Characterization of Dynamics of Stable Matchings: Attractors Mapped from Stable Matchings

Yoshiteru Ishida* and Takumi Sato
Department of Computer Science and Engineering, Toyohashi University of Technology Tempaku, Toyohashi 441-8580, Japan

Abstract

This note characterizes the dynamic structure of stable matchings of the stable marriage problem (SMP). The characterization focuses on the dynamic process of how the stable matching will be attained. To this end, the discrete problem of the SMP will be mapped to nonlinear dynamical models whose attractors include the counterparts of the stable matchings in the original SMP. A simple measure of decision-making difficulty is introduced. We use two types of diagram: a cross section diagram (and its 3D imaging) of a lattice to visualize the decision-making difficulty, and a radiation diagram to visualize the time evolution to a matching. Both diagrams are used to examine the dynamic structure of the neighborhood of an attractor and its basin as well as the region between basins in the dynamical model mapped from the SMP.

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1. Introduction

Mapping discrete problems to continuous dynamical models often gives an insight on the original discrete problem. We have been studying matching problems such as the stable marriage problem (SMP) [3, 4, 9, 11], the stable roommate problem (SRP) [7], and an assignment problem by mapping them to dynamical models such as a self-recognition model (SRM) [8] and the Lotka-Volterra model [5]. This note focuses on the difficulty in decision-making in relation to symmetries of preference structure. The difficulty in decision-making falls into two categories: a local one and a global (or structural) one. The local category can be further

* Corresponding author. Tel/fax: +81-532-44-6895.
E-mail address: ishida@cs.tut.ac.jp.
divided into intra-agent and inter-agent cases; in the intra-agent case such as the marriage problem, a man cannot decide in his mind which woman he likes more, while in the inter-agent case a man is liked and ranked in the same order by two women. In both cases, we need a tie break. The intra-agent difficulty is intrinsically local, whereas the inter-agent difficulty can be global if the number of agents involved becomes larger.

We often experience difficulty in making decisions not only when the options are complicated but also when the options are symmetric and cannot be distinguished from each other. We will focus on the latter difficulty, that is, the difficulty due to symmetry. On the one hand, in a discrete problem of the stable marriage problem (SMP), more-stable matchings emerge the more symmetric the preferences are. On the other hand, in the mapped dynamical model, the computational time required to reach the corresponding attractors increases as the model structures become more symmetric.

From the viewpoint of game theory, a mechanism design [10] and strategic questions [11] on matching problems are important but difficult problems. The present work focuses on the decision-making process of matching problems by investigating the dynamics of dynamical models mapped from the matching problems. This requires a detailed investigation on the stable manifold of the dynamical model, which in turn requires new diagrams and visualization techniques. This note uses the self-recognition model (SRM) [8] as a dynamical model.

Section 2 briefly explains both matching problems and dynamical models. The mapping framework is also presented with an example from the SMP to the SRM. Section 3 defines a measure of difficulty in decision-making. Two diagrams are also introduced to examine the global and local dynamics. Not only the correspondence between stable matchings in the SMP and attractors in the SRM, but also the correspondence between decision-making difficulty in the SMP due to symmetry and the border between basins in the SRM are presented.

### Nomenclature

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>SMP</td>
<td>stable marriage problem</td>
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<tr>
<td>SRP</td>
<td>stable roommate problem</td>
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<tr>
<td>SRM</td>
<td>self-recognition model</td>
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### 2. Mapping Matching Problems to Dynamical Models

#### 2.1 Matching Problems

This section introduces matching problems by presenting a naïve matching problem: the stable roommates problem (SRP) [7]. The SRP assumes $2N$ participants $\{p_i; i = 1 \ldots 2N\}$, each of whom has a strict (without tie) ordered preference over the other $2N-1$ participants. The SRP seeks complete matching consisting of $N$ pairs without being blocked. A matching is said to be blocked if a participant $p_i$ prefers a participant $p_j (i \neq j)$ to the current roommate in the matching and that preferred participant $p_j$ also prefers $p_i$ to the current partner in the matching. The pair $(p_i, p_j)$ is called a blocking pair.

