OPTIMAL BROADCASTING IN 2-DIMENSIONAL MANHATTAN STREET NETWORKS

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ABSTRACT
Broadcasting is the process of disseminating a message from a node of a communication network to all other nodes as quickly as possible. In this paper we consider Manhattan Street Networks (MSNs) which are mesh-structured, toroidal, directed, regular networks such that locally they resemble the geographical topology of the avenues and streets of Manhattan. Previous work on these networks has been mainly devoted to the study of the average distance and point-to-point routing schemes. Here we provide an algorithm which broadcasts optimally in a 2-dimensional $M \times N$ Manhattan Street Network ($M$ and $N$ even).

KEY WORDS
Manhattan Street Networks, broadcasting, communication networks.

1 Introduction

The study of a class of directed torus networks known as Manhattan Street Networks has received significant attention since they were introduced by Maxemchuk in 1985 [9] as a unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan. Most of the work has been devoted to the computation of the average distance and the generation of routing schemes for the 2-dimensional case [8, 10]. However the results are very often given as conjectures supported by computer simulations [10, 8]. The study of spanning trees [3] in a Manhattan Street Network (MSN) has allowed the computation of the diameter and the design of a multi-port broadcasting algorithm. More recently, Varvarigos in [11] evaluated the mean internodal distance, provided a shortest path routing algorithm and constructed edge-disjoint Hamiltonian cycles in the general 2-dimensional case. The multidimensional natural extension of the MSN has been considered by Banerjee et al., see [1, 2], with the determination of the average distance of a 3-dimensional MSN, and a conjecture for higher dimensions. Chung and Agrawal [4] studied the diameter and provided routing schemes for a 3-dimensional construction based on 2-dimensional MSNs, although the result is not strictly a 3-dimensional MSN.

In this paper we provide, for general 2-dimensional MSNs, an algorithm to broadcast in a Manhattan Street Network and a proof of its optimality.

2 Notation and Preliminary Results

We consider bidimensional $M \times N$ Manhattan Street Networks with $M \times N$ vertices. The vertices are labeled with pairs $(u_1, u_2)$ where $0 \leq u_1 \leq M - 1$ and $0 \leq u_2 \leq N - 1$. The edges of this graph are $\{(u_1, u_2) \rightarrow ((u_1 \pm 1) \mod M, u_2)\}$ and $\{(u_1, u_2) \rightarrow (u_1, (u_2 \pm 1) \mod N)\}$ where the sign is plus or minus depending on whether $u_1$ and $u_2$ are even or odd, see Figure 1. Therefore the MSNs considered in this study are modeled by directed regular graphs with indegree and outdegree two which are vertex symmetric.

Broadcasting in a graph is the process of spreading a message known initially by one vertex, subject to the following rules. The transfer of the message from one vertex to another (termed a call) takes one unit of time. A vertex can only call an adjacent vertex. A vertex can participate in at most one call per unit of time. A broadcast algorithm is a formal description of this process.

Given a connected digraph $G$ and a vertex $u$, the broadcast time of $u$, denoted $b(u)$, is the minimum number of time units required to broadcast a message originating at $u$. The broadcast time of the graph $G$ is defined $b(G) = \max\{b(u) | u \in G\}$. For any vertex $u$ in a connected graph with $|V|$ vertices, $b(u) \geq \lceil \log_2 |V| \rceil$, since...
during each time unit the number of vertices informed can at most double. For a vertex symmetric graph, the broadcast time is equal to the broadcast time of any of its vertices.

For grid graphs and in the undirected case, optimal broadcasting algorithms are known, for example, for 2- grids \((b(M_{m \times n}) = m + n - 2 = D)\), and 2-torus \((b(T_{m \times n}) = D,\) if \(m\) and \(n\) are even and \(b(T_{m \times n}) = D + 1\) otherwise), see [7]. Here we present the first optimal one-port broadcasting algorithms for general 2-dimensional Manhattan Street Networks.

The following result will be used in Section 4 to prove the optimality of the broadcast scheme introduced in this paper.

