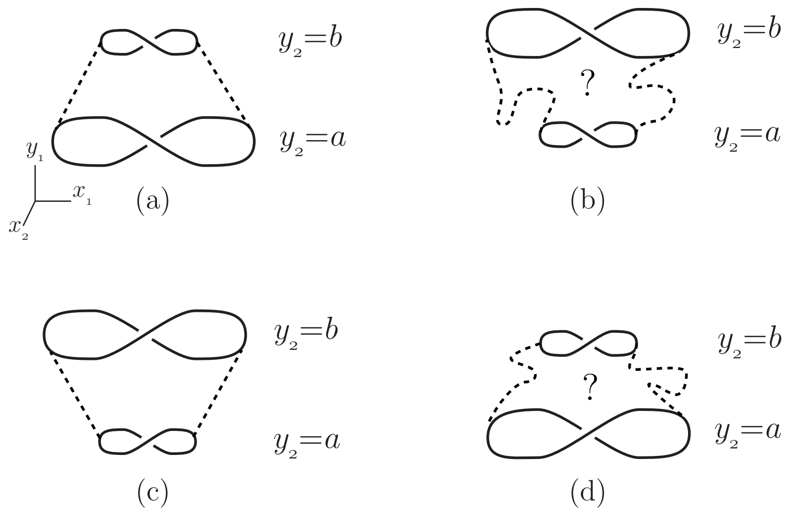

FIGURE 1. Parts (a) and (c) show two pairs of curves that can be joined by a Lagrangian cobordism. Is there a Lagrangian cobordism as in parts (b) and (d) when the ordering of the heights of these curves is changed?

of the cobordism:

$$\mathrm{wr}(L_b) - \mathrm{wr}(L_a) = \chi(L_{[a,b]}). \tag{1.1}$$

In this paper, we produce obstructions that go beyond these basic ones. These new obstructions can be viewed both qualitatively and quantitatively. For an example of the former, consider the following ∞-shaped curves in $\mathbb{R}^3$:

$$8^1_\pm(r) := \{(x_1, x_2, y_1) : x_1^2 + x_2^2 = r^2, \quad y_1 = \pm 2x_1x_2\}.$$

Let i_a denote the inclusion of $\mathbb{R}^3$ into $\{y_2 = a\}$. On one hand, it is possible to construct a Lagrangian cobordism between inclusions of $8^1_-(R)$ and $8^1_-(r)$ when the larger curve is included at a lower level: for example, as explicitly shown in Example 2.1, there exists a Lagrangian submanifold in $\mathbb{R}^4$ so that:

$$(L \cap \{y_2 = 1\}) = i_1(8^1_-(5)), \qquad (L \cap \{y_2 = 10\}) = i_{10}(8^1_-(4)).$$

On the other hand, a basic question probing the rigidity of Lagrangian cobordisms, posed by Y. Eliashberg, is whether or not there exists a Lagrangian cobordism between ∞-shaped curves with a negative crossing when the bottom end is smaller. This situation is summarized graphically in Figure 1.

The following theorem gives an answer to this question:

Theorem 1.1. *If $r \leq R$ and $a < b$, there does not exist a Lagrangian cobordism $L \subset \{a \leq y_2 \leq b\} \subset \mathbb{R}^4$ with*

$$(L \cap \{y_2 = a\}) = i_a(8^1_-(r)), \qquad (L \cap \{y_2 = b\}) = i_b(8^1_-(R)).$$

There are analogous statements about cobordisms between ∞-shaped curves with positive crossings: in this case, there does exist a Lagrangian cobordism when the bottom curve is smaller but not if the bottom curve is larger. This theorem illustrates an asymmetry in some Lagrangian cobordisms. There are many pairs of curves besides the ∞-shaped curves that can be realized as the ends of a Lagrangian cobordism in $\mathbb{R}^4$ but not if the ordering of their heights is changed. In particular, the asymmetry of Lagrangian cobordisms between curves extends to any Lagrangian cobordism between connected curves that can be extended to a planar Lagrangian that agrees with $\{y_1 = y_2 = 0\}$ outside a compact set. This is proved in [2], using the techniques of this paper, where it is shown that the set of connected negative (or positive) hyperplane slices of flat-at-infinity planar Lagrangians in $\mathbb{R}^4$ has the structure of a partially ordered set.

On the quantitative side, we show that it is possible to measure the "size" of a Lagrangian disk with boundary in a hyperplane. Consider a Lagrangian disk $L^2 \subset \mathbb{R}^4$ that transversally intersects $\{y_2 = a\}$ at its boundary $\partial L = i_a(8^1_-(r))$. The projection of this boundary curve to the x_1y_1-plane has two "lobes" having equals areas (of opposite signs); denote the absolute value of the area of one of these lobes by A. In fact, the lobe area A determines whether the Lagrangian disk can be squeezed into a rectangular cylinder $\mathbb{R}^2 \times R$, where R is a rectangle in the x_2y_2-coordinates; see Figure 2. More precisely, we have:

Theorem 1.2. *Suppose a Lagrangian disk $L \subset \mathbb{R}^4$ transversally intersects $\{y_2 = a\}$ and $(L \cap \{y_2 = a\}) = \partial L = i_a(8^1_\pm(r))$, for some $r > 0$. Suppose that each lobe of $\pi(\partial L)$ bounds a region with area of absolute value A. If the entire Lagrangian disk lies in the rectangular cylinder*

$$C = \{(x_1, x_2, y_1, y_2) \ : \ (x_2, y_2) \in I_{x_2} \times I_{y_2}\},$$

for some intervals $I_{x_2} \subset \mathbb{R}$ of length ℓ and $I_{y_2} \subset \mathbb{R}$ of length w, then:

$$\ell \cdot w \geq A.$$

To prove and generalize the above two theorems, we pass to Lagrangians without boundary, namely "unknotted planar Lagrangians." Let L_0 denote the zero-section of $T^*\mathbb{R}^n$. We say that a Lagrangian submanifold is **planar** if it is diffeomorphic to $\mathbb{R}^n$; a planar Lagrangian is **flat-at-infinity** if it agrees with L_0 outside a compact set of $\mathbb{R}^{2n}$, and is **unknotted** if it is Lagrangian isotopic to L_0 via a compactly supported symplectic isotopy of $\mathbb{R}^{2n}$. An unknotted planar Lagrangian will be flat-at-infinity; in $\mathbb{R}^4$, a flat-at-infinity planar Lagrangian is always unknotted [7]. This together with a gluing result allows us to avoid in the statements of Theorems 1.1 and 1.2 an "extendability" hypothesis that will appear in later theorems. As shall be described in Section 2, unknotted planar Lagrangians can be studied using the technique of generating families.

We now set some terminology. We say that a is a **generic height** for an unknotted planar Lagrangian L if L is transverse to the hyperplane $\{y_n = a\}$.

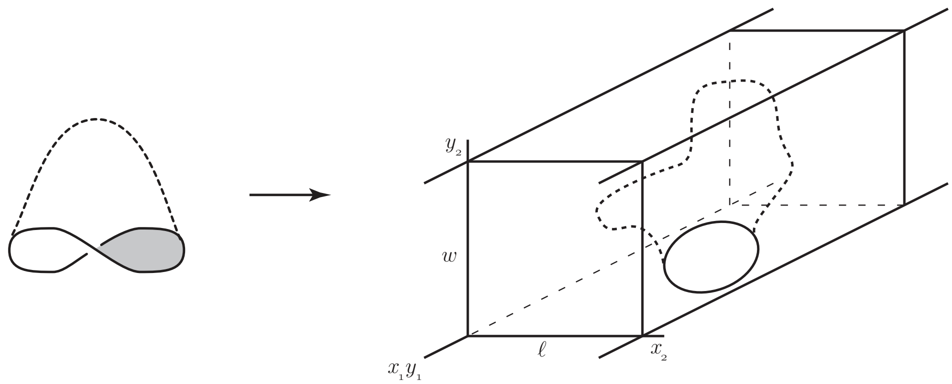

FIGURE 2. If the shaded area on the left is greater than the area $l \cdot w$, then the Lagrangian disk on the left cannot be squeezed into the rectangular cylinder on the right.

For a generic height a, the **slice** L_a is the intersection $L \cap \{y_n = a\}$. Such a slice is a submanifold of $\{y_n = a\} \simeq \mathbb{R}^{2n-1}$; the exactness of L easily implies that the projection $\pi(L_a)$ of L_a to $\mathbb{R}^{2n-2} = \{(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})\}$ is an exact Lagrangian immersion and, further, that the projection has trivial Maslov class. This generalizes the fact that in $\mathbb{R}^4$, the projection $\pi(L_a)$ bounds zero signed area and has winding number zero.

In higher dimensions, Theorem 1.2 generalizes to show the nonexistence of cobordisms that can be extended to unknotted planar Lagrangians. First, the curves $8^1_\pm(r)$ should be generalized to:

$$8^{n-1}_\pm(r) := \{(x_1, \dots, x_n, y_1, \dots, y_{n-1}) : \\ \sum_{i=1}^n x_i^2 = r^2, \quad y_i = \pm 2x_i x_n, \quad i = 1, \dots, n-1\}.$$

A construction similar to that in $\mathbb{R}^4$ provides an unknotted planar Lagrangian whose restriction gives a cobordism from $8^{n-1}_-(R)$ up to $8^{n-1}_-(r)$ when $r < R$. On the other hand:

Theorem 1.3. *If $r \leq R$ and $a < b$, there does not exist an unknotted planar Lagrangian whose restriction to $\{a \leq y_n \leq b\} \subset \mathbb{R}^{2n}$ gives a Lagrangian cobordism L with*

$$(L \cap \{y_2 = a\}) = i_a(8^1_-(r)), \qquad (L \cap \{y_2 = b\}) = i_b(8^1_-(R)).$$

An extension of non-squeezing phenomena in Theorem 1.2 to other boundary slices and to higher dimensions requires replacing the the area of one of the lobes with a generating family "capacity" of the slice. The next subsection describes this new theory of capacities.

1.2. **The Capacity Framework.** Proofs and extensions of Theorems 1.1, 1.2, and 1.3 involve a new notion of a capacity. The theory of capacities was preceded by Gromov's concept of a symplectic radius [9] and was originally developed through variational principles by Ekeland and Hofer [3, 4]. Viterbo [18] gave an alternative definition of a capacity for Lagrangian submanifolds of cotangent bundles using the finite-dimensional technique of generating families. In this paper, we develop a new type of (generating family) capacity. Our techniques build off of Viterbo's — especially those he uses for families of symplectic reductions in [18, §5] — but produce capacities that allow us to study slices of an unknotted planar Lagrangian. More precisely, for each slice L_a of a unknotted planar Lagrangian $L \subset \mathbb{R}^{2n}$ at a generic height a, we define two lower and two upper capacities:

$$c_{\pm}^{L,a} : H^*(L_a) \to (-\infty, 0], \qquad C_{\pm}^{L,a} : H^*(L_a) \to [0, \infty).$$

By analogy with the properties of capacities for subsets of $\mathbb{R}^{2n}$, we have the following theorem:

Theorem 1.4. *The capacities satisfy the following properties:*

Monotonicity: *Suppose $a < b$, $0 \notin [a,b]$, and a and b are generic heights of an unknotted planar Lagrangian L. Let $L_{[a,b]} = \bigcup_{t\in[a,b]} L_t$ be the cobordism between L_a and L_b given by L, and let $j_t : L_t \to L_{[a,b]}$ be the inclusion map. If $u \in H^*(L_{[a,b]})$ then:*

$$c_{+}^{L,a}(j_a^* u) \le c_{+}^{L,b}(j_b^* u), \qquad C_{+}^{L,a}(j_a^* u) \le C_{+}^{L,b}(j_b^* u),$$
$$c_{-}^{L,a}(j_a^* u) \ge c_{-}^{L,b}(j_b^* u), \qquad C_{-}^{L,a}(j_a^* u) \ge C_{-}^{L,b}(j_b^* u).$$

In any of the above relations, equality is possible only when both capacities equal 0.

