Power Allocation in MIMO Wireless Systems Subject to Long-Term, Short-Term, and Per-Antenna Power Constraints

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Abstract

We consider several fading channel models for which we aim to maximize ergodic capacity subject to different power constraints assuming that channel state information (CSI) is available at both the transmitter and the receiver. We characterize the optimal power allocation structure in the single-input-single-output (SISO), multiple-input-single-output (MISO), and multiple-input-multiple-output (MIMO) models subject to long- and short-term power constraints. The optimal power policy in each of the channel models depends upon the ratio of the two power constraints and the average signal-to-noise-ratio (SNR) of the system. We characterize the conditions for which the short-term power constraint can be eliminated without being violated in the optimal power policy. Additionally, we find suboptimal power allocation policies when the input power is subject to long-term and per-antenna power constraints and when the input power is subject to long-term, short-term, and per-antenna power constraints. Numerical results suggest that for the Rayleigh fading case, a short-term power constraint that is larger than a long-term power constraint does not significantly impact the ergodic capacity of the channel.

Index Terms

Adaptive transmission, channel state information (CSI), fading channels, multiple-input multiple-output (MIMO) systems, optimization methods, power control, resource allocation.

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I. Introduction

Adaptive power allocation for wireless systems in fading environments attempts to maximize a performance metric, e.g., the ergodic capacity, by allocating power based on the instantaneous channel state information (CSI) subject to limitations in terms of power constraints. In real-world wireless communication systems, there are three important limitations on the transmitted signal's power. The first limitation results from the battery life of the mobile, which is captured by long-term power constraints. The second limitation results from regulations and rules that prevent the transmitter from having an arbitrary power level due to environmental safety as well as interference prevention. According to the Federal Communication Commission (FCC), the transmit power in any time duration should not exceed by a certain amount depending on the application, frequency, height of the antenna, population of that area per square mile, and so on [1]. This regulatory constraint is captured by short-term power constraints. The third limitation results from the practical considerations for the design of the system, which is captured by the per-antenna power constraint and prevents the amplifier at each transmit antenna from nonlinearity or distortion effects by bounding the dynamic range of the power at each antenna [2]. In designing the system, all three power constraints should be taken into account.

A. Related Work

Some works consider long-term average power constraints only, where the average is taken over both the codewords and channel fading coefficients. For example, [3] considers single-antenna systems and shows that the optimal power allocation policy is water-filling in time. Additionally, multiple-antenna systems are considered in [4]. Both of these papers assume that CSI is available at both the transmitter and the receiver.

In other works, only short-term average power constraints are considered, where the average is taken only over the codewords and the constraint applies to each channel fading coefficient. In [5], the author studies the capacity of a multiple-input-multiple-output (MIMO) channel subject to a short-term power constraint for complete CSI, and for CSI at the receiver only. Another example of evaluating the capacity subject to only a short-term power constraint is [6], in which the authors provide an overview of the results on the Shannon capacity of MIMO channels under different assumptions on availability of CSI or Channel Distribution Information (CDI). We use terminology for the power constraints (long-term and short-term) from [7], in which the authors study the delay-limited capacity subject to long-term power constraint, short-term power constraint, or both.

Another power constraint that arises in this paper is a per-antenna power constraint. In [8], per-antenna power constraints are considered in MIMO wireless systems in the context of beamforming with incomplete CSI. Also, [9] studies optimal power allocation subject to ℓ_p-norm constrained eigenvalues, which can be viewed as a suboptimal power allocation policy under short-term and per-antenna power constraints. In general, per-antenna power constraints can be specified as either long- and short-term constraints. We only consider short-term per-antenna power constraints, since the other one is not important in practical system design. For succinctness, we refer to this power constraint as the per-antenna power constraint, which should not be confused with the short-term power constraint, by which we mean a short-term constraint on sum power.

In this paper, we focus on the single user case. Optimal multiuser power allocation subject to long-term power constraint has been studied in [10] and [11], while [12] has studied the optimal multiuser power allocation subject to both long- and short-term power constraints.

B. Summary of Contributions

We study the structure of optimal power allocation to maximize the ergodic capacity in single and multiple-antenna systems considering some or all of the power constraints described above with the assumption of CSI at both the transmitter and the receiver. This problem has been partially addressed by [13] for single-antenna systems subject to both long- and short-term power constraints, where a coding theorem is proved for the capacity of fading channels. We study this problem in more detail and extend the results to multiple-input-single-output (MISO) and MIMO channels. Furthermore, we study the suboptimal power allocation if the input power is subject to long-term and per-antenna power constraints and if the input power is subject to long-term, short-term, and per-antenna power constraints. We also characterize the situations for which one or more power constraints dominates and the others can be ignored.

