Dual-Diversity Square-Law Combining with Deterministic Weights in Correlated Rayleigh Fading

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Abstract — When a multi-branch diversity receiver has information on the channel statistics instead of the instantaneous channel state information and noncoherent reception is employed, a deterministically weighted square-law combining scheme can be used to improve the error performance over conventional square-law combining when the average branch signal-to-noise ratios are not equal. We consider such a scheme for a dual-diversity system with orthogonal signaling in correlated Rayleigh fading. The square-law output of each branch is multiplied by a deterministic weight and the weighted outputs are additively combined. The weights are chosen so as to minimize the symbol error probability (SEP). The SEP in terms of these weights is derived in closed form, and from it a method for computing the weights is obtained. The merit of this scheme when compared with conventional square-law combining is demonstrated through numerical examples.

Keywords: Correlated Rayleigh fading, deterministic weights, receive diversity, square-law combining, symbol error probability.

I. INTRODUCTION

For diversity receivers using coherent reception in the presence of fading, linear combining methods [1] like maximal-ratio combining (MRC) [2], equal-gain combining (EGC) [3, 4], and optimum combining [5, 6] effectively combat fading and ensure very good error performance. However, these combining methods require complete or partial channel state information (CSI), which is typically acquired by means of pilot symbols that precede a data symbol stream. Accurate channel estimation demands high pilot overhead, which, in turn, reduces the data rate.

In systems where the tolerable error rate is not too stringent, noncoherent reception can be employed. In addition, when the CSI is not available at the receiver, square-law combining (also referred to as postdetection combining) [7, 8] can be used along with noncoherent reception to combat the effects of fading. However, when a multi-branch diversity receiver has information on the channel statistics instead of the instantaneous CSI, a deterministically weighted square-law combining scheme (see, e.g., [9, 10]) can be used to improve the error performance when the average branch signal-to-noise ratios (SNRs) are not equal. We consider such a scheme for a dual-diversity system in correlated Rayleigh fading in which the square-law output of each branch is multiplied by a deterministic weight and the weighted outputs are additively combined. Orthogonal signaling is used. The weights are chosen so as to minimize the symbol error probability (SEP). A closed form expression for the SEP in terms of these weights is derived. From this expression, a method for computing the weights is obtained. The merit of this scheme when compared with conventional square-law combining is demonstrated through numerical examples.

II. SYSTEM MODEL

Consider a dual-diversity reception system in flat Rayleigh fading using symbol-by-symbol detection and M-ary orthogonal equienergy signaling with equal a priori probabilities. The $2 \times 1$ complex baseband received signal vector in a symbol interval is expressed as

$$r(t) = hs(t) + n(t), \quad 0 \leq t < T_s,$$

where $T_s$ is the symbol duration, $s(t)$ the information-bearing signal, $h$ the random complex fading gain vector having a $CN(0_2, K_h)$ distribution, where $0_2$ is the $2 \times 1$ vector of zeros (that is, $h$ is a complex circular Gaussian random vector with zero mean $fading\ covariance\ matrix$ $K_h$), and $n(t)$ the additive white Gaussian noise (AWGN) vector, which is a zero-mean complex circular vector random process independent of $h$ having covariance matrix

$$E[n(t_1)n^H(t_2)] = 2N_0\delta(t_1 - t_2)I_2,$$

with $E[\cdot]$ denoting the expectation operator, $(\cdot)^H$ the Hermitian (conjugate transpose) operator, $\delta(\cdot)$ the Dirac delta function, and $I_2$ the $2 \times 2$ identity matrix. We take the case of correlated fading, so that

$$K_h = \begin{bmatrix} K_{h_{11}} & K_{h_{12}} \\ K_{h_{21}} & K_{h_{22}} \end{bmatrix},$$

with $K_{h_{11}}, K_{h_{22}}$ distinct.