**Lemma 1** \(\text{For a graph } G \text{ with diameter } D, b(G) \geq D.\) Moreover, if there exist three vertices \(u, v_1\) and \(v_2\) such that \(v_1\) and \(v_2\) are at distance \(D\) of \(u\), then \(b(G) \geq D + 1\).

**Proof.** Given a graph \(G\) with diameter \(D\) let us assume that there exists a broadcasting protocol for it. By recurrence on \(i\), we see that at step \(i\) of this protocol, at most one vertex at distance \(i\) from the originator could be informed. Hence the first assertion. Moreover, if there exist two vertices at distance \(D\) from the originator, only one could be informed at time \(D\) and we would need at least one more step to finish the broadcasting process. Then, \(b(G) \geq D + 1.\) \(\Box\)

The diameter of a MSN has been computed in [3] by considering spanning trees. We confirm this result by deriving the diameter of a MSN from the comparison of the distribution of distances with an undirected torus of the same dimensions. We have also determined the distribution of vertices at each distance, which allows a computation of the average distance.

First we note that the graph is vertex symmetric and therefore the distribution of distances is the same from every node. We recall also that the diameter of the undirected torus with dimensions \(M \times N\) is \(|\lfloor \frac{M}{2}\rfloor + \lfloor \frac{N}{2}\rfloor|\).

In Figures 2 to 5 we represent the MSN with the vertex from which we start counting distances in a central position. The label of each vertex represents the distance variation for it between the MSN and the undirected subgraph. Figures correspond to each possible combination of dimensions: \(M = 0 \text{ mod } 4, N = 0 \text{ mod } 4; M = 0 \text{ mod } 4, N = 2 \text{ mod } 4; M = 2 \text{ mod } 4, N = 0 \text{ mod } 4\) and \(M = 2 \text{ mod } 4, N = 2 \text{ mod } 4\).

The vertices at maximum distance are:

\(M = 0 \text{ mod } 4, N = 0 \text{ mod } 4: (\frac{M}{2}, -\frac{N}{2} + 1), (-\frac{M}{2} + 1, \frac{N}{2})\)

\(M = 0 \text{ mod } 4, N = 2 \text{ mod } 4: (\frac{M}{2}, \frac{N}{2}), (-\frac{M}{2} + 1, 1, \frac{N}{2} - 1), (-\frac{M}{2} + 1, -\frac{N}{2} + 1), (\frac{M}{2}, -\frac{N}{2} + 2)\)

\(M = 2 \text{ mod } 4, N = 0 \text{ mod } 4: (\frac{M}{2} + \frac{N}{2}), (-\frac{M}{2} + 1, -\frac{N}{2} + 1), (\frac{M}{2} - 1, -\frac{N}{2} + 1), (-\frac{M}{2} + 2, \frac{N}{2})\)

\(M = 2 \text{ mod } 4, N = 2 \text{ mod } 4: (\frac{M}{2} + \frac{N}{2}), (-\frac{M}{2} + 1, -\frac{N}{2} + 1)\)
Figure 5. Distribution of the difference of distances (from vertex \((0, 0)\)) in a centered MSN with respect to the undirected 2-dimensional \(M \times N\) torus for \(M = 2\) mod 4 and \(N = 0\) mod 4. The originator and the more distant vertices are shown in a different size.

From this study we can also provide the distribution of vertices at each distance from a given vertex which is as follows (\(n_k\) denotes the number of vertices at distance \(k\) of a given vertex):

\[
M = 0 \text{ mod } 4, N = 0 \text{ mod } 4 \quad (M, N > 4):
\]

\[
n_1 = 2, n_2 = 4, n_3 = 8, n_4 = 11,
\]

\[
n_k = \begin{cases} 
4k-4 & \text{if } 5 \leq k \leq \frac{N}{4} \\
N-4+2k & \text{if } \frac{N}{4}+1 \leq k \leq \frac{N}{2}+2, M \neq N \\
2N & \text{if } \frac{N}{2}+3 \leq k \leq \frac{3N}{2}+2, M = N \\
2N-4 & \text{if } \frac{3N}{2}+3 \leq k \leq \frac{5N}{2}, M = N \\
M+2N-2k & \text{if } \frac{5N}{2}+1 \leq k \leq \frac{N}{2}+2, M \neq N \\
2M+2N+4-4k & \text{if } \frac{N}{2}+3 \leq k \leq \frac{3N}{2}+2 + \frac{N}{2} \\
\end{cases}
\]