Continuity: *Given $u \in H^*(L_{[a,b]})$, the function $c(t) = c^{L,t}(j_t^* u)$, where c is any one of the four capacities, is continuous on $[a,b]$ (with removable discontinuities at non-generic levels) and piecewise differentiable.*

Invariance: *If L^0 and L^1 are unknotted planar Lagrangians that are isotopic via a compactly supported symplectic isotopy that fixes the slice at a generic height a, then for any cohomology class $u \in H^*(L^0_a) = H^*(L^1_a)$, we have $c_{\pm}^{L^0,a}(u) = c_{\pm}^{L^1,a}(u)$ and $C_{\pm}^{L^0,a}(u) = C_{\pm}^{L^1,a}(u)$.*

Non-Vanishing: *For any generic, nonempty slice L_a of an unknotted planar Lagrangian L and for any nonzero $u \in H^*(L_a)$, at least one of the four capacities $c_{\pm}^{L,a}(u), C_{\pm}^{L,a}(u)$ is nonzero.*

Conformality: *If L is an unknotted planar Lagrangian and βL denotes the image of L under the dilation $(\mathbf{x},\mathbf{y}) \mapsto (\beta\mathbf{x}, \beta\mathbf{y})$ then, for any generic height a, any of the four capacities c, and any $u \in H^*(L_a) \simeq H^*(\beta L_{\beta a})$,*

$$c^{\beta L,\beta a}(u) = \beta^2 c^{L,a}(u).$$

Although these capacities depend on the entire Lagrangian, it is sometimes possible to compute these numbers only knowing the slice L_a. For example, suppose that the slice L_a agrees with one of our ∞-shaped curves $8^1_\pm(r)$ where the "lobes" each bound a region with area of absolute value A. Then for $0 \neq u \in H^0(L_a)$ and $0 \neq v \in H^1(L_a)$, the following calculation shows that these capacities carry geometric information about the slice:

	$c_+^{L,a}(u)$	$c_-^{L,a}(u)$	$C_+^{L,a}(v)$	$C_-^{L,a}(v)$
	$-A$	0	0	A
	0	$-A$	A	0

These calculations follow from Proposition 6.8.

A generalization of Theorem 1.2 to other shapes of boundary curves and to higher dimensions can be formulated in terms of capacities:

Theorem 1.5. *Let $L \subset \mathbb{R}^{2n}$ denote a Lagrangian ball with $\partial L = (L \cap \{y_n = a\})$ that has been obtained as the restriction of an unknotted planar Lagrangian to $\{y_n \geq a\}$ or to $\{y_n \leq a\}$. If the entire Lagrangian ball L lies in the symplectic cylinder $\mathbb{R}^{2n-2} \times I_{x_n} \times I_{y_n}$, for some intervals $I_{x_n} \subset \mathbb{R}$ of length ℓ and $I_{y_n} \subset \mathbb{R}$ of length w, then for any $u \in H^*(\partial L)$:*

$$\ell \cdot w \geq |c^{L,a}(j_a^* u)|,$$

where c is any of the four capacities.

After rearranging the inequality as $w \geq \frac{1}{\ell}|c^{L,a}(j_a^* u)|$, this theorem can be viewed as a "directionally sensitive" version for planar Lagrangians of the following result by Viterbo [19]: a Lagrangian that is Hamiltonian isotopic to the zero section of T^*T^n and is contained in the unit disk bundle for some metric g has a bounded generating family length $\gamma(L)$.

Beyond these qualitative and quantitative results, the theory of capacities has a number of other extensions and potential applications. For example, for a closed manifold B, it is possible to define capacities for slices of any Lagrangian submanifold of $T^*(B \times \mathbb{R})$ that has a generating family; the statements and proofs go through almost without modification. Further, as discussed in Section 6.4, the capacities can be thought of as fitting into Eliashberg, Givental, and Hofer's Symplectic Field Theory (SFT) framework, the latest generation of pseudo-holomorphic curve invariants [6]. Like a TQFT structure, an SFT-type invariant would assign a group to each slice and a homomorphism to each Lagrangian cobordism between the slices. The generating family capacities defined in this paper follow a similar structure: a capacity assigns a real number to each cohomology class of a slice, and Monotonicity implies that each Lagrangian cobordism gives rise to a relation between capacities. In fact, in the process of defining the capacities, we will have associated filtered groups to each slice. The maps used to prove Monotonicity will turn into filtered homomorphisms between these groups. This framework has possible applications to detecting knotting phenomena in Lagrangian surfaces with fixed boundary, a question parallel to recent progress in the study of Lagrangian cobordisms between Legendrian knots [1, 5].

The remainder of the paper is organized as follows: after reviewing the basics of the theory of generating families in Section 2, we construct and investigate the geometry of generating families for slices of unknotted planar Lagrangians in Section 3. We define the capacities for a slice in Section 4 using the Morse theory of a "difference function" associated to a generating family of a slice. In Section 5, we prove the properties of the capacities listed in Theorem 1.4. In Section 6, we apply the capacity framework to prove Theorems 1.1 – 1.3 and Theorem 1.5 and discuss how to fit the objects used to define the capacities into the framework of a field theory.

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2. Background on Generating Families

As described above, we will develop capacities for slices of generic, unknotted planar Lagrangians in $T^*(\mathbb{R}^n)$. The foundation of our construction is that all such Lagrangians have essentially unique generating families associated to them.

Given a function $f : M \to \mathbb{R}$, the graph of its differential is a Lagrangian submanifold of T^*M. The idea of generating families is to extend this construction by looking at real-valued functions on $M \times \mathbb{R}^N$, for some potentially large N. Suppose that we have a smooth function $F : M \times \mathbb{R}^N \to \mathbb{R}$ such that 0 is a regular value of the map $(\frac{\partial F}{\partial e_1}, \dots, \frac{\partial F}{\partial e_N}) : M \times \mathbb{R}^N \to \mathbb{R}^N$. We define Σ_F, the **fiber critical locus of F**, to be the m-submanifold

$$\Sigma_F := \left\{ (x, \mathbf{e}) \in M \times \mathbb{R}^N : \frac{\partial F}{\partial e_i}(x, \mathbf{e}) = 0, \text{for } i = 1, 2, \dots, N \right\}.$$

Define an immersion $i_F : \Sigma_F \to T^*M$ in local coordinates by

$$i_F(x, \mathbf{e}) = \left(x, \frac{\partial F}{\partial x}(x, \mathbf{e}) \right), \tag{2.1}$$

and let L be the image of i_F. It is easy to check that L is indeed Lagrangian. We say that F **generates** L, or F is a **generating family (of functions)** for L. Note that we can write L as:

$$L = \left\{ \left(x, \frac{\partial F}{\partial x}(x, \mathbf{e}) \right) : \frac{\partial F}{\partial \mathbf{e}}(x, \mathbf{e}) = 0 \right\}.$$

Although the Lagrangian generated by F may in general be immersed, we will start with an embedded Lagrangian and show that there exists a generating family for it.

Example 2.1. Via the theory of generating families, we can explicitly construct a flat-at-infinity (and thus unknotted) planar Lagrangian in $\mathbb{R}^4$ whose restriction gives the cobordisms between the curves in Figure 1(a), (c). To construct such a Lagrangian for the curves in (a), choose $0 < \varepsilon < \beta < \tau < K$, and consider $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x_1, x_2) = \ell(x_2)q(x_1) - d(x_1)c(x_2),$$

where ℓ, q, d and c are compactly supported functions such that, on the square $|x_1|, |x_2| \leq \sqrt{K - \varepsilon}$,

$$\ell(x_2) = x_2, \quad q(x_1) = K - x_1^2, \quad d(x_1) = 1, \quad c(x_2) = \frac{1}{3}x_2^3,$$

and, on the complement of this square,

$$\ell'(x_2) < 1, \quad q(x_1) < \varepsilon, \quad d(x_1) < 1, \quad -c'(x_2) < \beta - \varepsilon.$$

Then F generates the flat-at-infinity planar Lagrangian L satisfying

$$\begin{aligned} L = \{(x_1, y_1, x_2, y_2) : \quad & y_1 = \ell(x_2)q'(x_1) - d'(x_1)c(x_2), \\ & y_2 = \ell'(x_2)q(x_1) - d(x_1)c'(x_2) \ \}, \end{aligned}$$

and it is easy to verify that we get the desired 8^1_- curves as pictured in Figure 1(a): for $\beta \leq a \leq \tau$,

$$\begin{aligned} L \cap \{y_2 = a\} &= \{(x_1, x_2, -2x_1x_2, a) : x_1^2 + x_2^2 = K - a\}, \\ &= i_a(8^1_-(\sqrt{K - a})). \end{aligned}$$

To construct an unknotted planar Lagrangian between the 8^1_+ curves in Figure 1(c), simply consider the function $G = -F$, and the desired curves occur for $-\tau \leq a \leq -\beta$.

These constructions can be easily generalized to higher dimensions: we construct a Lagrangian of the form Γ_{dF} where $F : \mathbb{R}^n \to \mathbb{R}$ is compactly supported and is of the form

$$F(x_1, \dots, x_n) = \ell(x_n)q(x_1, \dots, x_{n-1}) - d(x_1, \dots, x_{n-1})c(x_n).$$

Now, on an n-dimensional cube,

$$\begin{aligned} &\ell(x_n) = x_n, \quad q(x_1, \dots, x_{n-1}) = K - x_1^2 - \cdots - x_{n-1}^2, \\ &d(x_1, \dots, x_{n-1}) = 1, \quad c(x_n) = \frac{1}{3}x_n^3, \end{aligned}$$

and outside this cube, the functions ℓ, q, d, c satisfy analogous conditions to those given in the $n = 2$ case above. Then Γ_{dF} is flat-at-infinity and, for $a > \beta$, the L_a slices agree with $8^{n-1}_-(\sqrt{K - a})$. Also note that Γ_{dF} is unknotted: it is not hard to explicitly construct a symplectic isotopy taking Γ_{dF} to the zero section. $\diamond$

If there is a generating family for a given Lagrangian L then it is easy to see that it is not unique: if $F : M \times \mathbb{R}^N \to \mathbb{R}$ generates L then so does, for example, $F' : M \times \mathbb{R}^{N+1} \to \mathbb{R}$, where $F'(x, e_1, \dots, e_N, e_{N+1}) = F(x, e_1, \dots, e_N) + e_{N+1}^2$. This is the first of three basic operations on a generating family that will not change the Lagrangian that is produced.

Definition 2.2. Two generating families $F_i : M \times \mathbb{R}^{N_i} \to \mathbb{R}$, $i = 1, 2$, are **equivalent** if they can be made equal after the operations of addition of a constant, fiber-preserving diffeomorphism, and stabilization, where these operations are defined as follows:

(1) Given a generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$, let $Q : \mathbb{R}^K \to \mathbb{R}$ be a non-degenerate quadratic function. Define $F \oplus Q : M \times \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ by $F \oplus Q(x, \mathbf{e}, \mathbf{e}') = F(x, \mathbf{e}) + Q(\mathbf{e}')$. Then $F \oplus Q$ is a **stabilization** of F.
(2) Given a generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$ and a constant $C \in \mathbb{R}$, $F' = F + C$ is said to be obtained from F by **addition of a constant**.
(3) Given a generating family $F : M \times \mathbb{R}^N \to \mathbb{R}$, suppose $\Phi : M \times \mathbb{R}^N \to M \times \mathbb{R}^N$ is a fiber-preserving diffeomorphism, i.e., $\Phi(x, \mathbf{e}) = (x, \phi_x(\mathbf{e}))$ for diffeomorphisms ϕ_x . Then $F' = F \circ \Phi$ is said to be obtained from F by a **fiber-preserving diffeomorphism**.