The information-bearing signal $s(t)$ is one of $M$ orthogonal equienergy signaling waveforms $S_0(t), \ldots, S_{M-1}(t)$
corresponding to symbols 0, . . . , M − 1, respectively, each having a support of [0, T_s) and energy 2E_s, implying
\[ \int_0^{T_s} S_l(t)S^*_l(t)dt = \begin{cases} 2E_s & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}, \quad i, j \in \{0, \ldots, M - 1\}, \] (4)
where \((\cdot)^*\) denotes the complex conjugate. The orthogonal signaling waveform set \(S\) is thus given by
\[ S = \{S_0(t), \ldots, S_{M-1}(t)\}. \] (5)

III. WEIGHTED SQUARE-LAW COMBINING

Let \(r_l(t)\) denote the complex baseband signal received over the \(l\)th diversity branch (the \(l\)th row of \(r(t)\) given by (1)), \(h_l\) the \(l\)th branch fading gain, and \(n_l(t)\) the \(l\)th branch AWGN. We can then rewrite (1) as
\[ r_l(t) = h_lS(t) + n_l(t), \quad l = 1, 2, \quad 0 \leq t < T_s. \] (6)

Let \(h = [h_1, h_2]^T\), where \((\cdot)^T\) is the transpose.

At the combiner, the following operations are performed:

1. Each branch received signal is passed through \(M\) squared-envelope detectors corresponding to the \(M\) signaling waveforms. The \(l\)th branch envelope detector output corresponding to signal \(S_l(t)\) is given by
   \[ \left| \int_0^{T_s} r_l(t)S^*_l(t)dt \right|^2. \]

2. For \(l = 1, 2\), the \(l\)th branch squared-envelope detector output corresponding to signal \(S_l(t)\) is multiplied by a deterministic positive weight \(g_l\), and the 2 weighted outputs are additively combined. We call this kind of combining weighted square-law combining. This results in the combiner output
   \[ \sum_{l=1}^{2} g_l \left[ \int_0^{T_s} r_l(t)S^*_l(t)dt \right]^2 \]
   corresponding to signal \(S_l(t)\), for \(i = 0, \ldots, M - 1\).

3. The decision device chooses the signal corresponding to the maximum of the \(M\) combiner outputs. Thus the decision rule is expressed as
   \[ \hat{s}(t) = \arg \left\{ \max_{s(t) \in S} \sum_{l=1}^{2} g_l \left[ \int_0^{T_s} r_l(t)s^*_l(t)dt \right]^2 \right\}, \] (7)
   where \(S\) is given by (5).

The weights \(g_1, g_2\) are chosen such that they minimize the SEP. Let \(g = [g_1, g_2]^T\) denote the weight vector.

Suppose \(S_m(t)\) is transmitted, implying \(r_l(t) = h_lS_m(t) + n_l(t)\). Using (4), the \(l\)th branch envelope detector output corresponding to signal \(S_l(t)\) is given by
\[ U_l(m, i) = \left| \int_0^{T_s} [h_lS_m(t) + n_l(t)]S^*_l(t)dt \right|^2 \]
\[ = \begin{cases} 2E_s|h_l|^2 & \text{if } i = m, \\ |z_l(i)|^2 & \text{if } i \neq m, \end{cases} \] (8)
where the random variable (r.v.) \(z_l(i)\) is expressed as
\[ z_l(i) = \int_0^{T_s} n_l(t)S^*_l(t)dt, \quad i = 0, \ldots, M - 1, \quad l = 1, 2. \] (9)

Due to the orthogonality of the signaling waveform set (see (4)) and the whiteness of the noise (see (2)), the r.v.s \(z_l(i), \quad i = 0, \ldots, M - 1, \quad l = 1, 2,\) are independent and identically distributed complex circular Gaussian variables, each having a \(CN(0, 4E_sN_0)\) distribution.

From (7), the decision statistic \(D(m, i)\) is given by
\[ D(m, i) = \sum_{l=1}^{2} g_lU_l(m, i) = \begin{cases} 2 \sum_{l=1}^{2} g_l |2E_s|h_l|^2 & \text{if } i = m, \\ 2 \sum_{l=1}^{2} g_l |z_l(i)|^2 & \text{if } i \neq m. \end{cases} \] (10)

Owing to the symmetry of the \(M\)-ary orthogonal constellation, the SEP (as a function of \(g\)), which we denote as \(P_e(g)\), is given by
\[ P_e(g) = 1 - \Pr \left( D(m, i) < D(m, m), \ i \neq m, \ i = 0, \ldots, M - 1 \right). \] (11)

Note that \(D(m, 0), \ldots, D(m, M - 1)\) are independent.

