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Stability of linear systems with time-varying delays: A direct frequency domain approach

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Abstract

The stability of linear systems with uncertain bounded time-varying delays (without any constraints on the delay derivatives) is analyzed. It is assumed that the system is stable for some known constant values of the delays (but may be unstable for zero delay values). The existing (Lyapunov-based) stability methods are restricted to the case of a single non-zero constant delay value, and lead to complicated and restrictive results. In the present note for the first time a stability criterion is derived in the general multiple delay case without any constraints on the delay derivative. The simple sufficient stability condition is given in terms of the system matrices and the lengths of the delay segments. Different from the existing frequency domain methods which usually apply the small gain theorem, the suggested approach is based on the direct application of the Laplace transform to the transformed system and on the bounding technique in L_2 . A numerical example illustrates the efficiency of the method.

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1. Introduction

Throughout the paper by stability we understand the asymptotic stability of the system. Robust stability of linear systems with uncertain constant or time-varying delays, where the nominal values of delays are zero, have been studied both, in the time domain and in the frequency domain (see e.g. [14,10,15–17,5,8,11] and the references therein). In the time domain, the main methods are based on the Lyapunov technique, while in the frequency domain on the application of the small gain theorem (see e.g. [8] and the references therein).

Systems with uncertain ‘non-small’ delays, where the *nominal delay values are non-zero* and constant appears in different applications such as internet networks, biological systems [13]. Such systems may be not stable for the zero values of the delays. Only few works have been devoted to stability analysis of such systems and all of these works were restricted to the case of a single non-zero nominal delay value: see [12,4,3] for time domain results and [11] for frequency domain conditions.

In the present note we consider the systems with a finite number of time-varying delays, where the nominal values of the delays may be uncommensurate. For the first time a stability criterion is derived in the multiple delay case

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without any constraints on the delay derivatives. For the sake of generality we consider the complex state vector and the complex system matrices. It should be noted that stability analysis of distributed retarded systems leads to ordinary differential-delay equations with complex, in general, coefficients, cf. [7]. Assuming that the system is stable for some known constant values of the delays, we derive sufficient stability conditions in terms of the system matrices and the sizes of the delay perturbations. We apply a new frequency domain approach which is based on the direct application of the Laplace transform to the transformed system, on the Parseval equality and on the bounding technique in L_2 . Similar approach has been introduced by [6, Chapter 10] for analysis of absolute stability of retarded systems with constant delays. In the present note we develop the method of [6] to the case of time-varying delays.

Notation. Below C^n denotes a space of complex column vectors with the Euclidean norm $\|\cdot\|$ and the unit matrix I . $\|A\|$ denotes the Euclidean norm of a $n \times n$ complex matrix A , which is equal to the maximum singular value of A . $C[-\eta, 0]$ is the space of continuous functions $\phi : [-\eta, 0] \rightarrow C^n$ with the norm $\|\phi\|_{C[-\eta, 0]} = \max_{t \in [-\eta, 0]} \|\phi(t)\|$. $L_2[0, \infty)$ is the space of square integrable functions $v : [0, \infty) \rightarrow C^n$ with the norm

$$\|v\|_{L_2[0, \infty)} := \left[\int_0^\infty \|v(t)\|^2 dt \right]^{1/2},$$

$L_2(-\infty, \infty)$, $L_2[0, l]$ ($l > 0$) are defined similarly. For a matrix-function $U(s)$, $s \in R$

$$\|U(s)\|_\infty := \sup_{-\infty < s < \infty} \|U(s)\|.$$

2. Stability

2.1. Problem formulation and main results

Consider the system

$$\dot{x}(t) = \sum_{k=1}^m A_k x(t - \tau_k(t)) \quad (t > 0), \quad (1)$$

$$x(t) = \phi(t) \quad (-\eta \leq t \leq 0, \phi \in C[-\eta, 0]), \quad (2)$$

where $x(t) \in C^n$, $\eta > 0$, A_k are constant complex-valued $n \times n$ -matrices; $\tau_k(t)$ are non-negative piecewise-continuous scalar functions defined on $[0, \infty)$ and satisfying the conditions

$$0 \leq h_k \leq \tau_k(t) \leq h_k + \mu_k, \quad k = 1, \dots, m; \quad t \geq 0, \quad (3)$$

where h_k are known constant (nominal) values of the delays, $h_k + \mu_k$ are known constant upper bounds of the delays and $h_k + \mu_k \leq \eta$, $k = 1, \dots, m$. For the existence results and the definitions of asymptotic stability of (1) see e.g. [9]. Consider also the nominal equation with the nominal values of the delays h_k :

$$\dot{v}(t) = \sum_{k=1}^m A_k v(t - h_k), \quad v(t) \in C^n, \quad (4)$$

$$v(t) = \phi(t), \quad -\eta \leq t \leq 0.$$

We assume that

A1. The nominal system (4) is asymptotically stable.