\[
\text{and } n_{\frac{M}{2} + \frac{N}{2} + 1} = 2
\]

\[
M = 0 \text{ mod } 4, N = 2 \text{ mod } 4 \quad (M, N > 4):
\]

\[
n_1 = 2, n_2 = 4, n_3 = 8, n_4 = 11,
\]

\[
n_k = \begin{cases} 
4k-4 & \text{if } 5 \leq k \leq \frac{N}{4}+1 \\
3N-2k & \text{if } \frac{N}{4}+2 \leq k \leq \frac{N}{2}+2, M \neq N \\
2N & \text{if } \frac{N}{2}+4 \leq k \leq \frac{3N}{2}+1, M = N \\
2M+2N+4-4k & \text{if } \frac{N}{2}+3 \leq k \leq \frac{3N}{2}+2 + \frac{N}{2} \\
\end{cases}
\]

\[
M = 2 \text{ mod } 4, N = 0 \text{ mod } 4 \quad (M, N > 4):
\]

\[
n_1 = 2, n_2 = 4, n_3 = 8, n_4 = 11,
\]

\[
n_k = \begin{cases} 
4k-4 & \text{if } 5 \leq k \leq \frac{N}{4}+1 \\
3N-2k & \text{if } \frac{N}{4}+2 \leq k \leq \frac{N}{2}+2 \\
2N & \text{if } \frac{N}{2}+4 \leq k \leq \frac{3N}{2}+1, M \neq N \\
2M+2N+4-4k & \text{if } \frac{N}{2}+3 \leq k \leq \frac{3N}{2}+2 + \frac{N}{2} \\
\end{cases}
\]

\[
M = 2 \text{ mod } 4, N = 2 \text{ mod } 4 \quad (M, N > 4):
\]

\[
n_1 = 2, n_2 = 4, n_3 = 8, n_4 = 11,
\]

\[
n_k = \begin{cases} 
4k-4 & \text{if } 5 \leq k \leq \frac{N}{4}+1 \\
2N & \text{if } \frac{N}{4}+3 \leq k \leq \frac{N}{2}+2, M \neq N \\
2M+2N+4-4k & \text{if } \frac{N}{2}+3 \leq k \leq \frac{3N}{2}+2 + \frac{N}{2} \\
\end{cases}
\]

Therefore we can state:

**Theorem 2** The diameter of a \(M \times N\) Manhattan Street Network, \(M, N > 4\), is equal to \(\frac{M}{2} + \frac{N}{2} + 1\) if \(M = 0\) mod 4 and \(N = 0\) mod 4. Otherwise the diameter is equal to \(\frac{M}{2} + \frac{N}{2}\).

From the distribution of vertices we can also compute the average distance \(D\) which results

\[
\begin{align*}
\frac{M+N+1}{2} - \frac{MN}{2} + 1 & \quad \text{if } M = 0 \text{ mod } 4, N = 0 \text{ mod } 4 \\
\frac{M+N+1}{2} - \frac{MN}{2} + 1 & \quad \text{if } M = 0 \text{ mod } 4, N = 2 \text{ mod } 4 \\
\frac{M+N+1}{2} - \frac{MN}{2} + 1 & \quad \text{if } M = 2 \text{ mod } 4, N = 0 \text{ mod } 4 \\
\frac{M+N+1}{2} - \frac{MN}{2} + 1 & \quad \text{if } M = 2 \text{ mod } 4, N = 2 \text{ mod } 4
\end{align*}
\]

in agreement with Varvarigos [11].
3 Optimal broadcasting in a MSN

We present in this section an algorithm to broadcast in any 2-dimensional MSN. We prove that this algorithm is optimal. The algorithm was suggested by the schemes produced using genetic programming. Optimal broadcasting algorithms, obtained by examination, are known for some important networks: cycles, cliques, grids, 2-dimensional undirected torus and hypercubes [7]. However, for other simple networks like the multi-dimensional undirected torus or the butterfly network, existing broadcasting algorithms are not optimal. In these cases, genetic programming has proven useful by generating new and better broadcasting schemes, see [6]. In [5], and using genetic programming, we obtained efficient broadcasting schemes for 2-dimensional and 3-dimensional Manhattan Street Networks. They suggest the following algorithm which we show it is optimal in Theorem 3.