Finally, we specialize the results for Rayleigh fading and provide numerical results that illustrate the SNR regimes, in which one or more of the power constraints can be ignored. For each channel model we consider in this paper, we quantify these regimes. Furthermore, numerical results for the Rayleigh fading case suggest that if the input power is subject to long- and short-term power constraints, a short-term power constraint that is larger than a long-term power constraint does not significantly impact the ergodic capacity of the channel, for large average SNRs in MISO systems and for all average SNRs in MIMO systems. In other words, if the short-term power constraint is larger than the long-term power constraint, the difference between the capacity of a Rayleigh fading channel with optimal power allocation subject to the long-term power constraint only and the capacity under both long- and short-term power constraints is relatively small. This observation suggests that the rules imposed by the FCC do not constrain the system design in the SNR regimes that are relevant in practice.

The remainder of the paper is organized as follows. In Section II, we describe the channel model and problem formulation. In Section III, we study the optimal power allocation for the SISO channel and for the MIMO channel subject to two or more of the three power constraints. In Section IV, we study the Rayleigh fading channel analytically and numerically. We conclude the paper in Section V.

II. Channel Model and Problem Statement

The baseband-equivalent discrete-time input-output relationship in our MIMO channel model is

$$\mathbf{y}(i) = \mathbf{H}(i)\mathbf{x}(i) + \mathbf{n}(i), \tag{1}$$

where i is the time index, $\mathbf{y}(i)$ is a complex vector of N_R received signals, $\mathbf{x}(i)$ is a complex vector of N_T transmit signals. $\mathbf{H}(i)$ and $\mathbf{n}(i)$ are random sequences capturing the effect of multipath fading and additive noise, respectively. The noise $\mathbf{n}(i)$ is a vector of N_R zero-mean, circularly symmetric, complex Gaussian random variables with $\mathbb{E}[\mathbf{n}(i)\mathbf{n}(i)^H] = N_0\mathbf{I}_{N_R}$, and $\mathbf{n}(i)$, $i = 1, 2, ...$ is a sequence of independent random vectors. The multipath fading $\mathbf{H}(i)$ at each time is an $N_R \times N_T$ matrix of complex fading coefficients. We assume that the matrix fading process is stationary and ergodic, and that it varies slowly enough that CSI is available to the receiver and transmitter. For the case of Rayleigh fading in Section IV, we assume that the entries of $\mathbf{H}(i)$ are independent and identically distributed (i.i.d.) complex Gaussian random variables with mean zero and variance 1/2 per real dimension.

The general optimization problem for maximizing the ergodic capacity subject to long-term, short-term, and per-antenna power constraints is

$$\max_{\mathbf{Q}(\mathbf{H})} \mathbb{E}_{\mathbf{H}} \left[\log\det\left(1 + \frac{1}{N_0}\mathbf{H}\mathbf{Q}(\mathbf{H})\mathbf{H}^H\right)\right], \tag{2a}$$

$$\text{subject to} \quad \mathbb{E}_{\mathbf{H}}\left[\mathrm{tr}(\mathbf{Q}(\mathbf{H}))\right] \le \bar{P}, \tag{2b}$$

$$\forall \mathbf{H} : \mathrm{tr}(\mathbf{Q}(\mathbf{H})) \le P_{max}, \tag{2c}$$

$$\forall \mathbf{H} : q_{kk}(\mathbf{H}) \le \hat{P}, k = 1, 2, ..., N_T, \tag{2d}$$

where $\mathbf{Q}$ is the input covariance matrix, which needs to be maximized as a function of the instantaneous channel and q_{kk} is the k^{th} diagonal entry of the matrix $\mathbf{Q}$. The expectations in (2a) and (2b) are with respect to the distribution of $\mathbf{H}$. The power constraints are described in (2b), (2c), and (2d); $\bar{P}$ represents the long-term power constraint, P_{max} represents the short-term power constraint, and $\hat{P}$ represents the per-antenna power constraint. For simplicity, we drop the time index i. This channel model and the optimization problem simplify to a scalar/vector problem for the SISO/MISO cases considered in the sequel.

