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Noise Amplification in Echo-Enabled Harmonic Generation (EEHG)*

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Abstract

Two essential elements of a seeded FEL based on the echo-enabled harmonic generation (EEHG) are the undulator-modulators, in which a laser beam modulates the beam energy. We study how the interaction of electrons in these undulators changes the noise properties of the beam. This paper is based on the method of noise analysis developed in Ref. [1] and extends it for the case of EEHG.

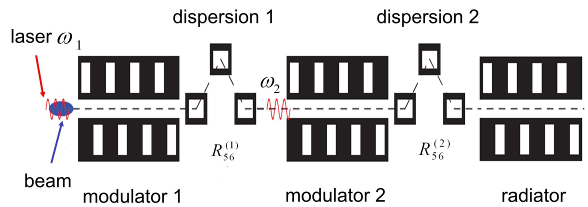

Figure 1: Schematic of the EEHG FEL.

INTRODUCTION

The echo-enabled harmonic generation was proposed in Refs. [2, 3]. The echo scheme has a remarkable up-frequency conversion efficiency and allows for generation of high harmonics with a relatively small energy modulation of the beam.

As was pointed out in Ref. [4], all seeding methods introduce additional noise to the beam. A specific mechanism of noise propagation and amplification in HGHG was studied in Ref. [5]. This mechanism takes into account interaction of electrons in the undulator-modulator and the resulting evolution of the bunching factor of the beam in the vicinity of the HGHG harmonics. In Ref. [1], the analysis of [5] was extended by explicitly considering the energy exchange of the electrons in the undulator-modulator caused by the electron interaction via undulator radiation. In this paper we apply the approach of [1] to the case or EEHG seeding.

THE PROBLEM

The schematic of the EEHG FEL is shown in Fig. 1. The EEHG FEL consists of two modulators, two dispersion sections and one radiator. Similar to the classic HGHG scheme, a laser pulse of frequency ω_1 is used to modulate the beam energy in the first undulator with the relative energy amplitude $\Delta\eta_1$. The beam then passes through the first dispersion section $R_{56}^{(1)}$. Then the beam energy is modulated with the laser of frequency ω_2 in the second modulator, with the amplitude $\Delta\eta_2$, and then passes through the second dispersion section $R_{56}^{(2)}$.

In this paper we consider the case when the laser frequencies in both modulators are the same, $\omega_1 = \omega_2 = \omega_0$ and use notation $k_0 = \omega_0/c = 2\pi/\lambda_0$. Following [1] we assume helical undulator-modulators and characterize each particle in the beam by its longitudinal coordinate z and relative energy deviation η. The values of these variables for particle i at the entrance of the seeding system are z_i and η_i, their values after the first chicane are $z_i^{(1)}$ and $\eta_i^{(1)}$, and the values at the exit (after the second chicane) are $\hat{z}_i$ and $\hat{\eta}_i$. Assuming an rms (relative) energy spread of the beam σ_η we introduce dimensionless variables [3] $\zeta = k_0 z$, $p = \eta/\sigma_\eta$, $A_{1,2} = \Delta\eta_{1,2}/\sigma_\eta$, $B_{1,2} = k_0 R_{56}^{(1,2)}\sigma_\eta$. The map from initial coordinates ζ_i, p_i at the entrance to the seeding system to the final ones $\hat{\zeta}_i$, $\hat{p}_i$ can be written in two steps

$$
\begin{aligned}
p_i^{(1)} &= p_i + A_1 \sin\zeta_i + \sum_{j\neq i} \mathcal{H}_1(\zeta_i - \zeta_j) \qquad (1)\\
\zeta_i^{(1)} &= \zeta_i + B_1\left[p_i + A_1 \sin\zeta_i + \sum_{j\neq i} \mathcal{H}_1(\zeta_i - \zeta_j)\right]\\
\hat{p}_i &= p_i^{(1)} + A_2 \sin\zeta_i^{(1)} + \sum_{j\neq i} \mathcal{H}_2(\zeta_i^{(1)} - \zeta_j^{(1)})\\
\hat{\zeta}_i &= \zeta_i^{(1)} + B_2\left[p_i^{(1)} + A_2 \sin\zeta_i^{(1)} + \sum_{j\neq i} \mathcal{H}_2(\zeta_i^{(1)} - \zeta_j^{(1)})\right]
\end{aligned}
$$