Algorithm Broadcast-MSN

1. The originator sends the message vertically

2. If a vertex receives the message from a vertical neighbor it first sends it vertically and then, in the following round, horizontally.

3. If a node receives the message from a horizontal neighbor it first sends it horizontally and then, in the following round, vertically.

To obtain an optimal broadcasting time, some informed vertices must change their broadcasting order. For a centered 2-dimensional MSN these vertices are:

\[ M = 0 \mod 4, N = 0 \mod 4: \left( -\frac{M}{2} + 1, 1 \right), \left( 0, \frac{N}{2} \right) \]

\[ M = 0 \mod 4, N = 2 \mod 4: \left( -\frac{M}{2} + 1, 1 \right), \left( 0, \frac{N}{2} \right), \left( \frac{M}{2} - 1, 0 \right), \left( \frac{M}{2} - \frac{N}{2} + 1, \frac{M}{2} - 1, -\frac{N}{2} + 1 \right) \]

\[ M = 2 \mod 4, N = 0 \mod 4: \left( 0, \frac{N}{2} \right), \left( \frac{M}{2}, -\frac{N}{2} + 1 \right), \left( \frac{M}{2}, 0 \right), \left( -\frac{M}{2} + 2, -\frac{N}{2} + 1 \right), \left( -\frac{M}{2} + 1, \frac{N}{2} - 1 \right), \left( -\frac{M}{2} + 1, \frac{N}{2} \right) \]

\[ M = 2 \mod 4, N = 2 \mod 4: \left( -\frac{M}{2} + 2, 1 \right), \left( 0, \frac{N}{2} \right), \left( \frac{M}{2}, 0 \right) \]

This algorithm differs noticeably from the standard broadcasting algorithm for the undirected case in which, after a first change of direction there can be new changes of direction.

On the other hand, extensive tests using genetic programming to generate broadcasting schemes for a MSN have always produced an equivalent algorithm (in some cases with the horizontal and vertical calls reversed or swapped), see [6, 5].

In figures 6-9, we show the execution of the algorithm for a 2-dimensional MSN \( M \times N \) with \( M \geq N, M, N > 4 \), and for each possible relation of dimensions: \( M = 0 \mod 4, N = 0 \mod 4; M = 0 \mod 4, N = 2 \mod 4; M = 2 \mod 4, N = 0 \mod 4 \). The nodes whose broadcast order have been changed are shown in a different size.

\[ 2 \mod 4, N = 0 \mod 4 \text{ y } M = 2 \mod 4, N = 2 \mod 4. \]

Theorem 3 The broadcast time for a Manhattan Street Network, \( M \times N \) vertices, \( M, N > 4 \), is as follows (and it is optimal)

\[
\begin{align*}
\frac{M}{2} + \frac{N}{2} + 2(= D + 1) & \text{ if } M = 0 \mod 4, N = 0 \mod 4 \\
\frac{M}{2} + \frac{N}{2} + 1(= D + 1) & \text{ if } M = 0 \mod 4, N = 2 \mod 4 \\
\frac{M}{2} + \frac{N}{2} + 1(= D + 1) & \text{ if } M = 2 \mod 4, N = 0 \mod 4 \\
\frac{M}{2} + \frac{N}{2} + 1(= D + 1) & \text{ if } M = 2 \mod 4, N = 2 \mod 4
\end{align*}
\]