By construction, these generating families are defined on non-compact domains. Analytically, it is convenient to consider functions that are well-behaved outside of a compact set. A common convention has been to consider generating families $F : M \times \mathbb{R}^N \to \mathbb{R}$ that are **quadratic-at-infinity**. This means that outside of a compact set in $M \times \mathbb{R}^N$, $F(x, \mathbf{e}) = Q(\mathbf{e})$, where Q is a non-degenerate quadratic function. See, for example, Viterbo [18] and Théret [14]. Another useful concept is to consider generating families $F : M \times \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ that are **linear-quadratic-at-infinity**: this means that outside a compact set in $M \times \mathbb{R}^N \times \mathbb{R}^K$, $F(x, \mathbf{l}, \mathbf{e}) = J(\mathbf{l}) + Q(\mathbf{e})$, where J is a nonzero linear function of $\mathbf{l}$ and Q is a non-degenerate quadratic function of $\mathbf{e}$; see, for example, [10].

The following theorem gives valuable existence and uniqueness results for the unknotted planar Lagrangians considered in this paper. The existence portion was proved by Sikorav in [13] via the "broken geodesic method" of Laudenbach and Sikorav [11]; existence can also be proved by a formula devised by Chekanov; see, for example, [16]. The uniqueness portion was proved by Viterbo [18] with precise details given by Théret [14].

Theorem 2.3 (Existence and Uniqueness of Generating Families)**.** *For $t \in [0, 1]$, let $L_t \subset T^*\mathbb{R}^n$ be the image of the zero section, L_0, under a compactly supported Hamiltonian isotopy ϕ_t of $T^*\mathbb{R}^n$. Then:*

(1) *There exists a quadratic-at-infinity generating family for L_t: if F is any quadratic-at-infinity generating family for L_0 then there exists a path of quadratic-at-infinity generating families $F_t : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ for $L_t = \phi_t(L_0)$ so that F_0 is a stabilization of F and $F_t = F_0$ outside a compact set.*
(2) *Any two quadratic-at-infinity generating families for L_t are equivalent.*

Remark 2.4. In fact, the existence portion of Theorem 2.3 can be formulated more generally as a Serre fibration [14] and applies to $L_t = \phi_t(L)$, where L is any Lagrangian that has a quadratic-at-infinity generating family.

Remark 2.5. The results quoted above are actually stated for Hamiltonian isotopies of the zero-section of T^*M where M is compact. Since we are dealing with compactly supported isotopies, we may think of our setting as T^*S^n.

In our terminology, Theorem 2.3 says that any unknotted planar Lagrangian in $\mathbb{R}^{2n}$ has a unique (up to equivalence) quadratic-at-infinity generating family associated to it. In $\mathbb{R}^4$, Theorem 2.3 applies to any flat-at-infinity planar Lagrangian since Eliashberg and Polterovich proved that such a Lagrangian must be unknotted [7].

3. Generating Families and Difference Functions for Lagrangian Slices

In this section, we discuss generating families F_a for slices L_a; the Morse theory of the difference functions associated to such generating families will be used in the next section to construct the capacities. The inspiration for the construction and use of a difference function — as opposed to Viterbo's direct use of the generating family in his capacities — comes from the second author's study of two-component Legendrian links [17]. There, the idea was to associate generating families $F_1(x, \mathbf{e}_1)$ and $F_2(x, \mathbf{e}_2)$ to each component of the link and then study topological invariants of the "difference" Δ of these generating families, where $\Delta : M \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ is given by $\Delta(x, \mathbf{e}_1, \mathbf{e}_2) = F_1(x, \mathbf{e}_1) - F_2(x, \mathbf{e}_2)$. The critical points of the difference function pick out intersections of the Lagrangian projections of the components of the link. Here, we will use a similar tactic to study the geometry of the slices L_a using differences between F_a and itself.

3.1. **Generating Families for Lagrangian Slices.** Let $L \subset \mathbb{R}^{2n}$ be an unknotted planar Lagrangian. By Theorem 2.3, L has a quadratic-at-infinity generating family $F : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ that is unique up to stabilization, fiber diffeomorphism, and an overall constant. Write the coordinates on the domain of F as $(\mathbf{x}, x_n, \mathbf{e})$, where $\mathbf{x} \in \mathbb{R}^{n-1}$.

To study a slice L_a of the Lagrangian, we first consider a new generating family $F_a : \mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \to \mathbb{R}$ given by

$$F_a(\mathbf{x}, x_n, \mathbf{e}) = F(\mathbf{x}, x_n, \mathbf{e}) - a\, x_n.$$

For F_a, we are considering x_n and $\mathbf{e}$ as fiber variables, and so F_a generates a Lagrangian in $T^*(\mathbb{R}^{n-1}) = \mathbb{R}^{2n-2}$ which will, in fact, be a projection of our slice. Let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$ denote the projection $\pi(\mathbf{x}, x_n, \mathbf{y}, y_n) = (\mathbf{x}, \mathbf{y})$.

Proposition 3.1. *If L is transverse to the hypersurface $\{y_n = a\}$ then the function F_a is a linear-quadratic-at-infinity generating family for the exact Lagrangian immersion $\pi(L_a)$.*

Proof. To see that F_a is linear-quadratic-at-infinity, note that since F is equal to a non-degenerate quadratic $Q(\mathbf{e})$ outside a compact set in $\mathbb{R}^{n-1} \times \mathbb{R}^{1+N}$, we have $F_a = -ax_n + Q(\mathbf{e})$ there as well. The fact that 0 is a regular value of $\left(\frac{\partial F}{\partial e_1}, \dots, \frac{\partial F}{\partial e_N}\right) : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$, and the hypothesis that L is transverse to $\{y_n = a\}$ will guarantee that 0 is a regular value of $\left(\frac{\partial F_a}{\partial x_n}, \frac{\partial F_a}{\partial e_1}, \dots, \frac{\partial F_a}{\partial e_N}\right) : \mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}$, and hence that F generates *some* Lagrangian. To see that F_a generates the claimed Lagrangian, we compute that F_a generates the following set:

$$\left\{ \left(\mathbf{x}, \frac{\partial F_a}{\partial \mathbf{x}}(\mathbf{x}, x_n, \mathbf{e}) \right) : \frac{\partial F_a}{\partial x_n}(\mathbf{x}, x_n, \mathbf{e}) = 0, \frac{\partial F_a}{\partial \mathbf{e}}(\mathbf{x}, x_n, \mathbf{e}) = 0 \right\}.$$

Rewriting this set using the fact that F generates L leaves us with:

$$\{(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, x_n, \mathbf{y}, a) \in L\},$$

which is simply the projection $\pi(L_a)$. □

Remark 3.2. Since $\pi(L_a)$ is an exact Lagrangian immersion into $\mathbb{R}^{2n-2}$, it lifts to an immersed Legendrian submanifold Λ_a in $J^1(\mathbb{R}^{n-1})$ with its usual contact structure.

3.2. The Difference Function.

Define the **difference function**

$$\Delta_a : \mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \times \mathbb{R}^{1+N} \to \mathbb{R}$$

by

$$\Delta_a(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) = F_a(\mathbf{x}, x_n, \mathbf{e}) - F_a(\mathbf{x}, \tilde{x}_n, \tilde{\mathbf{e}}).$$

We will see that, for generic F, Δ_a is a Morse-Bott function. A **generic Lagrangian** will be one that has a generating family whose difference function satisfies a Morse-Bott condition. In this and the next section, we will always be assuming that we are working with a generic Lagrangian. The results stated in Theorems 1.1 – 1.3 and Theorem 1.5 will hold for arbitrary Lagrangian cobordisms since any Lagrangian can be perturbed into a generic Lagrangian.

The capacities of a slice L_a will be constructed using the Morse theory of Δ_a, so it is important to identify its critical points.

Lemma 3.3. *The critical points of Δ_a are of two types:*

1. *For each double point $(\mathbf{x}, \mathbf{y})$ of $\pi(L_a)$, there are two critical points $(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}})$ and $(\mathbf{x}, \tilde{x}_n, \tilde{\mathbf{e}}, x_n, \mathbf{e})$ whose critical values are either both 0 or are $\pm v$, for some $v \neq 0$.*
2. *The set*

$$C_a = \left\{ (\mathbf{x}, x_n, \mathbf{e}, x_n, \mathbf{e}) \;:\; (\mathbf{x}, x_n, \mathbf{e}) \in \Sigma_{F_a} \right\}$$

is a critical submanifold of Δ_a with critical value 0.

For generic F, these critical points and submanifolds are non-degenerate and C_a has index $1 + N$.

Proof. It is straightforward to calculate that at a critical point $(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}})$ of Δ_a, the points $(\mathbf{x}, x_n, \mathbf{e})$ and $(\mathbf{x}, \tilde{x}_n, \tilde{\mathbf{e}})$ both lie in Σ_F. Further, their images in L both lie in L_a and have the same $\mathbf{y}$ coordinate. Thus, the critical points of Δ_a correspond to pairs of points in L of the form $(\mathbf{x}, x_n, \mathbf{y}, a)$ and $(\mathbf{x}, \tilde{x}_n, \mathbf{y}, a)$. The existence of the two types of critical points asserted in the lemma follows: a set coming from double points of the projection $\pi(L_a)$, and a set C_a coming from any point in L_a paired with itself.

To understand the non-degeneracy claim, note that if $(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}})$ is a critical point of Δ_a then $(\mathbf{x}, x_n, \mathbf{e}), (\mathbf{x}, \tilde{x}_n, \tilde{\mathbf{e}}) \in \Sigma_{F_a}$. So, after a fiber-preserving diffeomorphism of $\mathbb{R}^{n-1} \times \mathbb{R}^{1+N}$, we may assume that in a neighborhood of a critical point $(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}})$ of Δ_a, there exist functions $g(\mathbf{x})$, $h(x_n, \mathbf{e})$, $\tilde{g}(\mathbf{x})$, and $\tilde{h}(\tilde{x}_n, \tilde{\mathbf{e}})$ such that

$$F_a(\mathbf{x}, x_n, \mathbf{e}) = g(\mathbf{x}) + h(x_n, \mathbf{e}), \quad F_a(\mathbf{x}, \tilde{x}_n, \tilde{\mathbf{e}}) = \tilde{g}(\mathbf{x}) + \tilde{h}(\tilde{x}_n, \tilde{\mathbf{e}}).$$

If A (resp. $\tilde{A}$) is the Hessian d^2h at $(x_n, \mathbf{e})$ (resp. $d^2\tilde{h}$ at $(\tilde{x}_n, \tilde{\mathbf{e}})$) then:

$$d^2\Delta_a(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) = \begin{bmatrix} d^2g - d^2\tilde{g}(\mathbf{x}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A(x_n, \mathbf{e}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\tilde{A}(\tilde{x}_n, \tilde{\mathbf{e}}) \end{bmatrix}. \tag{3.1}$$

Since F_a is a generating family, A and $\tilde{A}$ are non-degenerate. At an isolated critical point, for generic F, the upper left entry is non-degenerate, so $d^2\Delta_a$ is non-degenerate. For a point in C_a, the kernel of $d^2\Delta_a$ is spanned by vectors of the form $[\mathbf{v}, \mathbf{0}, \mathbf{0}]$, which coincides with TC_a; the non-degeneracy follows, as does the claim about the index of the critical submanifold. □

Remark 3.4. We can similarly define a difference function for the generating family F for L by:

$$\delta(\mathbf{x}, x_n, \mathbf{e}, \tilde{\mathbf{e}}) = F(\mathbf{x}, x_n, \mathbf{e}) - F(\mathbf{x}, x_n, \tilde{\mathbf{e}}).$$

By the same proof as above, the assumption that L is embedded implies that all critical points of δ have critical value zero.