The short-term power constraint described above mathematically is consistent with FCC rules as we can see from the Electronic Code of Federal Regulations (e-CFR) [1].

III. General Structure of Optimal Power Allocation

It is conceptually and notationally appealing to treat SISO and MIMO channels separately, since in the SISO case the short-term power constraint coincides with the per-antenna power constraint, because there is only one antenna at the transmitter. On the other hand, the optimization problem (2) would be more intricate in the MIMO case.

A. SISO Channels

We first obtain the optimal power allocation in a SISO system subject to both long- and short-term power constraints. The channel model is the scalar form of (1) with $N_T = N_R = 1$.

Let $\gamma := \frac{\bar{P}|h|^2}{N_0}$ denote the instantaneous received SNR without power adaptation, where h is the scalar fading coefficient, and let $f(\gamma)$ denote the probability density function (pdf) of γ. Let $P(\gamma)$ denote the power policy capturing the second moment of the input signal as a function of γ. Then the received SNR with power adaptation is $P(\gamma)\gamma/\bar{P}$, and the ergodic capacity is [3]

$$C = \mathbb{E}_\gamma \left[\log \left(1 + \frac{P(\gamma)\gamma}{\bar{P}} \right) \right]. \tag{3}$$

Based upon the coding theorem in [13], the ergodic capacity is the solution to the following optimization problem:

$$\max_{P(\gamma)} \int \log \left(1 + \frac{P(\gamma)\gamma}{\bar{P}} \right) f(\gamma)\, d\gamma, \tag{4}$$

$$\text{subject to} \quad \mathbb{E}_\gamma \left[P(\gamma) \right] \le \bar{P},$$

$$\forall \gamma : P(\gamma) \le P_{max},$$

which corresponds to a scalar version of (2). To determine the general power policy, let $\alpha := P_{max}/\bar{P}$ be the ratio of the short-term power constraint to the long-term power constraint, and let α_{max} be a constant to be specified shortly, and consider the following three cases:

Case 1, $\alpha \leq 1$: In this case, the short-term power constraint is always satisfied with equality and the long-term power constraint can be ignored. Therefore, the optimal power policy in this case is

$$P(\gamma) = P_{max}, \forall \gamma. \tag{5}$$

Case 2, $\alpha_{max} \leq \alpha$: In this case, it turns out we can ignore the short-term power constraint, and the power policy is water-filling in time [3]

$$P(\gamma) = \begin{cases} 0, & \gamma < \gamma_0' \\ \frac{\bar{P}}{\gamma_0'} - \frac{\bar{P}}{\gamma}, & \gamma \geq \gamma_0' \end{cases}, \tag{6}$$

where the threshold γ_0' is determined from substituting above power allocation into

$$\mathbb{E}_\gamma \left[P(\gamma) \right] = \bar{P}. \tag{7}$$

Case 3, $1 < \alpha \leq \alpha_{max}$: In this case both long- and short-term power constraints play a role, and the power allocation in [13] provides the solution to the optimization problem. The power allocation in this case becomes

$$P(\gamma) = \begin{cases} 0, & \gamma < \gamma_0 \\ \frac{\bar{P}}{\gamma_0} - \frac{\bar{P}}{\gamma}, & \gamma_0 \leq \gamma < \frac{\gamma_0}{1-\alpha\gamma_0} \\ P_{max}, & \frac{\gamma_0}{1-\alpha\gamma_0} \leq \gamma \end{cases}, \tag{8}$$

where the threshold γ_0 is determined by substituting the above power allocation into (7). This solution comes from the KKT conditions [14]. As we mentioned previously, [13] determines a similar solution, which is valid only in this case (Case 3), since the threshold γ_0 in (8) has a valid solution only in this regime. In Section IV, for the Rayleigh fading channel, we show that the threshold γ_0 is uniquely determined from (7) if $1 < \alpha \leq \alpha_{max}$ (Case 3).