Using results on the characteristic function (c.f.) of quadratic forms of Gaussian random variables [11], the conditional c.f. of \(D(m, m)\), conditioned on the fading gain vector \(h\), is given by
\[ \Psi_{D(m,m)}(h; \omega) = E \left[ e^{j\omega D(m,m)} | h \right] \]
\[ = \exp \left\{ j\omega 4E_s \sum_{l=1}^{2} g_l|h_l|^2 \right\} \prod_{l=1}^{2} \left( 1 - j\omega 4E_sN_0g_l \right), \] (12)
while for \(i \neq m\), the c.f. of \(D(m, i)\) is given by
\[ \Psi_{D(m,i)}(\omega) = \frac{1}{\prod_{l=1}^{2} \left( 1 - j\omega 4E_sN_0g_l \right)}. \] (13)
Now the joint c.f. of $|h_1|^2, |h_2|^2$ is expressed as
\[
\Psi_{|h_1|^2, |h_2|^2}(j\omega_1, j\omega_2) = \mathbb{E} \left[ e^{j\omega_1|h_1|^2 + j\omega_2|h_2|^2} \right].
\] (14)

Denoting the vector $\omega$ as $\omega = [\omega_1, \omega_2]^T$ and using results on the c.f. of quadratic forms of Gaussian random variables, we can rewrite this joint c.f. as
\[
\Psi_{|h_1|^2, |h_2|^2}(j\omega) = \frac{1}{\det(I_2 - j\text{diag}(\omega)K_h)}.
\] (15)

Substituting (3) in (15) we get
\[
\Psi_{|h_1|^2, |h_2|^2}(j\omega) = \frac{1}{(1 - j\omega_1^2K_{h11} - j\omega_2^2K_{h22}) - j(\omega_1^2K_{h12} + \omega_2^2K_{h21}) - \omega_1\omega_2[(K_{h11} - |K_{h21}|^2)]}.
\] (16)

We define the average SNR of the $i$th branch as $\Gamma_i$, given by
\[
\Gamma_i \triangleq \frac{E_s}{N_0}K_{h_{ii}}, \quad i = 1, 2,
\] (17a)
and the fading correlation coefficient as $\rho$, given by
\[
\rho \triangleq \frac{|K_{h_{21}}|}{\sqrt{K_{h_{11}}K_{h_{22}}}}.
\] (17b)

In addition, we define the quantities $a_1, a_2$, and $\Delta$ as
\[
a_i \triangleq 2 + \Gamma_i = 2 + \frac{E_s}{N_0}K_{h_{ii}}, \quad i = 1, 2,
\]
\[
\Delta \triangleq (2 + \Gamma_1)(2 + \Gamma_2) - \rho^2\Gamma_1\Gamma_2 = a_1a_2 - \left(\frac{E_s}{N_0}\right)^{2}|K_{h_{21}}|^2.
\] (18)

Averaging (12) over the statistics of $h$ using (16) results in
\[
\Psi_{D(m,m)}(j\omega) = \frac{1}{\prod_{l=1}^M(1 - j\omega^2E_sN_0b_l)}
\] (19),
where $b_1$ and $b_2$ are given by
\[
b_1 = B + \sqrt{B^2 - 4A}, \quad b_2 = B - \sqrt{B^2 - 4A},
\] (20a)

where
\[
A = b_1b_2 = g_1g_2(\Delta - a_1 - a_2 + 1),
\]
\[
B = b_1 + b_2 = g_1(a_1 - 1) + g_2(a_2 - 1).
\] (20b)

It can be shown from (13) that the probability density function (p.d.f.) of $D(m,i)$ for $i \neq m$ is given by
\[
f_{D(m,i)}(x) = \exp\left(-\frac{x}{4E_sN_0b_1}\right) - \exp\left(-\frac{x}{4E_sN_0b_2}\right),
\] (21)

while from (19) the p.d.f. of $D(m,m)$ is given by
\[
f_{D(m,m)}(x) = \frac{\exp\left(-\frac{x}{4E_sN_0b_1}\right) - \exp\left(-\frac{x}{4E_sN_0b_2}\right)}{4E_sN_0(g_1 - g_2)},
\] (22)

where $b_1$ and $b_2$ are given by (20). Thus (24) is a closed form expression for the SEP in terms of the weights $g_1$ and $g_2$.