The stable marriage problem (SMP) [4, 11] assumes $N$ women $\{w_i; i = 1 \ldots N\}$ and $N$ men $\{m_i; i = 1 \ldots N\}$, each of whom has a strict (without tie) ordered preference (rank) over the opposite sex. In contrast to the SRP, the SMP seeks complete matching between women and men without being blocked.

As in the example shown in Fig. 1, the man $m_2$ has a preference $(w_1 w_3 w_2)$, which means that $m_2$ likes $w_1$ best, and prefers $w_1$ to $w_3$, and $w_3$ to $w_2$.

The preference of each agent is also expressed by a preference matrix $\{a_{ij}\}$ where the element $a_{ij}$ in the $i^{th}$ row and $j^{th}$ column is defined to be $R(m_i, w_j)/R(w_j, m_i)$ where $R(m_i, w_j)$ is the rank of $m_i$ to $w_j$. In the left part of
Table 1, for example, $a_{ij} = R(m_i, w_j)/R(w_j, m_i) = 2/1$ indicates that the man $m_1$ likes the woman $w_1$ second-best while the woman $w_1$ likes the man $m_1$ best. Figure 1 depicts a bipartite graph indicating one matching: {$\{m_1, w_3\}$, $\{m_2, w_1\}$, $\{m_3, w_2\}$}, while the preference (rank) of each person is indicated besides the nodes representing the person. In the matching, the persons such as $m_2$ and $w_3$ could be happier otherwise: $\{m_2, w_3\}$. Hence, the pair $(m_2, w_3)$ blocks the current matching as in Fig. 1. This means that the matching $\{\{m_1, w_3\}, \{m_2, w_2\}, \{m_3, w_1\}\}$ is not stable.

To visualize the global structure of all the matchings in an SMP instance, we used a diagram [8] (named affinity space diagram) similar to a phase space diagram which is often drawn for the continuous dynamical model. Affinity is used as a conventional measure of the happiness (happiness cannot be measured, of course) of a person in a pair. $A(m_i, w_j)$ is the man $m_i$’s affinity for the woman $w_j$, and $A(w_j, m_i)$ is the woman $w_j$’s affinity for the man $m_i$. The affinity matrix is also defined by the preference matrix as: $A(m_i, w_j) = N + 1 - R(m_i, w_j)$, varying from $N$ to 1 as the preference (rank) changes from 1 to $N$. The right part of Table 1 shows the affinity matrix corresponding to the preference matrix in the left part of Table 1. In the affinity space diagram, the following total affinity for women and men is used to arrange all the possible matchings $\mu$ in two-dimensional coordinates:

$$H_w(\mu) = \sum_{(w_j, m_i) \in \mu} A(w_j, m_i), \quad H_m(\mu) = \sum_{(w_j, m_i) \in \mu} A(m_i, w_j)$$

where women’s total affinity $H_w$ is plotted on the vertical axis and men’s total affinity $H_m$ on the horizontal axis (shown as Fig. 2 (a)).
2.2. Dynamical Models as Problem-Solving Mechanism

Dynamical models, when mapped properly from discrete problems, allow us to focus on the dynamics of the original problem, which could not be analyzed within a discrete problem setting.

First, the solutions of the original problem must be mapped to attractors of the dynamical model, so that the time evolution will permit the model to solve the original problem. The constraints and assumptions of the original problem should be properly reflected in the dynamical models. This way of mapping is a standard and natural way of mapping, as can be found, for example, in DNA computing [1] and Hopfield Network [6] in solving the combinatorial problem of the traveling salesman problem (TSP).

Second, time dependent variables of the dynamical models should be designed in an appropriate level of granularity consistent with the original problem. In matching problems such as the SMP and SRP, for example, the time dependent variables would be those representing pairs (or matchings) where a full selection of a pair (or a matching) may be normalized as the value 1 and a full exclusion as the value 0 of the variable.