Assumption A1 may be verified by applying the existing frequency domain criteria (see e.g. [16,12]).

Let $G(t)$ be a fundamental solution of (4), i.e. a matrix-valued function that satisfies (4) and the initial conditions $G(0+) = I$, $G(t) = 0$ ($t < 0$). Since (4) is stable, $v \in L_2[0, \infty)$ and

$$\|v(t)\| \leq M e^{-\alpha t} \|\phi\|_{C[-\eta, 0]},$$

$$\|G(t)\| \leq M e^{-\alpha t}, \quad \|\dot{G}(t)\| \leq M e^{-\alpha t}, \quad M \geq 1, \quad \alpha > 0. \quad (5)$$

1 Denote

$$K(z) = zI - \sum_{k=1}^m A_k e^{-zh_k}. \quad (6)$$

3 A1 means that the roots of the determinant $\det K(z)$ are in the open left half-plane. $K(z)$ is the Laplace transform of G , while $zK^{-1}(z) - I$ is the Laplace transform of \dot{G} . We have

$$\begin{aligned} \|K(is)\| &= \left\| \int_0^\infty e^{-ist} G(t) dt \right\| \leq \int_0^\infty \|G(t)\| dt \\ &\leq \int_0^\infty M e^{-\alpha t} dt \leq M/\alpha, \quad \forall s \in R, \end{aligned}$$

5 and thus

$$7 \quad \|K^{-1}(is)\|_\infty < \infty. \quad (7)$$

Similarly

$$9 \quad \|sK^{-1}(is)\|_\infty < \infty. \quad (8)$$

To formulate the main result (Theorem 2.1) put

$$11 \quad \theta(K) := \left\| \begin{bmatrix} \mu_1 s I_n \\ \dots \\ \mu_m s I_n \end{bmatrix} K^{-1}(is) [A_1 \dots A_m] \right\|_\infty.$$

Due to (8), $\theta(K) \leq \|sK^{-1}(is)\|_\infty \sum_{k=1}^m \mu_k \|A_k\| < \infty$.

13 **Theorem 2.1.** Under A1 (1) is asymptotically stable for all time-varying delays satisfying (3) if

$$\theta(K) < 1. \quad (9)$$

15 Proof of Theorem 2.1 is based on the following lemma:

17 **Lemma 2.1.** Assume that A1 and (9) hold. Then there exists $m_0 > 0$ such that for all time-varying delays satisfying (3) a solution x of (1), (2) satisfies the following estimate:

$$\|\dot{x}\|_{L_2[0,\infty)} \leq m_0 \|\phi\|_{C[-\eta,0]}.$$

19 Proofs of Theorem 2.1 and Lemma 2.1 are given in the next section. The following Corollary follows immediately from Theorem 2.1:

21 **Corollary 2.1.** Under A1 for all small enough μ_k , $k = 1, \dots, m$ and piecewise-continuous delays satisfying (3) the system (1) is asymptotically stable.

23 2.2. Proofs

Proof of Lemma 2.1. Representing (1) in the form

$$25 \quad \dot{x}(t) = \sum_{i=1}^m A_i x(t - h_i) + \sum_{i=1}^m A_i f_i(t), \quad (10)$$

where

$$27 \quad f_k(t) = x(t - \tau_k(t)) - x(t - h_k),$$

1 and applying to (10) the variation of constants formula, we find that (1), (2) is equivalent to the equation

$$x(t) = v(t) + \int_0^t G(t-t_1) \sum_{k=1}^m A_k f_k(t_1) dt_1 \quad (t \geq 0), \quad (11)$$

3 where v is a solution of (4). Differentiating (11) in t we arrive at the equation

$$\dot{x}(t) = \dot{v}(t) + \sum_{k=1}^m \left[\int_0^t \dot{G}(t-t_1) A_k f_k(t_1) dt_1 + A_k f_k(t) \right]. \quad (12)$$

5 Take into account that $G(0) = I$ and $zK^{-1}(z) - I$ is the Laplace transform of \dot{G} . Then due to the property of the convolution, we can assert that the Laplace transform of the expression

$$7 \int_0^t \dot{G}(t-t_1) A_k f_k(t_1) dt_1 + A_k f_k(t)$$

is $zK^{-1}(z)A_k F_k(z)$, where $F_k(z)$ is the Laplace transform of $f_k(t)$. We have

$$\begin{aligned} & \left\| \begin{bmatrix} sI_n \\ \dots \\ sI_n \end{bmatrix} K^{-1}(is) [\mu_1 A_1 \dots \mu_m A_m] \text{diag} \left\{ \frac{F_1(is)}{\mu_1}, \dots, \frac{F_m(is)}{\mu_m} \right\} \right\|_{L_2(-\infty, \infty)} \\ & \leq \theta(K) \max_{k=1, \dots, m} \left\| \frac{F_k(is)}{\mu_k} \right\|_{L_2(-\infty, \infty)}. \end{aligned}$$