Example 3.5. For the generating family $F : \mathbb{R}^2 \to \mathbb{R}$ constructed in Example 2.1, it is straightforward to check that for $\beta < a < \tau$, all critical points of Δ_a occur when $|x_1|, |x_2|, |\tilde{x}_2| < \sqrt{K - \varepsilon}$. In this region, we have $F(x_1, x_2) \equiv x_2(K - x_1^2) - \frac{1}{3}x_2^3$, so the isoloated critical points are:

$$q_- = \left(0, \sqrt{K-a}, -\sqrt{K-a}\right), \qquad \text{and}$$
$$q_+ = \left(0, -\sqrt{K-a}, \sqrt{K-a}\right),$$

while the critical submanifold is given by $\big(s, t, t\big)$, where $s^2 + t^2 = K - a$. Using the explicit definition of F, we find that the critical values are:

$$\Delta_a(q_-) = \frac{4}{3}(K-a)^{\frac{3}{2}} > 0, \quad \Delta_a(q_+) = -\frac{4}{3}(K-a)^{\frac{3}{2}} < 0,$$

while the indices are given by

$$\mathrm{Ind}(q_-) = 3, \quad \mathrm{Ind}(q_+) = 0.$$

Furthermore, it can be explicitly seen that $C_a = \{(s,t,t) : s^2 + t^2 = K - a\}$ is a non-degenerate critical submanifold of critical value 0 and index 1. $\diamond$

3.3. **Morse Theory for Δ_a on Split Domains.** In order to detect information about the sign of the crossings $\pi(L_a)$, the capacities will be defined using intersections of the sublevel sets of Δ_a with "positive" and "negative" half-spaces of the domain. Since the boundary between these half-spaces is not a level set of Δ_a, we cannot directly use the usual Morse-theoretic constructions, but we shall see that certain techniques still work.

To set notation, denote the sublevel sets of Δ_a by:

$$\Delta_a^\lambda = \big\{(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) \in \mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \times \mathbb{R}^{1+N} \ : \ \Delta_a(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) \leq \lambda\big\}.$$

The distinction between positive and negative capacities comes from a splitting of the domain $\mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \times \mathbb{R}^{1+N}$ into positive and negative pieces as follows:

$$\mathcal{P}_+ = \big\{(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) \ : \ x_n \leq \tilde{x}_n\big\},$$
$$\mathcal{P}_- = \big\{(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) \ : \ x_n \geq \tilde{x}_n\big\}.$$

We divide sublevel sets into positive and negative pieces as well:

$$\Delta_{a,\pm}^\lambda = \Delta_a^\lambda \cap \mathcal{P}_\pm. \tag{3.2}$$

The motivation for this splitting comes from the fact that

$$\Delta_a(\mathbf{x}) - \Delta_b(\mathbf{x}) = (a-b)(\tilde{x}_n - x_n). \tag{3.3}$$

In particular, on $\mathcal{P}_+$, $a < b$ implies $\Delta_b \geq \Delta_a$, and thus $\Delta_b^\lambda \subset \Delta_a^\lambda$; while on $\mathcal{P}_-$, $a < b$ implies $\Delta_a^\lambda \subset \Delta_b^\lambda$. Thus, for $a < b$ and for any λ, we have

$$\Delta_{b,+}^\lambda \subset \Delta_{a,+}^\lambda \text{ and } \Delta_{a,-}^\lambda \subset \Delta_{b,-}^\lambda. \tag{3.4}$$

This type of containment will be instrumental in the proof of Monotonicity for the capacities.

The intersections of sublevel sets with half-spaces obey a central Morse-theoretic lemma: if there is no critical value of Δ_a in $[\sigma, \tau]$ then each half of the sublevel set at σ is a deformation retract of the corresponding half of the sublevel set at τ. The proof of the following lemma relies crucially on the assumption that L is embedded.

Lemma 3.6. *If there are no critical points of Δ_a in $\mathcal{P}_+$ with critical value in the interval $[\sigma, \tau]$ then $\Delta_{a,+}^\sigma$ is a deformation retract of $\Delta_{a,+}^\tau$. A similar statement holds with "$+$" replaced by "$-$".*

Proof. Let $\mathcal{P}_0 = \{(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) : x_n = \tilde{x}_n\}$. The idea of the proof is to modify the gradient of Δ_a to an integrable vector field X such that:

1. The derivative $X(\Delta_a)$ is positive and uniformly bounded away from 0 on $\Delta_a^{-1}([\sigma, \tau])$, and
2. X is tangent to $\mathcal{P}_0$ (and hence its flow preserves $\mathcal{P}_\pm$).

Following the negative flow of such a vector field clearly gives the desired deformation retract.

To construct the vector field X, first decompose $\nabla\Delta_a$ into a component X_T parallel to $\mathcal{P}_0$ and a component X_N normal to $\mathcal{P}_0$. To define X, let $\beta_\epsilon(t)$ be a smooth nonnegative function that is equal to 1 outside of $(-\epsilon,\epsilon)$ for some positive ϵ to be chosen, does not exceed 1, and vanishes at 0. Then let:

$$X = X_T + \beta_\epsilon(\tilde{x}_n - x_n)X_N.$$

It is clear by construction that X is tangent to $\mathcal{P}_0$. To prove that $X(\Delta_a)$ is uniformly bounded away from 0 on $\Delta_a^{-1}([\sigma,\tau])$, split the domain into an ϵ-neighborhood of $\mathcal{P}_0$ and its complement. We will show that on each piece, there exists an $r > 0$ such that $X(\Delta_a) \geq r$.

First consider an ϵ-neighborhood of $\mathcal{P}_0$, with ϵ still to be determined. It is easy to check that the restriction of Δ_a to $\mathcal{P}_0$ is the difference function δ for the original Lagrangian L as in Remark 3.4. Thus, along $\mathcal{P}_0$, we have $X_T = \nabla\delta$. The embeddedness of L implies that δ has no critical values in $[\sigma,\tau]$. Further, since δ is the difference of two functions that are quadratic-at-infinity, it is straightforward to prove that there is an $r > 0$ such that $\nabla\delta(\Delta_a) = \|\nabla\delta\|^2 \geq 2r$ on $\delta^{-1}([\sigma,\tau])$. Choose $\epsilon > 0$ so that $\|X_T\|^2 \geq r$ on an ϵ-neighborhood of $\mathcal{P}_0$; this immediately implies that $\|X\|^2 \geq r$ there as well. Outside that neighborhood, we have $X(\Delta_a) = \|\nabla\Delta_a\|^2$. As with δ, since there are no critical values of Δ_a in $[\sigma,\tau]$ and Δ_a is the difference of two functions that are linear-quadratic at infinity, it is straightforward to prove that there is an $r > 0$ such that $\|\nabla\Delta_a\|^2 \geq 2r$ on $\Delta_a^{-1}([\sigma,\tau])$. □

4. Capacities for Lagrangian Slices

The goal of this section is to associate to a generic slice L_a and each element $u \in H^k(L_a)$ four real numbers $c_\pm$, $C_\pm$. These capacities will be defined using Morse-theoretic techniques applied to the split sublevel sets $\Delta^\lambda_{a,\pm}$ and will satisfy the five properties listed in Theorem 1.4.

4.1. **Definition of the Capacities.**

To define the capacities, we will need to examine maps between the relative cohomology groups of the sublevel sets $\Delta^\lambda_{a,\pm}$ and the cohomology groups of L_a. As has been the convention, assume that the domain of Δ_a is $\mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \times \mathbb{R}^{1+N}$. At some point, we will need to assume that $N \geq n-1$, a condition that can be guaranteed by stabilization. Suppose throughout this section that $\lambda < -\eta < 0 < \eta < \Lambda$ and that η has been chosen small enough so that 0 is the only critical value of the difference function Δ_a in $[-\eta,\eta]$. By Lemma 3.6, the precise choice of η is immaterial. For all $k \in \mathbb{Z}_{\geq 0}$, we will define maps $i^*_\pm$, p^λ_a, D^Λ_a, and $\mathcal{I}$

whose domains and ranges are related as follows:

$$\begin{array}{ccccc} & & H^k(L_a) & & \\ & & \big\downarrow \mathcal{I} & & \\ & & H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}) & & \\ & \swarrow^{p_a^\lambda} & & \searrow^{D_a^\Lambda} & \\ H^{k+N+1}(\Delta_a^\eta, \Delta_a^\lambda) & & & & H^{k+N+2}(\Delta_a^\Lambda, \Delta_a^\eta) \\ \big\downarrow i_\pm^* & & & & \big\downarrow i_\pm^* \\ H^{k+N+1}(\Delta_{a,\pm}^\eta, \Delta_{a,\pm}^\lambda) & & & & H^{k+N+2}(\Delta_{a,\pm}^\Lambda, \Delta_{a,\pm}^\eta). \end{array}$$

The maps $i_\pm^*$ come from a Mayer-Vietoris sequence. One reason that the splitting using $\mathcal{P}_\pm$ is nice is that the relative cohomology of two sublevel sets is completely determined by the relative cohomologies of their splittings.

Lemma 4.1. *Suppose that $0 \notin [\sigma, \tau]$. Then, for any $k \in \mathbb{Z}_{\geq 0}$, the natural inclusions induce an isomorphism:*

$$(i_+^*, i_-^*) : H^k(\Delta_a^\tau, \Delta_a^\sigma) \to H^k(\Delta_{a,+}^\tau, \Delta_{a,+}^\sigma) \oplus H^k(\Delta_{a,-}^\tau, \Delta_{a,-}^\sigma).$$

Proof. This will follow from a Mayer-Vietoris argument. As in the proof of Lemma 3.6, let $\mathcal{P}_0 = \{(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) : x_n = \tilde{x}_n\}$ and let $\Delta_{a,0}^\lambda = \Delta_{a,+}^\lambda \cap \Delta_{a,-}^\lambda$; that is, $\Delta_{a,0}^\lambda = \Delta_a^\lambda \cap \mathcal{P}_0$. It suffices to prove that $H^k(\Delta_{a,0}^\tau, \Delta_{a,0}^\sigma) = 0$. In fact, we will show that $\Delta_{a,0}^\sigma$ is a deformation retract of $\Delta_{a,0}^\tau$. Since $\Delta_a|_{\mathcal{P}_0}$ coincides with the difference function δ for F introduced in Remark 3.4, and since the embeddedness of L implies since δ has no nonzero critical values, we can use $-\nabla\delta$ to deformation retract $\Delta_{a,0}^\tau$ to $\Delta_{a,0}^\sigma$. □

The maps p_a^λ and D_a^Λ are defined from examining long exact sequences of triples:

Definition 4.2. For each $k \in \mathbb{Z}_{\geq 0}$, in the long exact sequence of the triple $(\Delta_a^\eta, \Delta_a^{-\eta}, \Delta_a^\lambda)$, let p_a^λ be the projection homomorphism:

$$p_a^\lambda : H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}) \to H^{k+N+1}(\Delta_a^\eta, \Delta_a^\lambda).$$

Similarly, in the long exact sequence of the triples $(\Delta_a^\Lambda, \Delta_a^\eta, \Delta_a^{-\eta})$, let

$$D_a^\Lambda : H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}) \to H^{k+N+2}(\Delta_a^\Lambda, \Delta_a^\eta)$$

be the connecting homomorphism.

Lastly, we use a Gysin sequence to define the map $\mathcal{I}$.

Lemma 4.3. *For $k \in \mathbb{Z}_{\geq 0}$, by stabilizing if necessary, assume that $k + N \geq n - 1$. Then there exists an injective map*

$$\mathcal{I} : H^k(L_a) \to H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}).$$

Moreover, $\mathcal{I}$ is an isomorphism when there are no non-degenerate critical points with critical value 0 *and index* $k+N+1$.