We choose the $2 \times 1$ weight vector $g$ so as to minimize $P_e(g)$. We find from (24) that $P_e(g)$ depends only on the ratio $g_2/g_1$. To find the weight vector components $g_1$ and $g_2$, we impose the constraint
\[
g_1 + g_2 = 1
\] (25)
and put
\[
g_1 = \frac{1}{1+\epsilon}, \quad g_2 = \frac{\epsilon}{1+\epsilon},
\] (26)
so that $\epsilon = g_2/g_1$. Substituting (26) in (24), we get the SEP as a function of $\epsilon$, given by
\[
P_e(\epsilon) = \sum_{k=1}^{M-1} \binom{M-1}{k} \left(-\frac{1}{1+\epsilon}\right)^{k+1}
\]
\[
\times \sum_{l=0}^{k} \binom{k}{l} (-1)^l \frac{g_2^l}{g_1}
\]
\[
\times \frac{1}{(k-l)^2 + (k-l)^3 + \epsilon^2 + \epsilon^3}.
\] (27)
The value of $\epsilon$ is chosen so as to minimize $P_e(\epsilon)$.

Let the function $G_{k,l}(\epsilon, \Delta, a_1, a_2)$ denote the denominator on the right-hand side of (27), implying

$$G_{k,l}(\epsilon, \Delta, a_1, a_2) = \left( \Delta - a_1 - a_2 + 1 \right) \left( l + (k-l)\epsilon \right)^2 + \left( a_1 - 1 + \epsilon(a_2 - 1) \right) \times \left( l + (k-l)\epsilon \right) + \epsilon.$$  

(28)

Differentiating the right-hand side of (27) with respect to $\epsilon$ and putting the result to zero we get the equation

$$\sum_{k=1}^{M-1} \left( \frac{M-1}{k} \right) (-1)^{k+1} \frac{(-1)^{k+1}}{(1-\epsilon)^{k+1}} \times \sum_{l=0}^{k} \left( \frac{k^l}{l!} \right) (-1)^l \epsilon \frac{H_{k,l}(\epsilon, \Delta, a_1, a_2)}{[G_{k,l}(\epsilon, \Delta, a_1, a_2)]^2} = 0,$$  

(29)

where

$$H_{k,l}(\epsilon, \Delta, a_1, a_2) = \left( l \times (1-k)\epsilon \right) G_{k,l}(\epsilon, \Delta, a_1, a_2) + (1-\epsilon) \times \left( (\Delta - a_1 - a_2 + 1) \left( l^2 - (k-l)^2\epsilon^2 \right) + \left( a_1 - 1 \right) l - (a_2 - 1)(k-l)^2\epsilon^2 \right).$$  

(30)

The left-hand side of (29) is a weighted sum of rational functions of $\epsilon$, implying that the value of $\epsilon$ is the root of a polynomial in $\epsilon$. It can be shown using the second derivative of $P_e(\epsilon)$ in (27) that $\epsilon$ given by (29) results in a minimum. The equation (29) can be solved numerically to obtain the value of $\epsilon$, and correspondingly the values of $g_1$ and $g_2$ using (26).

**A. Case of $M = 2$**

In the case of $M = 2$ (orthogonal binary signaling, such as orthogonal binary frequency-shift keying), (29) results in the quadratic equation

$$(\Delta - a_1)^2(\Delta - 2a_2)\epsilon^2 + 2(\Delta - a_1)(\Delta - a_2)\epsilon - (\Delta - a_2)^2(\Delta - 2a_1) = 0.$$  

(31)

On solving the quadratic equation (31) for $\epsilon$, we obtain

$$\epsilon = \frac{(\Delta - a_2)(\Delta - 2a_1)}{(\Delta - a_1)(\Delta - 2a_2)};$$  

(32)

where $a_1$, $a_2$, and $\Delta$ are given by (18).