where the first two equations describe a phase-space transformation from initial state to the position before the second modulator, and the second pair transforms the variables to the exit of the seeding system. In the equations above we use $\mathcal{H}_{1,2} = h_{1,2}/\sigma_\eta$, where the functions $h_{1,2}$ are responsible for the interaction of particle in the first and the second undulators respectively. These functions are defined in [1] (note that ζ in [1] is equal to ζ/k_0 in this paper)

$$
h_{1,2}(\zeta) = -a_{1,2}\left(1 - \frac{\zeta}{2\pi N_{u1,2}}\right)\cos\zeta \qquad (2)
$$

for $2\pi N_{u1,2} > \zeta > 0$ and $h(\zeta) = 0$ otherwise, with the indexes 1 and 2 referring to the first and the second undulator,

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respectively. The parameters $a_{1,2}$ are

$$a_{1,2} = 2\pi \frac{e^2 K_{1,2}^2 N_{u1,2} \lambda_{u1,2}^2}{S_{1,2}\gamma^3 mc^2 \lambda_0}, \quad (3)$$

where $N_{u1,2}$ is the number of undulator periods, λ_0 the undulator radiation wavelength, $K_{1,2}$ the undulator parameters, $S_{1,2}$ the transverse area of the beam in the undulators, and $\lambda_{u1,2}$ the undulator periods (in what follows we will either use the indexes 1 and 2 to indicate the first and the second modulator, or do not use indexes when we refer to both).

Our goal is to compute the noise amplification factor in the final state, which we define as [1]

$$\langle F(k)\rangle = \frac{1}{N}\langle \sum_{j\neq l} e^{ik(\hat{z}_j-\hat{z}_l)}\rangle \approx N\langle e^{i\nu(\hat{\zeta}_1-\hat{\zeta}_2)}\rangle, \quad (4)$$

with $\nu = k/k_0$. The averaging in this equation is performed with an N-particle distribution function of the beam at the entrance to the system

$$\langle F(k)\rangle = \frac{N}{(\sqrt{2\pi}k_0 L)^N}\int_{-\zeta_0}^{\zeta_0} d\zeta_1 dp_1 e^{-p_1^2/2} \quad (5)$$
$$\int_{-\zeta_0}^{\zeta_0} d\zeta_2 dp_2 e^{-p_2^2/2}\dots\int_{-\zeta_0}^{\zeta_0} d\zeta_N dp_N e^{-p_N^2/2} e^{i\nu(\hat{\zeta}_1-\hat{\zeta}_2)},$$

where $\zeta_0 = k_0 L/2$. In this equation we assumed a Gaussian energy distribution, and a uniform spacial distribution over the bunch length L. We also assumed that there is no correlation between the particles at the entrance, which allows us to use the product of 1-particle distribution functions for the N-particle distribution.

Note that dropping the terms with $i = j$ in (4) we discarded the shot noise in $\langle F(k)\rangle$. For an uncorrelated positions of the particles in the beam $\langle F(k)\rangle = 0$; the interaction of particles in undulators and dispersion in the chicanes makes it nonzero.

Using (1), we express the final $\hat{\zeta}_i$ in terms of the initial coordinates and momenta

$$\hat{\zeta}_i = \zeta_i + (B_1+B_2)(p_i + A_1\sin\zeta_i) + B_2 A_2 \quad (6)$$
$$\times \sin\left(\zeta_i + B_1\left[p_i + A_1\sin\zeta_i + \sum_{j\neq i}\mathcal{H}_1(\zeta_i-\zeta_j)\right]\right)$$
$$+ (B_1+B_2)\sum_{j\neq i}\mathcal{H}_1(\zeta_i-\zeta_j) + B_2\sum_{j\neq i}\mathcal{H}_2(\zeta_i^{(1)}-\zeta_j^{(1)}).$$