Proof. Recall that Δ_a always has a non-degenerate critical submanifold C_a diffeomorphic to L_a with critical value 0 and index $N+1$; suppose that there are m non-degenerate critical points, $z_1, \dots, z_m$, of critical value 0 and index $k+N+1$. Let $W^-(C_a)$ denote the descending disk bundle of the the critical set C_a. Standard Morse-Bott theory says that the homotopy type of Δ_a^η is obtained from that of $\Delta_a^{-\eta}$ by attaching one $(k+N+1)$-cell for each z_i and attaching the (oriented) disk bundle $W^-(C_a)$ along its bounding sphere bundle. Thus we have an isomorphism:

$$H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}) \simeq H^{k+N+1}(W^-(C_a), \partial W^-(C_a)) \oplus \mathbb{Z}^m.$$

Let $\rho : H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}) \to H^{k+N+1}(W^-(C_a), \partial W^-(C_a))$ denote the associated (surjective) projection map with ρ^{-1} the associated inclusion. Lastly, to identify $H^{k+N+1}(W^-(C_a), \partial W^-(C_a))$ with $H^k(L_a)$, we consider the Gysin sequence

$$\cdots \to H^{k+N+1}(C_a) \to H^{k+N+1}(W^-(C_a), \partial W^-(C_a)) \xrightarrow{\pi_*} H^k(C_a) \to H^{k+N+2}(C_a) \to \cdots .$$

Since C_a is diffeomorphic to the $(n-1)$-dimensional L_a, our hypothesis that $N \geq n-1$ implies that $k+N+1 > n-1$, and thus π_* is an isomorphism. The desired injective map $\mathcal{I}$ is defined to be $(\rho)^{-1} \circ (\pi_*)^{-1}$. □

We now have the maps necessary for the definition of the capacities:

$$\varphi_{a,\pm}^\lambda : H^k(L_a) \to H^{k+N+1}(\Delta_{a,\pm}^\eta, \Delta_{a,\pm}^\lambda)$$

is given by $\varphi_{a,\pm}^\lambda = i_\pm^* \circ p_a^\lambda \circ \mathcal{I}$, while

$$\Phi_{a,\pm}^\Lambda : H^k(L_a) \to H^{k+N+2}(\Delta_{a,\pm}^\Lambda, \Delta_{a,\pm}^\eta)$$

is given by $\Phi_{a,\pm}^\Lambda = i_\pm^* \circ D_a^\Lambda \circ \mathcal{I}$.

Definition 4.4. For $u \in H^k(L_a)$, the **positive and negative lower capacities**, $c_\pm^{L,a}(u) \in (-\infty, 0]$, are defined to be

$$c_\pm^{L,a}(u) = \sup\{\lambda < 0 : \varphi_{a,\pm}^\lambda(u) = 0\};$$

if the set on the right hand side is empty then $c_\pm^{L,a}(u) = 0$. For $u \in H^k(L_a)$, the **positive and negative upper capacities**, $C_\pm^{L,a}(u) \in [0, \infty)$, are defined to be

$$C_\pm^{L,a}(u) = \inf\{\Lambda > 0 : \Phi_{a,\pm}^\Lambda(u) \neq 0\};$$

if the set on the right hand side is empty then $C_\pm^{L,a}(u) = 0$.

4.2. **Foundational Properties.** In this section, we prove two important technical properties, the first of which is that the capacities always occur at the critical values of the difference function Δ_a.

Lemma 4.5. *For all $u \in H^k(L_a)$, $c_{\pm}^{L,a}(u)$ and $C_{\pm}^{L,a}(u)$ are critical values of Δ_a.*

Proof. We will prove the proposition for the capacities $c_{\pm}^{L,a}(u)$, as the proof for the capacities $C_{\pm}^{L,a}(u)$ is entirely similar.

First notice that if $\varphi_{a,\pm}^{\lambda}(u) \neq 0$, for all λ, then by definition $c_{\pm}^{L,a}(u) = 0$, and by Lemma 3.3, 0 is always a critical value of Δ_a. Next, we will show that if $\varphi_{a,\pm}^{\lambda}(u) = 0$ for some λ that is a non-critical value of Δ_a, then $\lambda < c_{\pm}^{L,a}(u)$. Suppose there are no critical values of Δ_a in $[\lambda, \nu]$. By Lemma 3.6, the inclusion map of pairs

$$i : (\Delta_{a,\pm}^{\eta}, \Delta_{a,\pm}^{\lambda}) \to (\Delta_{a,\pm}^{\eta}, \Delta_{a,\pm}^{\nu})$$

induces an isomorphism on relative cohomology. The commutativity of the diagram

$$\begin{array}{ccc} & & H^{k+N+1}(\Delta_{a,\pm}^{\eta}, \Delta_{a,\pm}^{\lambda}) \\ & \overset{\varphi_{a,\pm}^{\lambda}}{\nearrow} & \\ H^k(L_a) & & \Big\downarrow (i^*)^{-1} \\ & \underset{\varphi_{a,\pm}^{\nu}}{\searrow} & \\ & & H^{k+N+1}(\Delta_{a,\pm}^{\eta}, \Delta_{a,\pm}^{\nu}) \end{array}$$

shows that if $\varphi_{a,\pm}^{\lambda}(u) = 0$ then $\varphi_{a,\pm}^{\nu}(u) = 0$. Thus, we obtain $\lambda < c_{\pm}^{L,a}(u)$. $\square$

Corollary 4.6. *Suppose $0 \neq u \in H^k(L_a)$. Then $c_{\pm}^{L,a}(u) = 0$ if and only if $\varphi_{a,\pm}^{\lambda}(u) \neq 0$ for all λ, and $C_{\pm}^{L,a}(u) = 0$ if and only if $\Phi_{a,\pm}^{\Lambda}(u) = 0$ for all Λ.*

Proof. We will prove this for $c_{\pm}^{L,a}(u)$; the argument for $C_{\pm}^{L,a}(u)$ is analogous.

By definition, if $\varphi_{a,\pm}^{\lambda}(u) \neq 0$, for all λ, then $c_{\pm}^{L,a}(u) = 0$. To show the converse, first let V^- denote the set of negative critical values of Δ_a, and let $m = \sup V^-$. Note that $m < 0$ since the set of critical values of Δ_a is discrete (as Δ_a is linear-quadratic-at-infinity). We will show that if $0 \neq u \in \ker \varphi_{a,\pm}^{\lambda}$ for some λ, then $c_{\pm}^{L,a}(u) \leq m$. If not, then we can assume $u \in \ker \varphi_{a,\pm}^{\lambda}$ for some $m < \lambda < -\eta$. But then by examining the long exact sequence of the triple $(\Delta_a^{\eta}, \Delta_a^{-\eta}, \Delta_a^{\lambda})$, we find that the map p_a^{λ} is injective, and thus $0 \neq u \in H^k(L_a)$ cannot be in the kernel of $\varphi_{a,\pm}^{\lambda}$. $\square$

The second important technical property is that the capacities are independent of the choice of generating family F of L.

Lemma 4.7. *For all $u \in H^k(L_a)$, the capacities $c_{\pm}^{L,a}(u)$ and $C_{\pm}^{L,a}(u)$ are independent of the generating family F used to define L.*

Proof. It suffices to show that the capacities are unchanged if F is altered by the addition of a constant, a fiber-preserving diffeomorphism, or a stabilization.

If F' is obtained from F by the addition of a constant then the corresponding difference function Δ'_a agrees with the difference function Δ_a, and thus the capacities are unchanged.

Next, suppose that F' is obtained from F by a fiber-preserving diffeomorphism. Then $F'_a(\mathbf{x}, x_n, \mathbf{e}) = F(\mathbf{x}, x_n, \phi_{(\mathbf{x},x_n)}(\mathbf{e})) - ax_n$. It follows that there is a $\mathcal{P}_\pm$-preserving diffeomorphism of $\mathbb{R}^{n-1} \times \mathbb{R}^{1+N} \times \mathbb{R}^{1+N}$ taking the sublevel sets Δ_a^λ to $(\Delta'_a)^\lambda$, for all λ. The naturality of the long exact sequences then implies that the capacities are unchanged.

Lastly, suppose F' is obtained from F by stabilization, i.e., $F'(\mathbf{x}, x_n, \mathbf{e}, \mathbf{e}') = F(\mathbf{x}, x_n, \mathbf{e}) + Q(\mathbf{e}')$, for some non-degenerate quadratic function $Q : \mathbb{R}^q \to \mathbb{R}$. It then follows that the associated difference function Δ'_a is a stabilization of Δ_a:

$$\Delta'_a(\mathbf{x}, x_n, \mathbf{e}, \mathbf{e}', \tilde{x}_n, \tilde{\mathbf{e}}, \tilde{\mathbf{e}}') = \Delta_a(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) + Q(\mathbf{e}') - Q(\tilde{\mathbf{e}}').$$

In general, for any function $G : \mathbb{R}^m \to \mathbb{R}$ and any non-degenerate quadratic Q, we will show that for $a < b$, there is a natural isomorphism:

$$H^*(G^b, G^a) \simeq H^{*+\operatorname{Ind} Q}((G \oplus Q)^b, (G \oplus Q)^a). \tag{4.1}$$

It suffices to consider $Q : \mathbb{R} \to \mathbb{R}$ with either $Q(e) = e^2$ or $Q(e) = -e^2$. For the first case, it is easy to see that the pair $(G \oplus Q)^b, (G \oplus Q)^a)$ deformation retracts to (G^b, G^a). For the second case, the isomorphism in Equation (4.1) can be seen as follows: after a homeomorphism and deformation retract, $((G \oplus Q)^b, (G \oplus Q)^a)$ becomes

$$\begin{aligned}\big((\mathbb{R}^m \times (-\infty, -1] \cup [1, \infty)) \cup G^b \times [-1, 1],& \\ (\mathbb{R}^m \times (-\infty, -1] \cup [1, \infty)) \cup G^a \times [-1, 1]\big).&\end{aligned} \tag{4.2}$$

By excision, the cohomology groups of the pair in Equation (4.2) are isomorphic to:

$$H^*\big(G^b \times [-1, 1], (G^b \times \{\pm 1\}) \cup (G^a \times [-1, 1])\big), \tag{4.3}$$

which can be regarded as the cohomology groups of a suspension of (G^b, G^a), as desired. □

5. Properties of the Capacities

With the capacities now defined, we are ready to prove the five properties of these capacities enumerated in Theorem 1.4.

5.1. **Monotonicity.** Let L_a, L_b be generic slices of a planar Lagrangian with $0 \notin [a, b]$. If $L_{[a,b]} = \cup_{t \in [a,b]} L_t$ and if $j_t : L_t \to L_{[a,b]}$ is the inclusion map then the goal of monotonicity is to compare the capacities of the cohomology classes $j_a^*(u) \in H^k(L_a)$ and $j_b^*(u) \in H^k(L_b)$, for some $u \in H^k(L_{[a,b]})$.

The purpose of splitting the domain of the difference function Δ_a into $\mathcal{P}_\pm$ was to obtain the containments $\Delta^\lambda_{b,+} \subset \Delta^\lambda_{a,+}$ and $\Delta^\lambda_{a,-} \subset \Delta^\lambda_{b,-}$; see equations and relations (3.2) – (3.4). This is the key fact in the proof of Monotonicity. We begin with a lemma:

Lemma 5.1. *Consider $a < b$ with $0 \notin [a,b]$. Then the following diagram (and its analogues in the cases of the negative lower capacity and the upper capacities) commutes:*

$$\begin{array}{ccccc} & & H^k(L_a) & \xrightarrow{\varphi^\lambda_{a,+}} & H^{k+N+1}(\Delta^\eta_{a,+}, \Delta^\lambda_{a,+}) \\ & \overset{j_a^*}{\nearrow} & & & \downarrow i^* \\ H^k(L_{[a,b]}) & & & & \\ & \underset{j_b^*}{\searrow} & & & \\ & & H^k(L_b) & \xrightarrow[\varphi^\lambda_{b,+}]{} & H^{k+N+1}(\Delta^\eta_{b,+}, \Delta^\lambda_{b,+}). \end{array}$$

Assuming the lemma for now, we prove Monotonicity:

Proof of Monotonicity. To prove the inequality $c_+^{L,a}(j_a^* u) \leq c_+^{L,b}(j_b^* u)$, note that Lemma 5.1 shows that if $\varphi^\lambda_{a,+}(j_a^* u) = 0$ then $\varphi^\lambda_{b,+}(j_b^* u) = 0$. Thus, we have:

$$\begin{aligned} c_+^{L,a}(j_a^* u) &= \sup\{\lambda : \varphi^\lambda_{a,+}(j_a^* u) = 0\} \\ &\leq \sup\{\lambda : \varphi^\lambda_{b,+}(j_b^* u) = 0\} \\ &= c_+^{L,b}(j_b^* u). \end{aligned}$$

When comparing $c_-^{L,a}(j_a^* u)$ and $c_-^{L,b}(j_b^* u)$, the fact that $\Delta_{a,-} \subset \Delta_{b,-}$ causes the map i^* to reverse direction, and hence reverses the inequality. The proofs for the upper capacities are analogous.