Third, the initial value setting in the mapped dynamical model can play a significant role in studying the dynamic process of the original problem. Again, in matching problems for example, the initial value setting can be used to represent active agents (proposing agents in the SMP) as well as to set an initial matching (or initial pairing) to observe which stable matching will be eventually selected and how it will be selected. It will be revealed that the initial value in the region between two basins corresponds to impossible situations in decision-making due to symmetry and because of the definition of basins.

2.3. Example of Mapped Dynamical Model

The SMP has been mapped to dynamical models such as self-recognition models (or mutual recognition networks) [8] and generalized Lotka-Volterra models [5]. In both types of mapping, a pair is expressed as a time dependent variable varying from 0 (full exclusion of the pair from a matching) to 1 (full inclusion of the pair in a matching). Let us concentrate on the mapping to the self-recognition model. The self-recognition model uses the time dependent variables $r_i(t)$ and their normalized ones $R_i(t)$:

$$\frac{dR_i(t)}{dt} = \sum_j T_{ij}^+ R_j(t)$$

where

$$R_i(t) = \frac{1}{1 + \exp(-r_i(t))}.$$  

$$T_{ij}^+ = \begin{cases} 
T_{ij} + T_{ji} - 1 & \text{if there are interactions between } i \text{ and } j, \\
0 & \text{if there is no interaction between } i \text{ and } j. 
\end{cases}$$

The parameter $T_{ij} = 1 (-1)$ when the variable $R_i$ is designed to stimulate (inhibit) $R_j$, and $T_{ji} = 0$ otherwise (no interactions). Inhibitions are used to exclude violations of the assumptions in the SRM such as one man and one woman must be paired; and to exclude unstable matchings by inhibiting variables corresponding to blocking pairs. Figure 2 shows an example of the SRM mapped from an instance of the SMP shown in Table 1.

In the phase space of the dynamical model, attractors correspond to stable matchings and a basin of the attractor to initial settings from which the decision will be attracted to the stable matching corresponding to the
attractor. However, there can be cases in which some attractors do not correspond to any stable matching, depending on the mapping and the dynamical model. Also, a border between basins specifies the initial values from which a decision on any matching is impossible.

3. Asymmetric Characterization of a Lattice formed by Stable Matchings

3.1 Difficulty in Decision-Making due to Symmetry

We experience difficulty in decision-making when the available options are symmetric (including no information at all) and choosing one option makes no difference from not choosing any option. In conducting computer simulations to investigate the correspondence between stable matchings and attractors, we observed that simulations starting from certain initial values did not terminate or took a huge number of simulation steps. We assume that similar situations in terms of symmetry occur for a computer (a simulator).

The dynamical model as a problem-solving mechanism of the original SMP can provide not only solutions (as attractors) but also dynamics of how a certain matching can be attained, or not attained, depending on the initial point from which the simulation started. Thus, stable matchings correspond to attractors, and impossible decision-making corresponds to a border between the basins of the attractors. Symmetry breaking (a tie-break) is required in order to decide a matching.

3.2. Diagrams to Investigate Global and Local Symmetry

Since difficulty in decision-making depends on symmetry, we used several diagrams to observe the symmetry. Figure 3 (a) is an affinity space diagram [8], which amounts to a lattice of matchings placed in a
coordinate of the affinity. Each node of the lattice represents a matching consisting of three woman-man pairs where women’s total affinity is used for the horizontal axis and men’s total affinity for the vertical axis in placing the node. A straightforward way to observe decision-making is to measure the computational time required to reach the attractor. Figure 3 (b) shows the number of simulation steps required to reach the attractors. Figure 4 shows the number of simulation steps, indicated by the color gradation in the 2D diagram in (a), and by the height in the 3D diagram in (b). Because we can measure the required steps from any initial values, the diagrams shown in Figs. 4 (a) (b) may be analogically regarded as an “MRI” of the lattice by focusing on the preference of only man (vertical axis) and that of only woman (horizontal axis). The diagrams show that there is a border between basins where the required number of simulation steps starting from the border is huge, indicating that decisions starting from these initial values are very difficult or even impossible to make due to the symmetric structure. Figure 3 (b) shows a cross section along the line from the bottom-right corner to the top-left corner in Fig. 4 (a). Figures 4 (a) and (b) indicate that decisions close to the origin are also difficult to make.