In what follows we will neglect the interaction term in the argument of $\mathcal{H}_2$ using

$$\zeta_i^{(1)} \approx \zeta_i + B_1(p_i + A_1\sin\zeta_i). \quad (7)$$

As in [1], we use the the approximation of week interaction

$$a_{1,2} k R_{56}^{(1,2)} \ll 1, \quad (8)$$

and expand exponentials in (5) in Taylor series in powers of $\mathcal{H}_1$ and $\mathcal{H}_2$. After several straightforward transformations, we find from (5)

$$\langle F(k)\rangle = \frac{N}{(\sqrt{2\pi}k_0 L)^2}\sum_{k,m,k',m'}$$
$$\times\int d\zeta_1 dp_1 d\zeta_2 dp_2 e^{-p_1^2/2-p_2^2/2} G_{m,m'} R_{m,m'}$$
$$\times e^{ip_1[\nu(B_1+B_2)+mB_1]-ip_2[\nu(B_1+B_2)+m'B_1]}$$
$$\times e^{i\nu(\zeta_1-\zeta_2)} e^{ik\zeta_1-ik'\zeta_2} J_k\left(\nu A_1\left(B_1+B_2\right)\right)$$
$$\times J_{k'}\left(\nu A_1\left(B_1+B_2\right)\right) J_m\left(\nu A_2 B_2\right) J_{m'}\left(\nu A_2 B_2\right)$$
$$\times e^{im(A_1B_1\sin\zeta_1+\zeta_1)-im'(A_1B_1\sin\zeta_2+\zeta_2)}, \quad (9)$$

where $R_{m,m'}$ is obtained from the Taylor expansion and contains terms liner in $\mathcal{H}$:

$$R_{m,m'} = 1 + i\Big[\nu(B_1+B_2)\left[\mathcal{H}_1(\zeta_1-\zeta_2) - \mathcal{H}_1(\zeta_2-\zeta_1)\right]$$
$$+ B_1\left[m\mathcal{H}_1(\zeta_1-\zeta_2) - m'\mathcal{H}_1(\zeta_2-\zeta_1)\right]$$
$$+ \nu B_2\left[\mathcal{H}_2(\zeta_1^{(1)}-\zeta_2^{(1)}) - \mathcal{H}_2(\zeta_2^{(1)}-\zeta_1^{(1)})\right]\Big].$$

The function $G_{m,m'}$ is

$$G_{m,m'} = \Bigg(\int_{-\zeta_0}^{\zeta_0}\frac{d\zeta dp}{\sqrt{2\pi}k_0 L} e^{-p^2/2}$$
$$\times\exp\Big[i\nu(B_1+B_2)[\mathcal{H}_1(\zeta_1-\zeta) - \mathcal{H}_1(\zeta_2-\zeta)]$$
$$+ iB_1[m\mathcal{H}_1(\zeta_1-\zeta) - m'\mathcal{H}_1(\zeta_2-\zeta)]$$
$$+ i\nu B_2[\mathcal{H}_2(\zeta_1^{(1)}-\zeta^{(1)}) - \mathcal{H}_2(\zeta_2^{(1)}-\zeta^{(1)})]\Big]\Bigg)^{N-2}.$$

Calculation of (9) are straightforward but extremely cumbersome. We study the limit when

$$n_0 k^2 (R_{56}^{(2)})^2 a_2^2 N_{u2}\lambda_0 \ll 1,$$
$$n_0 k^2 (R_{56}^{(1)} - nR_{56}^{(2)})^2 a_1^2 N_{u1}\lambda_0 \ll 1, \quad (10)$$

where n is the EEHG harmonic number, assuming $n \gg 1$. The first of these two inequalities is analogous to the one used [1]. In is interesting that the second one involves the difference $R_{56}^{(1)} - nR_{56}^{(2)}$; while both quantities $R_{56}^{(1)}$ and $nR_{56}^{(2)}$ are large, their difference, for an EEHG scheme optimized for the nth harmonic, turns out to be of order of $R_{56}^{(2)}$ [3]. Hence, when $N_{u1} \approx N_{u2}$ and $a_1 \approx a_2$, both inequalities in (10) are about the same.

THE RESULT

The final result of the calculations involves the Fourier transformation of functions $\mathcal{H}_{1,2}$:

$$\hat{\mathcal{H}}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\mathcal{H}(x). \quad (11)$$

In the limit $N_u \gg 1$, which we assume here, the function $\hat{\mathcal{H}}(u)$ has a narrow peak (of width $\sim 1/N_u$) in the vicinity of $u = \pm 1$. For what follows, we will need the imaginary part of $\hat{\mathcal{H}}$ and its absolute value in a small vicinity of unit value of the argument, $u = s + 1$, with $s \sim N_u^{-1} \ll 1$. They are given by

$$\begin{aligned} \operatorname{Im} \hat{\mathcal{H}}(s+1) &= N_u v(2\pi N_u s), \\ |\hat{\mathcal{H}}(s+1)|^2 &= N_u^2 w(2\pi N_u s), \end{aligned} \tag{12}$$

with

$$\begin{aligned} v(x) &= \frac{x - \sin x}{2x^2}, \\ w(x) &= \frac{2 + x^2 - 2\cos(x) - 2x \sin x}{4x^4}. \end{aligned} \tag{13}$$