To show that the inequalities are strict if one of the capacities involved is nonzero, we work as in [18, Lemma 4.7]: note that for all but finitely many a, we may assume that there exists an ϵ so that if $t \in (a-\epsilon, a+\epsilon)$ then Δ_t has k distinct nonzero critical values $c_1(t), \ldots, c_k(t)$ coming from k non-degenerate critical points. Using arguments as in the proof of Lemma 3.3, these critical values come from smooth paths of Morse critical points $q_1(t), \ldots, q_k(t)$, which, in turn, come from double points of $\pi(L_t)$. We compute that:

$$\begin{aligned} c_i'(t) = \partial_t(\Delta_t(q_i(t))) &= (\partial_t \Delta_t)(q_i(t)) + d\Delta_t(q_i(t))\partial_t(q_i(t)) \\ &= (\partial_t \Delta_t)(q_i(t)) \\ &= x_n(t) - \tilde{x}_n(t). \end{aligned}$$

Since L_t is embedded, the x_n-heights of the double points of $\pi(L_t)$ must be different, and hence $c_i'(t) \neq 0$, as desired. □

Proof of Lemma 5.1. The scheme of the proof is to introduce an **extended difference function** to express the cobordism $L_{[a,b]}$ in terms of level sets.

The extended difference function is defined by

$$\Delta_{ab}(t, \mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) = \Delta_t(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}), \quad t \in [a, b].$$

The function and its level sets have the following properties, as can be seen by direct computation and the techniques of Lemma 3.3:

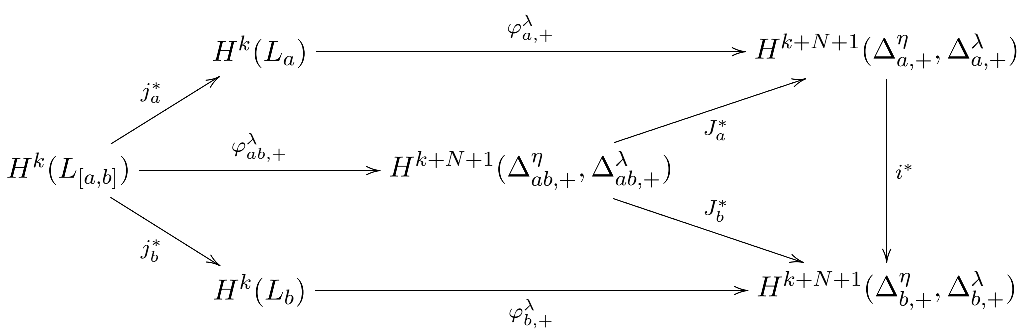
(1) The sublevel sets of Δ_{ab} satisfy $\Delta_{ab}^\lambda = \bigcup_{t\in[a,b]} \Delta_t^\lambda$ and interact with $\mathcal{P}_\pm$ in the same way;
(2) Δ_{ab} is Morse-Bott with a critical submanifold of index $N+1$ that can be identified with $L_{[a,b]}$ via the fiber critical set Σ_F.

The same constructions as in Section 4.1 then allow us to define a map

$$\varphi_{ab,+}^\lambda : H^k(L_{[a,b]}) \to H^{k+N+1}(\Delta_{ab,+}^\eta, \Delta_{ab,+}^\lambda).$$

Similar constructions yield maps $\varphi_{ab,-}^\lambda$ and $\Phi_{ab,\pm}^\Lambda$.

To finish the proof, we need only note that the naturality of the Mayer-Vietoris sequence, the long exact sequence of a triple, and the Gysin sequence shows that the top and bottom parallelograms of the following diagram commute, as does the rightmost triangle:



□

5.2. **Continuity.** By fixing a cohomology class in $H^*(L_{[a,b]})$, not only can we compare capacities at different levels, but we also create a continuous, piecewise differentiable function from each capacity.

Proof of Continuity. Suppose that a and b are generic levels with $0 < a < b$ (the proof is entirely similar for negative levels). Using a fiber-preserving diffeomorphism of $\mathbb{R}^{n-1} \times \mathbb{R}^{N+1}$ that sends x_n to $\frac{a}{t} x_n$, we may modify the functions F_t, $t \in [a,b]$, to functions $\bar{F}_t$ that agree outside a compact set. An application of the argument in Lemma 4.7 shows that these modified functions yield difference functions $\bar{\Delta}_t$ that give the same capacities as before. It follows that for every $\epsilon > 0$, there is a $\delta > 0$ so that if $|s - t| < \delta$, then $\|\bar{\Delta}_s - \bar{\Delta}_t\| < \epsilon$. In particular, we obtain the inclusions:

$$\bar{\Delta}_{s,\pm}^{\lambda-\epsilon} \subset \bar{\Delta}_{t,\pm}^{\lambda} \subset \bar{\Delta}_{s,\pm}^{\lambda+\epsilon}.$$

These inclusions lead to the following commutative diagram for the lower capacities:

$$\begin{array}{ccc}
 & & H^{k+N+1}(\bar{\Delta}^{\eta}_{s,\pm}, \bar{\Delta}^{\lambda-\epsilon}_{s,\pm}) \\
 & \nearrow^{\varphi^{\lambda-\epsilon}_{t,\pm}\circ j_t^*} & \uparrow i^* \\
H^k(L) & \xrightarrow{\varphi^{\lambda}_{s,\pm}\circ j_s^*} & H^{k+N+1}(\bar{\Delta}^{\eta}_{t,\pm}, \bar{\Delta}^{\lambda}_{t,\pm}) \\
 & \searrow_{\varphi^{\lambda+\epsilon}_{s,\pm}\circ j_s^*} & \uparrow i^* \\
 & & H^{k+N+1}(\bar{\Delta}^{\eta}_{s,\pm}, \bar{\Delta}^{\lambda+\epsilon}_{s,\pm}).
\end{array}$$

It follows immediately from the bottom and top triangles, respectively, that:

$$c(s) - \epsilon \leq c(t) \leq c(s) + \epsilon,$$

as required. The proof for the upper capacities is similar.

The fact that $c(t)$ is monotone shows that its one-sided limits exist at a non-generic level t, and the proof above with $a = t - \epsilon$ and $b = t + \epsilon$ shows that these two limits are equal. Thus, the non-generic levels are removable singularities.

Finally, piecewise differentiability follows from the arguments at the end of the proof of Monotonicity in the previous section. □

5.3. **Invariance.** The capacities for a slice L_a depend on the entire Lagrangian L since they are defined in terms of a generating family for L. In this section, however, we will show that isotoping the Lagrangian while keeping the slice L_a unchanged will not change the capacities. Specifically, let ψ_t, $t \in [0, 1]$, be a compactly supported symplectic isotopy of $\mathbb{R}^{2n}$. Let $L(t) = \psi_t(L)$ be the image of L under the isotopy, and suppose that the slice $L_a(t) = L(t) \cap \{y_n = a\}$ is always equal to L_a.

The proof of Invariance, namely that the capacities of L_a do not depend on $L(t)$, follows easily from the fact that the capacities lie in the discrete set of the critical values of the difference function (see Lemma 4.5) and from following proposition about the continuity of the capacities, whose proof is a slight variation on the proof of continuity given above:

Proposition 5.2. *With the notation above, for $u \in H^k(L_a)$, the capacities $c^{L(t),a}_{\pm}(u)$ and $C^{L(t),a}_{\pm}(u)$ are continuous in t.*

Proof. As noted in Remark 2.4, there is a 1-parameter family $F(t)$ of quadratic-at-infinity generating families for $L(t)$. Now use the same proof as in the previous section. □

5.4. **Non-Vanishing.** Next, we will prove Non-Vanishing, which asserts that, for $u \neq 0$, at least one of the four capacities $c^{L,a}_{\pm}(u)$, $C^{L,a}_{\pm}(u)$ is nonzero. The scheme of the proof is to understand the cohomology of $(\Delta^{\theta}_a, \Delta^{-\theta}_a)$ for large θ, and then to use this to relate the capacities.

Lemma 5.3. *For $\theta \gg 0$, $H^*(\Delta^{\theta}_a, \Delta^{-\theta}_a) = 0$.*

A component in the proof of this lemma is the following "critical non-crossing" lemma, whose proof can be found, for example, almost word-for-word in the proof of Lemma 3.10 of [17].

Lemma 5.4. *Consider a smooth* 1*-parameter family of difference functions* $\Delta_a(t)$ *that arises from a* 1*-parameter family of quadratic-at-infinity generating families* $F(t)$ *and levels* $a(t)$*. Suppose* $\tau, \sigma : [0,1] \to \mathbb{R}$ *are continuous paths such that* $\tau(t)$ *and* $\sigma(t)$ *are regular values of* $\Delta_a(t)$ *with* $\tau(t) < \sigma(t)$ *for all* t*. Then for any* $s, t \in [0,1]$*, there is an isomorphism*

$$H^*\left(\Delta_a(s)^{\sigma(s)}, \Delta_a(s)^{\tau(s)}\right) \simeq H^*\left(\Delta_a(t)^{\sigma(t)}, \Delta_a(t)^{\tau(t)}\right).$$

Proof of Lemma 5.3. Suppose $a > 0$; the proof for $a < 0$ is entirely similar. Choose θ large enough so that the magnitudes of the critical values of Δ_b are less than θ, for all $b > a$. Choose $b > a$ such that the slice L_b is empty; this is possible since L is flat-at-infinity. Since L_b is empty, Lemma 3.3 shows that the difference function Δ_b has no critical points, and hence that $H^*(\Delta_b^\theta, \Delta_b^{-\theta}) = 0$. We can then conclude, using Lemma 5.4 with the constant paths $\pm\theta$, that:

$$H^*(\Delta_a^\theta, \Delta_a^{-\theta}) \simeq H^*(\Delta_b^\theta, \Delta_b^{-\theta}) = 0.$$

□

We now leverage Lemma 5.3 to relate the spaces involved in the definition of the capacities.

Lemma 5.5. *For* $\theta \gg 0$*,* $\ker p_a^{-\theta} = \ker D_a^\theta$.