9

Therefore, due to the Parseval equality, we obtain

$$\begin{aligned} & \left\| \int_0^t \begin{bmatrix} \dot{G}(t-t_1) \\ \dots \\ \dot{G}(t-t_1) \end{bmatrix} [\mu_1 A_2 \dots \mu_m A_m] \text{diag} \{f_1(s), \dots, f_m(s)\} \right. \\ & \left. + \begin{bmatrix} I_n \\ \dots \\ I_n \end{bmatrix} [\mu_1 A_1 \dots \mu_m A_m] \text{diag} \left\{ \frac{f_1(t)}{\mu_1}, \dots, \frac{f_2(t)}{\mu_2} \right\} \right\|_{L_2[0, \infty)} \\ & = \frac{1}{2\pi} \left\| \begin{bmatrix} sI_n \\ \dots \\ sI_n \end{bmatrix} K^{-1}(is) [\mu_1 A_1 \dots \mu_m A_m] \text{diag} \left\{ \frac{F_1(is)}{\mu_1}, \dots, \frac{F_m(is)}{\mu_m} \right\} \right\|_{L_2(-\infty, \infty)} \\ & \leq \theta(K) \max_{k=1, \dots, m} \left\| \frac{F_k(is)}{\mu_k} \right\|_{L_2(-\infty, \infty)} = \theta(K) \max_{k=1, \dots, m} \frac{\|f_k\|_{L_2[0, \infty)}}{\mu_k}. \end{aligned} \quad (13)$$

11

The following holds:

$$13 \quad \|f_k\|_{L_2[0, \infty)}^2 = \|f_k\|_{L_2[0, l]}^2 + \|f_k\|_{L_2[l, \infty)}^2$$

15 with a fixed finite $l > 2\eta$. Since (1) is a linear equation, there is a constant $m_1 = m_1(l)$, such that $\|x(t)\|_{C[0, l]} \leq m_1 \|\phi\|_{C[-\eta, 0]}$. Hence,

$$\|f_k\|_{L_2[0, l]} \leq m_2 \|\phi\|_{C[-\eta, 0]} \quad (m_2 = m_2(l) \equiv \text{const}). \quad (14)$$

17 Take into account that

$$f_k(t) = x(t - \tau_k(t)) - x(t - h_k) = - \int_{t-\tau_k(t)}^{t-h_k} \dot{x}(s) ds.$$

1 Then, applying the Schwarz inequality and changing the order of integration, we find

$$\begin{aligned}
 \|f_k\|_{L_2[l,\infty)}^2 &= \int_l^\infty \left\| \int_{t_k-\tau_k(t)}^{t-h_k} \dot{x}(s) \, ds \right\|^2 dt \leq \int_l^\infty \left(\int_{t-h_k-\mu_k}^{t-h_k} \|\dot{x}(s)\| \, ds \right)^2 dt \\
 &\leq \int_l^\infty \mu_k \int_{t-h_k-\mu_k}^{t-h_k} \|\dot{x}(s)\|^2 \, ds \, dt = \int_l^\infty \mu_k \int_0^{\mu_k} \|\dot{x}(t-h_k-s_1)\|^2 \, ds_1 \, dt \\
 &= \mu_k \int_0^{\mu_k} \int_l^\infty \|\dot{x}(t-s_1-h_k)\|^2 \, dt \, ds_1 = \mu_k \int_0^{\mu_k} \int_{l-s_1-h_k}^\infty \|\dot{x}(t_1)\|^2 \, dt_1 \, ds_1 \\
 &\leq \mu_k^2 \int_0^\infty \|\dot{x}(t_1)\|^2 \, dt_1.
 \end{aligned} \tag{15}$$

3 Since (4) is stable, there is a constant m_2 , such that $\|\dot{v}\|_{L_2[0,\infty)} \leq m_2 \|\phi\|_{C[-\eta,0]}$, cf. Theorem 8.4.3 from Gil (1998). Therefore (12)–(15) imply

$$\left\| \begin{bmatrix} \dot{x}(t) \\ \dots \\ \dot{x}(t) \end{bmatrix} \right\|_{L_2[0,\infty)} \leq m_3 \|\phi\|_{C[-\eta,0]} + \theta(K) \left\| \begin{bmatrix} \dot{x} \\ \dots \\ \dot{x} \end{bmatrix} \right\|_{L_2[0,\infty)} \quad (m_3 \equiv \text{const}). \tag{16}$$

Hence, condition (9) yields the required result.