As was pointed out, we assume that the EEHG seeding is optimized for the nth harmonic of the laser frequency ($n \gg 1$) and calculate the noise in the vicinity of the frequency $n\omega_0$ with $\Delta\nu = \omega/\omega_0 - n$. We first define the coefficients

$$\begin{aligned} b = & e^{-(\Delta\nu B_1 + \nu B_2 - B_1)^2} J_{1+n} \left(\nu A_2 B_2\right)^2 \\ & \times [J_0 \left(\nu A_1 B_2 - (1-\Delta\nu) A_1 B_1\right)^2 \\ & + J_2 (\nu A_1 B_2 - (1-\Delta\nu) A_1 B_1)^2] \end{aligned} \tag{14}$$

and

$$\begin{aligned} c = & e^{-(\nu B_2 - B_1)^2} J_1 \left(\nu A_1 B_2 - B_1 A_1\right)^2 \\ & \times [J_n \left(\nu A_2 B_2\right)^2 + J_{2+n} (\nu A_2 B_2)^2]. \end{aligned} \tag{15}$$

We then introduce the following quantities

$$\begin{aligned} F_1 &= 2 n_0 \lambda_0 \alpha_1 N_{u1} (\Delta\nu B_1 + \nu B_2 - B_1) v(x_1) \\ P_1 &= n_0^2 \lambda_0^2 \alpha_1^2 N_{u1}^2 (\Delta\nu B_1 + \nu B_2 - B_1)^2 w(x_1) \\ F_2 &= 2 n_0 \lambda_0 \alpha_2 N_{u2} \nu B_2 v(x_2) \\ P_2 &= n_0^2 \lambda_0^2 \alpha_2^2 N_{u2}^2 \nu^2 B_2^2 w(x_2) \end{aligned} \tag{16}$$

where $x_{1,2} = 2\pi N_{u1,2} \Delta\nu$ and $\alpha_{1,2} = a_{1,2}/\sigma_\eta$. The noise factor is equal to

$$\langle F(k) \rangle = b(F_1 + P_1) + c(F_2 + P_2), \tag{17}$$

where the term $b(F_1 + P_1)$ on the right-hand side is the contribution of the first undulator, and the term $c(F_2 + P_2)$ the contribution from the second one.

It is important to emphasize here that a high-harmonic EEHG requires $B_1 \gg 1$ (a strong first chicane), however optimization of the seeding at harmonic $n \gg 1$ leads to the condition that the difference $nB_2 - B1 \sim 1$. It follows from this observation that the arguments in the above expressions that involve the difference $\nu B_2 - B_1$ being a difference of two large numbers remain of the order of one in the vicinity of the resonant frequency $\nu = n$.

NUMERICAL SIMULATIONS

To verify our analytical solution we carried out a series of numerical simulations of noise in EEHG. An ensemble of particles with random coordinates z_i uniformly distributed on the interval $[0, L]$ and momenta p_i randomly chosen from a Gaussian distribution was subjected to the coordinate transformation (1). The sum $N^{-1} \sum_{j \neq l} e^{ik(\hat{z}_j - \hat{z}_l)}$ was calculated and averaged over N_{av} repetitions each time with a new random seed. To suppress the edge effects near the ends of the beam, we used a periodic boundary condition for z_i treating z_i and $z_i + lL$ as identical, where l is an integer. Typical parameters for the simulations were $N \approx 10^4$, $N_{\text{av}} \approx 5000$ and $L = 100\lambda_0$. Such simulations are too slow to be able to reproduce real parameters of an experiment (which are characterized by very small interaction strength and very large number of particles per unit length of the beam, see next Section), but they are useful as a test of the analytical expressions derived in the previous section.