Proof. The connecting homomorphisms in the exact sequences of the triples $(\Delta_a^\theta, \Delta_a^\eta, \Delta_a^{-\eta})$ and $(\Delta_a^\theta, \Delta_a^\eta, \Delta_a^{-\theta})$ together with the map $p_a^{-\theta}$ in the exact sequence of the triple $(\Delta_a^\eta, \Delta_a^{-\eta}, \Delta_a^{-\theta})$ fit together in the following commutative diagram:

$$\begin{array}{ccc} & \overset{p_a^{-\theta}}{\nearrow} & H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\theta}) \\ H^{k+N+1}(\Delta_a^\eta, \Delta_a^{-\eta}) & & \downarrow d^* \\ & \underset{D_a^\theta}{\searrow} & H^{k+N+2}(\Delta_a^\theta, \Delta_a^\eta). \end{array}$$

By Lemma 5.3, the connecting homomorphism d^* at the right side of the diagram is an isomorphism, so $\ker D_a^\theta = \ker d^* \circ p_a^{-\theta} = \ker p_a^{-\theta}$, as desired. □

Proof of Non-Vanishing. Suppose that $0 \neq u \in H^k(L_a)$ and $c_+^{L,a}(u) = c_-^{L,a}(u) = 0$. Then, by Corollary 4.6, for all λ, $u \notin \ker p_a^\lambda$. In particular, for $\theta \gg 0$, Lemma 5.5 implies that $u \notin \ker p_a^{-\theta} = \ker D_a^\theta$. So, again by Corollary 4.6, either $C_+^{L,a}(u) \neq 0$ or $C_-^{L,a}(u) \neq 0$. □

5.5. **Conformality.** The proof of conformality follows immediately from two facts:

(1) If F generates L then

$$\bar{F}(\mathbf{x}, x_n, \mathbf{e}) = \beta^2 F(\frac{1}{\beta}\mathbf{x}, \frac{1}{\beta}x_n, \mathbf{e})$$

generates βL.

(2) The following sublevel sets are diffeomorphic:

$$\Delta_a^\lambda \simeq \bar{\Delta}_{\beta a}^{\beta^2 \lambda}.$$

6. Applications and Extensions

6.1. **Obstructions to Cobordisms.** In order to prove Theorems 1.1 and 1.3, we begin with two useful lemmas. The first is a gluing result:

Lemma 6.1 ([2])**.** *Let $L, L' \subset \mathbb{R}^{2n}$ be two Lagrangians that are transverse to and agree on $\{y_n = a\}$. For all $\epsilon > 0$, there exists a Lagrangian L'' such that:*

(1) $L'' \cap \{y_n < a - \epsilon\} = L \cap \{y_n < a - \epsilon\}$ *and*
(2) $L'' \cap \{y_n > a + \epsilon\} = L' \cap \{y_n > a + \epsilon\}$.

Note that the proposition in [2] is stated for Lagrangians in $\mathbb{R}^4$, but the proof applies to higher dimensions. In addition, in [2], the Lagrangians are assumed to be flat-at-infinity and planar, but the proof can be carried out near $L_a = L'_a$ in a standard neighborhood of either one of the Lagrangians.

The second is a local computation of critical values of Δ_a from the geometry of $\pi(L_a)$:

Lemma 6.2. *Suppose that $q = (\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}})$ is a non-degenerate critical point of Δ_a and that there exists a path $\gamma_q : [0,1] \to \Sigma_{F_a}$ that begins at $(\mathbf{x}, \tilde{x}_n, \tilde{\mathbf{e}})$ and ends at $(\mathbf{x}, x_n, \mathbf{e})$. Let $i_a : \Sigma_{F_a} \to \pi(L_a)$ denote the inclusion as in equation (2.1). Then the critical value of $(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}})$ is:*

$$\Delta_a(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{\mathbf{e}}) = \int_{i_a \circ \gamma_q} \mathbf{y}\, d\mathbf{x}.$$

Proof. A direct computation using equation (2.1) shows that $i_a^*(\mathbf{y}\, d\mathbf{x}) = dF_a$, and hence we have:

$$\Delta_a(\mathbf{x}, x_n, \mathbf{e}, \tilde{x}_n, \tilde{e}) = \int_{\gamma_q} dF_a = \int_{i_a \circ \gamma_q} \mathbf{y}\, d\mathbf{x}.$$

□

Note that by Stokes' Theorem, this critical value can be thought of as the negative symplectic area of a 2-chain bounded by the loop $i_a \circ \gamma_q$. In particular, the critical values of Δ_a are determined by the slice L_a.

Example 6.3. In Example 3.5, we calculated the critical values of the non-degenerate critical points of an explicit generating family for a Lagrangian with slices agreeing $8^1_-(r)$. Using Lemma 3.3, we see that for *any* difference function Δ_a associated to *any* Lagrangian having a slice L_a agreeing with $8^1_-(r)$, the non-degenerate critical points come from the double point of $\pi(L_a)$: one with $x_2 > \tilde{x}_2$ and another with $x_2 < \tilde{x}_2$. The loop $i_a \circ \gamma_q$ of Lemma 6.2 associated to the point in $\mathcal{P}_-$ (where $x_2 > \tilde{x}_2$) will be oriented clockwise, and thus this critical point has critical value $v > 0$ equal to the area of one of the lobes of our projected figure-8 curve. The loop $i_a \circ \gamma_q$ associated to the critical point in $\mathcal{P}_+$ will be the same curve (but now oriented counterclockwise), and so the critical point will have critical value $-v < 0$. $\diamond$

Proof of Theorem 1.3. First notice that, for any $s > 0$, if there is an unknotted planar Lagrangian L so that $L_a = i_a(8^{n-1}_-(s))$, then $a > 0$; otherwise, it would be possible to construct an (exact) embedded Lagrangian sphere as follows. If $a < 0$, then there exists a Lagrangian disk in $\{y_n \leq a\}$ with boundary $i_a(8^{n-1}_-(s))$. By the construction in Example 2.1, for some $a' > 0$, there exists a Lagrangian disk $\{y_n \geq a'\}$ with boundary $i_{a'}(8^{n-1}_-(s))$. Using Lemma 6.1, translations of these disks can be glued to form an embedded (exact) Lagrangian sphere, which is impossible for $n > 1$ by a result of Gromov [9].

Next notice that, for any $s > 0$, if there is an unknotted planar Lagrangian L so that $L_a = i_a(8^{n-1}_-(s))$, then $C^{L,a}_+(u) = 0$ and $c^{L,a}_-(u) = 0$, for all $u \in H^k(L_a)$. This follows from Monotonicity and the facts that $a > 0$ and that for $\tau \gg 0$, we have $L_\tau = \emptyset$.

In addition, notice that if there exists an unknotted planar Lagrangian $L \subset \mathbb{R}^{2n}$ so that $L_a = i_a(8^{n-1}_-(s))$, then the critical values of Δ_a are $0, \pm v_s$ where $v_s > 0$ and does not depend on the height a. The fact that there are three critical values of Δ_a follows from Lemma 3.3 since $\pi(L_a)$ has a single double point; if all critical values were 0 then there would be a contradiction to Non-Vanishing. The fact that the critical value $v_s > 0$ does not depend on the height a follows from Lemma 6.2.

For fixed $r < R$, let v_r and v_R be the positive critical values associated to the difference functions for heights giving slices $8^{n-1}_-(r)$ and $8^{n-1}_-(R)$, respectively. We can then deduce that $v_r < v_R$ as follows. By the construction in Example 2.1, there exists an unknotted planar Lagrangian $L \subset \mathbb{R}^{2n}$ and $a < b$ so that

$$L_a = i_a(8^{n-1}_-(R)), \qquad L_b = i_b(8^{n-1}_-(r)).$$

Non-vanishing implies that for all non-trivial $u \in H^k(L_{[a,b]})$, $C_-(j_b^* u) = v_r$ or $c_+(j_b^* u) = -v_r$. If $C_-(j_b^* u) = v_r$, then by Monotonicity, $C_-(j_a^* u) = v_R > v_r$. Similarly, $c_+(j_b^* u) = -v_r$ implies, by Monotonicity, $c_+(j_a^* u) = -v_R < -v_r$, again implying $v_R > v_r$.

The desired result now follows by contradiction: suppose there exists an unknotted planar Lagrangian $L \subset \mathbb{R}^{2n}$ and $a < b$ so that

$$L_a = i_a(8_-^{n-1}(r)), \qquad L_b = i_b(8_-^{n-1}(R)), \qquad r \leq R.$$

Repeating the Non-vanishing and Monotonicity arguments as above, we find that $v_R < v_r$. This shows that $r = R$ is not possible and gives us a contradiction to the above paragraph when $r < R$. □

Remark 6.4. The above proof uses in an essential way that there are just two non-zero critical values of Δ_a. If, for example, it were true that every flat-at-infinity planar Lagrangian in $\mathbb{R}^{2n}$ is unknotted, then the proof of anti-symmetry in [2] would extend to higher dimensional cobordisms between many different pairs of submanifolds.

To prove Theorem 1.1, which has no extendibility hypothesis on the cobordism, we need the following geometric result, whose proof is a simple modification of [1, Theorem 6.4] with the writhe taking the place of the Thurston-Bennequin number.

Lemma 6.5. *If L be an embedded Lagrange cobordism between the slices L_a and L_b with $a < b$, then*

$$\chi(L) = \mathrm{wr}(L_b) - \mathrm{wr}(L_a),$$

where the writhe wr *is calculated with respect to the blackboard framing of the x_1y_1 projection.*

Proof of Theorem 1.1. Suppose that there exists a Lagrangian cobordism $L \subset \{a \leq y_2 \leq b\} \subset \mathbb{R}^4$ with $L_a = i_a(8_-^1(r))$ and $L_b = i_b(8_-^1(R))$. This cobordism can be extended to a flat-at-infinity planar Lagrangian as follows. After perhaps translating L, we know, by Example 2.1, that there exists a Lagrangian disk in $\{y_2 \geq b\}$ with boundary $i_b(8_-^1(R))$ and a flat-at-infinity planar Lagrangian with a disk removed that lies in $\{y_2 \leq a\}$ and has boundary $i_a(8_-^1(r))$. Using Lemma 6.1, we can glue the cobordism, the disk, and the punctured plane together to form a flat-at-infinity Lagrangian. Since $\mathrm{wr}(8_-^1(r)) = \mathrm{wr}(8_-^1(R))$, Lemma 6.5 shows that the Lagrange cobordism L must be topologically an annulus. Thus, the Lagrangian L' must be planar. By a result of Eliashberg and Polterovich [7], L' is unknotted, and we can apply the proof of Theorem 1.3 to show that no such L' exists. □

Remark 6.9, below, gives an alternate proof of Theorem 1.1 using explicit calculations of capacities.

6.2. **4-dimensional Computations.** We have already seen that the functorial properties of the capacities can be used to obtain obstructions to cobordisms. In order to get some understanding of what the capacities are measuring, we now explicitly calculate capacities of some slices. Recall that in general, capacities for a slice of a Lagrangian depend on the entire Lagrangian. However, we will show that when the slice is particularly simple, for example when the slice agrees with $8^1_\pm(r)$, the capacities only depend on the slice.