Proof. We recall that the graph is vertex symmetric, and the broadcast can be initiated from any vertex. Using the broadcast algorithm in a MSN with dimensions \( M \times N, M, N \geq 4 \), and starting from vertex \((0,0)\), for \( M = 0 \mod 4, N = 0 \mod 4 \), at step \( \frac{M}{2} + \frac{N}{2} + 2(= D + 1) \) the last vertices to be informed are \( \left( \frac{M}{2}, \frac{N}{2} + 3 \right), \left( \frac{M}{2}, \frac{N}{2} + 1 \right), \left( -\frac{M}{2} + 2, -\frac{N}{2} + 2 \right) \) and \( \left( -\frac{M}{2} + 3, \frac{N}{2} \right) \). For \( M = 0 \mod 4, N = 2 \mod 4 \), at step \( \frac{M}{2} + \frac{N}{2} + 1(= D + 1) \) the last vertices to be informed are \( \left( \frac{M}{2} - 1, \frac{N}{2} \right), \left( \frac{M}{2} - 1, \frac{N}{2} - 2 \right), \left( \frac{M}{2} - \frac{N}{2} + 2 \right), \left( -\frac{M}{2} + 2, -\frac{N}{2} + 2 \right), \left( -\frac{M}{2} + 2, -\frac{N}{2} + 3 \right), \left( -\frac{M}{2} + 1, -\frac{N}{2} + 1 \right) \) and \( \left( -\frac{M}{2} + 1, \frac{N}{2} - 1 \right) \). For \( M = 2 \mod 4, N = 0 \mod 4 \), at step \( \frac{M}{2} + \frac{N}{2} + 1(= D + 1) \) the last vertices to be informed are \( \left( \frac{M}{2}, \frac{N}{2} - 1 \right), \left( \frac{M}{2} - 1, -\frac{N}{2} + 1 \right), \left( -\frac{M}{2} + 3, -\frac{N}{2} + 1 \right), \left( -\frac{M}{2} + 1, -\frac{N}{2} + 1 \right), \left( -\frac{M}{2} + 1, -\frac{N}{2} + 2 \right), \left( -\frac{M}{2} + 2, -\frac{N}{2} + 2 \right) \).
Figure 7. One-port broadcasting in a 2-dimensional $M \times N$ MSN when $M = 0 \mod 4$ and $N = 2 \mod 4$. The nodes whose broadcast order have been changed are shown in a different size.

Figure 8. One-port broadcasting in a 2-dimensional $M \times N$ MSN when $M = 2 \mod 4$ and $N = 0 \mod 4$. The nodes whose broadcast order have been changed are shown in a different size.

Figure 9. One-port broadcasting in a 2-dimensional $M \times N$ MSN when $M = 2 \mod 4$ and $N = 2 \mod 4$. The nodes whose broadcast order have been changed are shown in a different size.

and $\left(-\frac{M}{2} + 4, \frac{N}{2}\right)$. For $M = 2 \mod 4$, $N = 2 \mod 4$, at step $\frac{M}{2} + \frac{N}{2} + 1 (= D + 1)$ the last vertices to be informed are $\left(\frac{M}{2} - 1, \frac{N}{2}\right)$, $\left(\frac{M}{2} - 1, -\frac{N}{2} + 2\right)$, $\left(\frac{M}{2} - 1, -\frac{N}{2} + 1\right)$, $\left(-\frac{M}{2} + 1, -\frac{N}{2} + 1\right)$ and $\left(-\frac{M}{2} + 1, -\frac{N}{2} + 2\right)$. As the graph can not be informed in time $D$ as there are at least two vertices at this distance from the originator, the broadcast is optimal (see Lemma 1).

This broadcast scheme, therefore, outperforms the algorithm presented in [1].

4 Conclusion

We provide in this paper a broadcasting algorithm for general 2-dimensional Manhattan Street Networks (even dimensions) and a proof of its optimality. The results found in [5] for 3-dimensional $M \times N \times P$ MSNs suggest a similar modification of the algorithm which we have also proved optimal for some values of the dimensions. Further work is necessary to extend the results to the general multidimensional case.

5 Acknowledgement

Research supported by the Secretaria de Estado de Universidades e Investigación (Ministerio de Educación y Ciencia), Spain, and the European Regional Development Fund (ERDF) under project TIC2002-00155.
References