7 **Proof of Theorem 2.1.** Let $v(t)$ be a solution of (4), $x(t)$ be a solution of (1), (2).

By Parseval's equality from (11) similarly to (13) we obtain

$$\begin{aligned}
 \|x\|_{L_2[0,\infty)} &\leq \|v\|_{L_2[0,\infty)} + \sum_{k=1}^m \left\| \int_0^t G(t-t_1) A_k f_k(s) \, ds \right\|_{L_2[0,\infty)} \\
 &\leq \|v\|_{L_2[0,\infty)} + \sum_{k=1}^m \|A_k\| \cdot \|K^{-1}(is)\|_\infty \|f_k\|_{L_2[0,\infty)}.
 \end{aligned} \tag{9}$$

Then from (14), (15), (7) and Lemma 2.1 it follows that $x \in L_2[0, \infty)$.

11 From (11), (5) we find

$$\begin{aligned}
 \|x(t)\| &\leq M \|\phi\|_{C[-\eta,0]} + \sum_{k=1}^m \int_0^t M e^{-\alpha(t-t_1)} \|A_k\| \|f_k(t_1)\| \, dt_1 \\
 &\leq M \|\phi\|_{C[-\eta,0]} + M \sum_{k=1}^m \|A_k\| \left(\int_0^t e^{-2\alpha(t-t_1)} \, dt_1 \right)^{1/2} \left(\int_0^t \|f_k(t_1)\|^2 \, dt_1 \right)^{1/2}, \quad t \geq 0.
 \end{aligned} \tag{17}$$

13 The latter inequality together with (14), (15) and Lemma 2.1 imply that for all $t \geq 0$ there exists $m_5 > 1$ such that
 15 $\|x(t)\| \leq m_5 \|\phi\|_{C[-\eta,0]}$. Moreover, $\dot{x}(t)$, given by the right-hand side of (1), is uniformly bounded. Then, $x(t)$ is uniformly continuous, $x \in L_2[0, \infty)$, and thus, by Barbalat's lemma [1], $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

2.3. Example

17 Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x(t - \tau_1(t)) + \begin{bmatrix} 0 & 0 \\ -0.4 & 0 \end{bmatrix} x(t - \tau_2(t)), \tag{18}$$

19 which was analyzed in [12] for $\tau_1 \equiv 0$ and constant τ_2 , where the following stability interval was found in the frequency domain $3.3791 < \tau_2 < 4.7963$. The non-delayed system (i.e. (18) with $\tau_i \equiv 0$, $i = 1, 2$) is not asymptotically
 21 stable. Hence, the existing methods via simple Lyapunov–Frasovskii functionals (LKF), such as [14,15,2,5,3], are not

1 applicable. Only complete LKF (which corresponds to necessary and sufficient conditions for stability of the nominal
 2 system (4)) may be used [8,12] and [4]. The conditions of discretized method of Gu [8] (which are sufficient only)
 3 are not feasible in this example for constant values of τ_2 . The Lyapunov-based conditions of [12] leads to interval of
 4 almost zero length even for *constant* delays τ_2 : $\tau_2 \in (3.999999, 4.000001)$.

5 For $\tau_1 \equiv 0$ and *fast-varying* τ_2 by recent complete descriptor LKF method of [4] the asymptotic stability interval is
 6 $3.98 \leq \tau_2(t) \leq 4.02$. By Theorem 2.1 the stability interval is wider: $\tau_2(t) \in [3.98, 4.11]$. For the case of two non-zero
 7 delays choosing $\tau_1(t) \in [0, 0.002]$, we obtain by Theorem 2.1 the stability interval $\tau_2(t) \in [3.998, 4.1]$, which is wider
 8 than the one $\tau_2(t) \in [3.998, 4.002]$ by [4].

9 3. Conclusions

10 A new simple stability criterion is derived for systems with time-varying delays from the given segments. No
 11 constraints are given on the delay derivatives. The system under consideration may be unstable without delay. This
 12 case in the time domain can be treated only via complete LKF. The existing Lyapunov-based methods are restricted to
 13 the case of a single non-zero nominal delay value, and lead to complicated conditions and restrictive results.

14 A new frequency domain method is suggested, which is based on the application of the Laplace transform to the
 15 transformed system, and on the bounding technique in L_2 . The sufficient conditions are formulated in terms of the
 16 system matrices and the lengths of the delay segments. The new criterion treats the case of multiple non-zero nominal
 17 delays, where the existing methods are not applicable. In the cases, where the Lyapunov-based results are applicable,
 18 the new criterion improves the results.

19 Further improvement of the results may be achieved by choosing h_k in the middle of the delay segment and by
 'scaling' of the condition (9) of Theorem 2.1.

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