A key step to doing explicit calculations is to determine the critical values and indices of critical points of Δ_a geometrically from the projection $\pi(L_a)$. The computation of critical values follows from Lemma 6.2 in the previous section where it is shown that the critical value associated to a critical point $q = (x_1, x_2, \mathbf{e}, \tilde{x}_2, \tilde{\mathbf{e}})$ of Δ_a is obtained by integrating along a path $i_a \circ \gamma_q \subset \pi(L_a)$. Similarly, the index of an isolated critical point q of Δ_a may also be calculated using the path γ_q. The path γ_q defines a path Γ_q of Lagrangian subspaces of $\mathbb{R}^2$ via

$$\Gamma_q(t) = T_{i_a \circ \gamma_q(t)} \pi(L_a).$$

Close this path to a loop $\bar{\Gamma}_q$ by rotating $\Gamma_q(1)$ clockwise until it coincides with $\Gamma_q(0)$. The index of Δ_a at q can then be calculated in terms of the Maslov index of $\bar{\Gamma}_q$:

Lemma 6.6. *If q is an isolated critical point of Δ_a and γ_q is a path in $\mathbb{R} \times \mathbb{R}^{1+N}$ as in Lemma 6.2, then:*

$$\operatorname{Ind}_q \Delta_a = -\mu(\bar{\Gamma}_q) + (N+1).$$

Proof. Equation (3.1) shows that:

$$\operatorname{Ind}_q \Delta_a = \operatorname{Ind}_{x_1}(g'' - \tilde{g}'') + (\operatorname{Ind} A - \operatorname{Ind} \tilde{A}) + (N+1). \tag{6.1}$$

To interpret this formula in terms of the Maslov index, consider that the path of Hessians $d^2F_a(\gamma_q(s))$ generates the path of Lagrangian subspaces Γ_q in the language of Théret [15, Appendix B]. Théret further shows that

$$\mu(\Gamma_q) = \operatorname{Ind}_{\gamma_q(0)} F_a - \operatorname{Ind}_{\gamma_q(1)} F_a.$$

(Note that Théret takes the opposite sign convention for the Maslov index than the one used here and in [12].) By construction, we may compute that

$$d^2F_a(\gamma_q(0)) = \begin{bmatrix} \tilde{g}'' & 0 \\ 0 & \tilde{A} \end{bmatrix}, \qquad d^2F_a(\gamma_q(1)) = \begin{bmatrix} g'' & 0 \\ 0 & A \end{bmatrix}.$$

Combining this with Equations (6.1) and (6.2), we obtain

$$\operatorname{Ind}_q \Delta_a = (N+1) - \mu(\Gamma_q) + \left[\operatorname{Ind}(\tilde{g}'' - g'') - \operatorname{Ind}(\tilde{g}'') + \operatorname{Ind}(g'')\right].$$

It is easy to check that the term $\operatorname{Ind}(\tilde{g}'' - g'') - \operatorname{Ind}(\tilde{g}'') + \operatorname{Ind}(g'')$ accounts precisely for the change in Maslov index engendered by closing Γ_q into $\bar{\Gamma}_q$. This completes the proof of the index formula for an isolated critical point. □

Example 6.7. In Example 3.5, we calculated the indices of the non-degenerate critical points of an explicit generating family (with $N = 0$) of a Lagrangian with slices agreeing with $8^1_-(r)$. Lemma 6.6 gives us a way to calculate the indices of the non-degenerate critical points of *any* difference function Δ_a of *any* Lagrangian having a slice agreeing with such a unknotted figure-8 curve with a negative crossing. Notice that the loop $\bar{\Gamma}_{q_-}$ associated to the critical point $q_- \in \mathcal{P}_-$ $(x_2(q_-) > \tilde{x}_2(q_-))$ will make one full rotation in the clockwise direction, and thus $\mu(\bar{\Gamma}_{q_-}) = -2$. So for any difference function Δ_a for L_a with domain $\mathbb{R} \times \mathbb{R}^{1+N} \times \mathbb{R}^{1+N}$, $\mathrm{Ind}_{q_-} \Delta_a = 2 + N + 1 = N + 3$. The loop $\bar{\Gamma}_{q_+}$ associated to $q_+ \in \mathcal{P}_+$ will make a half rotation in the counterclockwise direction, and so $\mathrm{Ind}_{q_+} \Delta_a = -1 + N + 1 = N$. ⋄

Now that we can calculate critical values and indices using only L_a, let us proceed to calculate the capacities of $8^1_\pm(r)$ or, more generally, any slice with a projection diagram agreeing with that of $8^1_\pm(r)$. To set up the geometric situation more precisely, let L_a and L'_a be slices that project to immersed curves in the (x_1, y_1)-plane with one double point q. Let q^+, q^- denote the preimages of q in L_a (resp. L'_a) with $x_2(q^+) > x_2(q^-)$, and assume that the crossing in L_a (resp. L'_a) is negative (resp. positive) so that the path $i_a \circ \gamma_q$ from q^+ to q^- in L_a (resp. L'_a) projects to a loop traversed in the counterclockwise (resp. clockwise) direction. Let A denote the absolute value of the area of the region bounded by $\pi \circ i_a \circ \gamma_q$. With this notation, we have the following calculation of capacities:

Proposition 6.8. *Let L, L' be any flat-at-infinity planar Lagrangians with slices L_a, L'_a as described above. Then, for $0 \neq u \in H^0(L_a)$ and $0 \neq v \in H^1(L_a)$,*

$$c_+^{L,a}(u) = -A, \quad c_-^{L,a}(u) = 0, \quad C_+^{L,a}(v) = 0, \quad C_-^{L,a}(v) = A.$$

For $0 \neq u \in H^0(L'_a)$ and $0 \neq v \in H^1(L'_a)$,

$$c_+^{L',a}(u) = 0, \quad c_-^{L',a}(u) = -A, \quad C_+^{L',a}(v) = A, \quad C_-^{L',a}(v) = 0.$$

Proof. Let $F : \mathbb{R}^2 \times \mathbb{R}^N \to \mathbb{R}$ be a quadratic-at-infinity generating family for any Lagrangian L whose slice at $y_2 = a$ is L_a. By Lemma 3.3, F_a must have two non-degenerate critical points and a non-degenerate critical submanifold. By Lemma 6.2, the loop $i_a \circ \gamma_q$ with clockwise orientation is associated with the critical point with positive critical value, and hence this critical point lies in $\mathcal{P}_- = \{x_2 > \tilde{x}_2\}$. By Lemmas 6.2 and 6.6, the critical value of this point is A, and its index must be $N + 3$. By a similar argument, the other critical point occurs in $\mathcal{P}_+ = \{x_2 < \tilde{x}_2\}$, has critical value $-A$ and index N. Calculations of the non-degenerate critical points, values, and indices for L'_a are analogous.

We will compute the lower capacities $c_\pm^{L,a}(u)$; calculations for the upper capacities $c_\pm^{L,a}(v)$ and all capacities of L'_a follow analogously. To show that $c_-^{L,a}(u) = 0$, notice that the index and critical value calculations above, combined with Lemma 3.6, show that $H^k(\Delta_{a,-}^{-\eta}, \Delta_{a,-}^{\lambda}) = 0$, for all k and for any $\lambda < -\eta$. Thus, examining the exact sequence of the triple $(\Delta_{a,-}^{\eta}, \Delta_{a,-}^{-\eta}, \Delta_{a,-}^{\lambda})$, we see that $\varphi_{a,-}^{\lambda}$ is an isomorphism for all λ, and so we are done by Corollary 4.6. Similar arguments using the fact that there are no critical points of index $N + 2$ with positive critical value show that the upper capacities $C_\pm^{L,a}(u)$ also vanish. Thus, by Non-Vanishing, it must be the case that

$c_+^{L,a}(u) \neq 0$. Since $-A$ is the only negative critical value of Δ_a, Lemma 4.5 shows that $c_+^{L,a}(u) = -A$, as desired. □

Remark 6.9. An alternate proof to Theorem 1.1 follows from gluing arguments, this explicit calculation of the capacities, and Monotonicity.

6.3. **Non-Squeezing Phenomena.** As in the proofs of the non-existence results, the non-squeezing results in Theorems 1.2 and 1.5 rely primarily on Monotonicity.

Proof of Theorem 1.5. We prove the inequality for C_-; the other proofs are entirely analogous. Fix $u \in H^*(L)$. Recall from Proposition 5.2 that $C_-(t) = C_-^{L,t}(j_t^* u)$ is a continuous, piecewise differentiable function of t. If the capacity comes from a piecewise continuous path of crossings $(\mathbf{x}(t), x_n(t), \tilde{x}_n(t))$, then the proof of Monotonicity shows that at all but finitely many levels t, $C_-'(t) = x_n(t) - \tilde{x}_n(t)$. Since the capacity $C_-(t)$ comes from a critical point in $\mathcal{P}_-$ and, by hypothesis, $x_n(t)$ and $\tilde{x}_n(t)$ lie in an interval of length ℓ, we have, for all but finitely many $t \geq a$:

$$-\ell \leq C_-'(t) \leq 0. \tag{6.2}$$

The theorem now follows from this bound on the derivative and the hypothesis that when $t > w + a$, $C_-(t) = 0$ (since $L_t = \emptyset$). □

Proof of Theorem 1.2. For any $r > 0$, a Lagrangian disk $L \subset \mathbb{R}^4$ with boundary $\partial L = (L \cap \{y_2 = a\}) = i_a(8^1_\pm(r))$ can be extended to a flat-at-infinity planar Lagrangian. This follows from the construction in Example 2.1 and a gluing argument of Lemma 6.1. The result then follows immediately as a corollary of Theorem 1.5 given the calculation of the capacities in Proposition 6.8. □

6.4. **Capacities and Field Theory.** To bring the capacities into an SFT-style framework, we use the relative cohomology groups of sublevel sets to assign a filtered cohomology theory — really, four filtered cohomology theories — to each slice. To a Lagrange cobordism between slices, we assign a filtered homomorphism for each of the filtered cohomologies. As we shall see, the direction of the homomorphism depends on whether we consider sublevel sets in $\mathcal{P}_+$ or in $\mathcal{P}_-$. Further, the capacities can be used to detect non-triviality of the homomorphisms.

Based on the definitions of the capacities, we define:

Definition 6.10. The **positive and negative lower filtered cohomology groups** of a generic slice of an unknotted planar Lagrangian L are

$$h^*_{\pm,\lambda}(L,a) = H^{*+N+1}(\Delta^{-\eta}_{a,\pm}, \Delta^{\lambda}_{a,\pm}),$$

where $\lambda < 0$ is the filtration level and η is any positive real number such that there are no critical values of Δ_a in $[-\eta, 0)$. The **positive and negative upper filtered cohomology groups** are defined similarly:

$$H^*_{\pm,\Lambda}(L,a) = H^{*+N+2}(\Delta^{\Lambda}_{a,\pm}, \Delta^{\eta}_{a,\pm}),$$

for all $\Lambda > 0$.

The proof of Lemma 4.7 shows that these cohomologies are independent of the generating family used to define L, up to an overall shift in degree. Further, Lemma 3.6 shows that the filtered cohomologies can only change values when the filtration level passes through a critical value of Δ_a.

For $a < b$, the inclusions $i_+ : \Delta^\lambda_{b,+} \hookrightarrow \Delta^\lambda_{a,+}$ and $i_- : \Delta^\lambda_{a,-} \hookrightarrow \Delta^\lambda_{b,-}$ from Equation (3.4) yield filtered homomorphisms between the cohomology groups of different slices:

$$i^*_+ : h^*_{+,\lambda}(L,a) \to h^*_{+,\lambda}(L,b), \quad i^*_+ : H^*_{+,\Lambda}(L,a) \to H^*_{+,\Lambda}(L,b),$$
$$i^*_- : h^*_{-,\lambda}(L,b) \to h^*_{-,\lambda}(L,a), \quad i^*_- : H^*_{-,\Lambda}(L,b) \to H^*_{-,\Lambda}(L,a).$$

As mentioned above, the capacities can give information about when these maps are nontrivial. For example, given $0 \notin [a,b]$, the analogue of the diagram in Lemma 5.1 for the upper capacities immediately implies:

Proposition 6.11. *With notation as in Section 5.1, suppose that* $C^{L,b}_+(j^*_b u) > 0$*, for some* $u \in H^k(W)$*. Then the map* $i^*_+ : H^k_{+,\Lambda}(L,a) \to H^k_{+,\Lambda}(L,b)$ *is nontrivial for* $\Lambda = C^{L,b}_+(j^*_b u)$.

Remark 6.12. For large enough Λ, the group $H^*_{+,\Lambda}(L,a) \oplus H^*_{-,\Lambda}(L,a)$ is equal to the generating family (co)homology of the lift of $\pi(L_a)$ to a Legendrian link in the standard contact $\mathbb{R}^3 = J^1\mathbb{R}$; see Fuchs and Rutherford [8], Traynor [17], or Jordan and Traynor [10] for more information on generating family homology for Legendrian knots and links. In fact, since L is exact, it lifts to a Legendrian cobordism between the lifts of $\pi(L_a)$ and $\pi(L_b)$. Thus, the field theory in this section suggests a structure for a field theory for Legendrian cobordisms between Legendrian knots defined by generating families. As Fuchs and Rutherford proved that the generating family cohomology of a Legendrian knot is isomorphic to the linearized contact homology (when these are defined), this remark justifies saying that the field theory defined above fits into an SFT framework.

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