Using the definitions in (17), we can rewrite (32) as

$$\epsilon = \frac{\left( [1 + \Gamma_1^{-1}] [1 + 2\Gamma_2^{-1}] - \rho^2 \right) \left( 1 + 2\Gamma_1^{-1} - \rho^2 \right)}{\left( [1 + \Gamma_2^{-1}] [1 + 2\Gamma_1^{-1}] - \rho^2 \right) \left( 1 + 2\Gamma_2^{-1} - \rho^2 \right)}.$$  

(33)

Substituting (32) in (27) with $M = 2$, we obtain

$$P_e(\epsilon) = \frac{(3\Delta^2 - 4[a_1 + a_2]\Delta + 4a_1a_2)}{\Delta^2(\Delta - a_1 - a_2)}.$$  

(34)

On the other hand, by putting $\epsilon = 1$ on the right-hand side of (27) with $M = 2$, we get the SEP for conventional square-law combining ($g_1 = g_2 = 1/2$), denoted as $P_e,\text{conv}$, which is given by

$$P_e,\text{conv} = \frac{3(\Delta - a_1 - a_2)}{\Delta^2}.$$  

(35)

Now (34) can be rewritten as

$$P_e(\epsilon) = \frac{3(\Delta - a_1 - a_2)}{\Delta^2} - \frac{(a_1 - a_2)^2}{\Delta^2(\Delta - a_1 - a_2)} = P_e,\text{conv} - \frac{(a_1 - a_2)^2}{\Delta^2(\Delta - a_1 - a_2)};$$  

(36)

which clearly implies that when $a_1 \neq a_2$, that is, $\Gamma_1 \neq \Gamma_2$, $P_e(\epsilon) < P_e,\text{conv}$.

**IV. NUMERICAL RESULTS**

We present numerical results for orthogonal signaling. The SEP $P_e(\epsilon)$ for weighted square-law combining is given by (27), while the SEP $P_e,\text{conv}$ for conventional square-law combining is given by $P_e(1)$, which is obtained by substituting

$$f_{D(m,i)}(x) = \frac{x \exp \left( -\frac{x}{2E_s\bar{N}_0} \right)}{(2E_s\bar{N}_0)^2}, \quad x \geq 0,$$  

(37)

in (23).

Fig. 1 shows the SEP of weighted square-law combining versus $g_2/g_1$, computed using (27), for $\Gamma_1 + \Gamma_2$, the total average SNR of the two branches, having a value of 25 dB with varying $\Gamma_2/\Gamma_1$ and varying fading correlation coefficient $\rho$. We find that a minimum exists for each curve which increases with $\Gamma_2/\Gamma_1$ as well as with $\rho$.

We also observe that the error performance improves with the decrease of branch SNR imbalance, that is, with the decrease of $\Gamma_2/\Gamma_1$. It can be shown from (29) that when $\Gamma_2/\Gamma_1 = 1$, the minimum occurs at $g_2/g_1 = 1$ irrespective of the value of $\rho$.

Fig. 2 shows the SEP, computed using (27), with $\epsilon$ chosen to minimize the SEP in case of weighted square-law combining and $\epsilon = 1$ for conventional square-law combining, versus the total average SNR $\Gamma_1 + \Gamma_2$ of the two branches for different values of the ratio $\Gamma_2/\Gamma_1$ and $\rho = 0.95$. We find from the plots that the while the performance improves with decreasing branch SNR imbalance, the performance gap between the weighted and the conventional square-law combining increases with increasing branch SNR imbalance. It is observed that weighted square-law combining can offer a reasonable SNR gain.
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For example, when $\Gamma_1 + \Gamma_2 = 25$ dB and $\Gamma_2/\Gamma_1 = 90$, an SNR gain of about 0.7 dB is obtained. This shows the merit of the weighted square-law combining scheme when a multi-branch diversity receiver has information on the channel statistics and noncoherent reception is employed.

An example of such a case is when an antenna, which has known lower gain than (and may have approximately known fading correlation in typical handset fading environments with) the main antenna, is added to provide multipath diversity to a handset that uses noncoherent combining, with the weights preset by the handset manufacturer.

V. Conclusion

For a dual-branch diversity receiver having information on the channel statistics instead of the instantaneous CSI in a correlated Rayleigh fading environment, a weighted square-law combining scheme using deterministic weights has been considered. Orthogonal equienergy signaling has been employed. We have derived a closed-form expression for the SEP in terms of the weights and chosen the weights to minimize the SEP. It has been found that when the average branch SNRs are unequal, the weighted square-law combining improves the error performance over conventional square-law combining.

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