Fig. 3. (a) Affinity space diagram showing matchings as nodes in a coordinate of the women’s total affinity $H_w$ (men’s total affinity $H_m$) on the vertical (horizontal) axis; (b) cross section diagram, with the number of simulation steps used plotted on the vertical axis and initial value setting on the horizontal axis ranging from 0 (woman-optimal) to 1 (man-optimal). Three valleys are attractors corresponding to woman-optimal stable matching (left), man-optimal stable matching (right) and third stable matching (middle). The two peaks indicate impossible decisions starting from a border between two basins.

Fig. 4. Diagrams showing the number of simulation steps required to reach attractors. The required time step is indicated by color gradation in 2D in (a), and by height in 3D in (b). Three attractors (valleys) as well as two border lines between attractors (hills) can be observed. Note that decision-making becomes more difficult closer to the origin (point of no information in the initial value).
Figure 5 shows time evolutions of the normalized variables $R_i(t)$ for every pair. These radiation diagrams can reveal local symmetry among pairs within a diagram as well as among matchings between diagrams. Within a diagram, it can be seen that decisions for one matching will be done synchronously and abruptly in the course of decision-making among promising pairs. Thus, for the decision-making in matching problems, decisions can be easier to make with more pairs in a synchronous fashion. It may be tempting to explain this synchronous phenomenon as “June bride”. Between diagrams, an exchange symmetry can be observed between the woman-optimal matching (Fig. 5 (a)) and the man-optimal matching (Fig. 5 (c)).

### 3.3. Correspondence between Stable Matchings and Attractors

By mapping the SMP to dynamical models, a correspondence between stable matchings in the SMP and attractors in the dynamical models has been studied. Since the mapping has not yet been examined exhaustively, we do not yet know whether every stable matching has its counterpart of attractors, nor whether distinct stable matchings are mapped to distinct attractors. However, we can show that there can be invariant sets that do not have corresponding stable matchings. Our mapping scheme indicates that symmetric structures in the preference matrix will be reflected on the symmetric structures of the dynamical models mapped. Thus, a symmetric structure such as a Latin SMP [2] will be mapped to the dynamical model exhibiting symmetric structure where variables cannot be distinguished. Hence, the model with an equal initial value assigned to all the variables will evolve to the equal value of all the variables afterward, forming an invariant set consisting of points with equal values of variables. In the SRM, the origin where all the variables are 0 is a fixed point.

This fact indicates that the algorithms for solving the SMP such as the Gale and Shapley (GS) algorithm [3] involve asymmetry in the process of solving. The GS algorithm, for example, has the temporal asymmetry (asynchronous) that some pairing must be formed in the first place even though all the members are under the symmetric situation as far as their own preference and others’ preferences are concerned. To reflect such asymmetry in the process of problem-solving in the trajectory of dynamical models, initial values for the dynamical model must be asymmetric, avoiding the fixed points and the invariant set mentioned above.
4. Conclusion

We have shown the detailed dynamics of how a matching will be attained, switched to another matching, or is difficult to attain by mapping the stable marriage problem to a dynamical model. It was also demonstrated that the static structure of a lattice formed by matchings including stable ones can be captured as a stable manifold that includes detailed information of the dynamic structure. Difficulty in decision-making reflecting symmetric structure can be measured by the number of simulation steps required. This type of analysis can shed light on the dynamic process of how matchings are attained. New types of diagrams may also play a role in revealing the spatio-temporal pattern of tightly coupled multiple variables.

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