An example of one of such simulations is shown in Fig. 2. In this simulation we used the following parameters

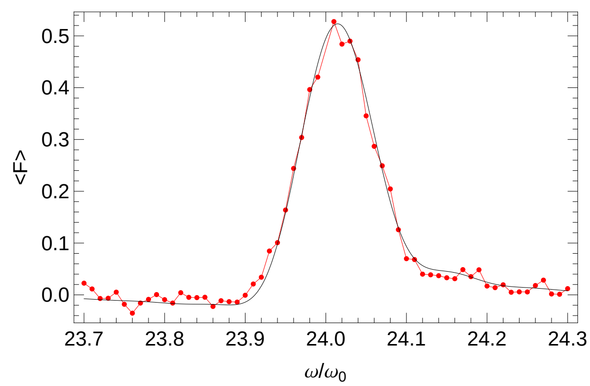

Figure 2: Noise amplification factor $\langle F(k) \rangle$. The red dots connected by red line are the results of computer simulation and the black curve is the theory (17).

for the EEHG, optimized for generation of 24th harmonic: $A_1 = 3.0$, $A_2 = 1.0$, $B_1 = 26.77$, and $B_2 = 1.14$. We assumed 10 undulator periods in both undulators and the same interaction strengths $a_1/\sigma_\eta = a_2/\sigma_\eta = 1.2 \cdot 10^{-3}$. One can see a very good agreement between the theory and the simulation.

PRACTICAL EXAMPLE

As a numerical example, let us consider the nominal parameters of the seeded FEL at the Fermi@Elettra project [6]. The electron beam energy is 1.2 GeV, the slice energy spread is 150 keV and the peak current is 800 A. We assume the wavelength of the seed laser $\lambda_0 = 240$ nm and consider generation of the 24th harmonic (the wavelength of 10 nm) using EEHG. One of the possible options for the seeding system was worked out in Ref. [7]: $A_1 = 3.0$, $A_2 = 3.0$, $B_1 = 8.6$, and $B_2 = 0.38$ which corresponds to $R_{56}^{(1)} = 2.6$ mm and $R_{56}^{(2)} = 0.12$ mm. We choose for the modulator-undulator parameters: $\lambda_{u1,2} = 15$ cm, $N_{u1,2} = 6$. We remind the reader that we use a helical undulator with the value of K, which can be inferred from the above parameters, $K = 1.84$. We assume that the transverse size of the beam in the modulator-undulator is $\sigma_x = \sigma_y = 100$ μm and use for the parameter S in (3) $S = 2\pi\sigma_z\sigma_y$.

First we estimate the parameters $a_{1,2}$ in (3) to obtain $a_1 = a_2 = 6.3 \times 10^{-11}$. We then find $a_1 k R_{56}^{(1)} \approx 1.0 \times 10^{-4}$ and $a_2 k R_{56}^{(2)} \approx 4.5 \times 10^{-6}$, which justifies our approximations (8). We also find that $n_0 \lambda_0 k^2 (R_{56}^{(2)})^2 a_2^2 N_{u2} = 7.5 \times 10^{-4}$ and $n_0 \lambda_0 k^2 (R_{56}^{(1)} - n R_{56}^{(2)})^2 a_1^2 N_{u1} = 1.4 \times 10^{-3}$ which shows that inequalities (10) are satisfied. The plot of the noise amplification factor calculated using (17) is shown in Fig. 3. Calculations

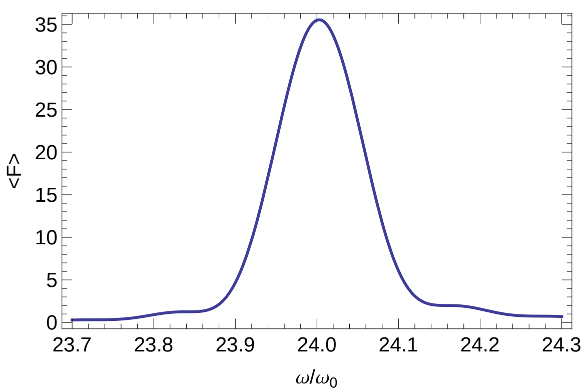

Figure 3: Noise amplification factor $\langle F(k) \rangle$ for parameters of FERMI.

show that the noise is dominated by the second undulator, and moreover, the nonlinear term P_2 in Eq. (17). Insensitivity of the EEHG noise to the parameters of the first modulator was previously found in Ref. [8]. Remarkably, P_2 scales as the fourth power of the undulator periods, N_{u2}^4, (because $a_2 \propto N_{u2}$), and hence can be controlled by the choice of N_{u2}. Of course, decreasing N_{u2} leads to increasing the laser power for producing a given amplitude of energy modulation, however, one of the advantages of EEHG is that it can be accomplish with relatively low amplitudes (equal to 2-3 energy spread in the beam).

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