Note that the threshold γ_0 in (8) and γ_0' in (6) are, in general, different. In order to eliminate the short-term power constraint, the power policy in (6) should always satisfy the short-term power constraint $P(\gamma) \leq P_{max}, \forall \gamma$, but the maximum value of $P(\gamma)$ in (6) occurs as $\gamma \to \infty$ and is equal to $\bar{P}/\gamma_0'$. Therefore,

$$\frac{\bar{P}}{\gamma_0'} \leq P_{max} \quad \Rightarrow \quad \frac{1}{\gamma_0'} \leq \alpha \quad \Rightarrow \quad \alpha_{max} = \frac{1}{\gamma_0'}. \tag{9}$$

In general, the values of γ_0' and α_{max} depend on the distribution of γ and the average SNR of the system. In Section IV, we find these values for a Rayleigh fading channel analytically and numerically. It is worth mentioning that the value of α_{max} is important in practical wireless communication systems, since it might be the case that the allowed α is larger than α_{max}. In that case, we can simply ignore the short-term power constraint, and the optimal power allocation policy is water-filling in time. From e-CFR [1], we find the following rule: “In measuring transmission in the 1710-1755 MHz and 2110-2155 MHz bands using an average power technique, the peak-to-average ratio (PAR) of the transmission may not exceed 13 dB.”In fact, as we will see from the numerical results in Section IV-B, the value of α_{max} is less than 13 dB for all the channel models we consider at an average SNR higher than -10 dB. Therefore, a system designer can simply ignore the short-term power constraint in those regimes and use water-filing in time as the power policy without violating the above rule. In other words, this FCC constraint is quite liberal.

If there are multiple antennas at the transmitter and a single antenna at the receiver, i.e., MISO, the problem of maximizing the ergodic capacity subject to both long- and short-term power constraints is the same as the one in the SISO case, since we can convert the MISO channel to an equivalent SISO channel. We consider two different conversions for MISO systems: singular value decomposition (SVD),which is optimal, and antenna selection at the transmitter, which is suboptimal. In Section IV, we derive the optimal power allocation policy for each case and obtain the thresholds for the optimization problems in Rayleigh fading.

B. MIMO Channels

In this section we assume that there are multiple antennas at the transmitter and the receiver. The channel model and problem statement are described in Section II. Let $n := \max(N_R, N_T)$ and $m := \min(N_R, N_T)$. The fading matrix $\mathbf{H}$ can be represented using SVD as

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H, \tag{10}$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices and $\mathbf{\Lambda}$ is a diagonal matrix with entries equal to the square roots of the eigenvalues of the Wishart matrix

$$\mathbf{W} = \begin{cases} \mathbf{H}\mathbf{H}^H, & \text{if} \quad N_R \leq N_T \\ \mathbf{H}^H\mathbf{H}, & \text{if} \quad N_R > N_T \end{cases}.$$

Denote the eigenvalues of $\mathbf{W}$ by $\lambda_k, 1 \leq k \leq m$. The equivalent channel model is [5]

$$\tilde{\mathbf{y}} = \mathbf{\Lambda}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}, \tag{11}$$

where we use the transformation $\tilde{\mathbf{y}} = \mathbf{U}^H\mathbf{y}$, $\tilde{\mathbf{x}} = \mathbf{V}^H\mathbf{x}$, and $\tilde{\mathbf{n}} = \mathbf{U}^H\mathbf{n}$. The equivalent channel consists of m parallel channels. Note that the trace of the input covariance matrix is invariant with respect to this transformation, e.g., $\mathrm{tr}(\mathbf{Q}_\mathbf{x}) = \mathrm{tr}(\mathbf{Q}_{\tilde{\mathbf{x}}})$.

Let $\mathbf{\Lambda}' := \sqrt{\frac{\bar{P}}{mN_0}}\mathbf{\Lambda}$ be the normalized channel model with diagonal entries equal to square root of $\lambda'_k = \frac{\bar{P}}{mN_0}\lambda_k, 1 \le k \le m$. In the SVD equivalent channel model, the power allocation policy is a function of $\mathbf{\Lambda}'$. Therefore, the covariance matrix can be given as $\mathbf{Q}(\mathbf{\Lambda}')$ or $\mathbf{Q}(\underline{\lambda}')$, where $\underline{\lambda}' := [\lambda'_1, \lambda'_2, ..., \lambda'_m]$. To maximize the ergodic capacity, the covariance matrix should be diagonal [5]. Let $P_k(\underline{\lambda}'), 1 \le k \le m$ denote the k^{th} diagonal entry of the covariance matrix. Note that each P_k is a function of the vector $\underline{\lambda}'$, or in other words, a function of all λ'_l's, $1 \le k, l \le m$.

In the following three subsections, we obtain the power allocation that maximizes the ergodic capacity subject to various combinations of the three power constraints described earlier. Optimal power allocations are obtained in Section III-B1, but suboptimal solutions are obtained in Sections III-B2 and III-B3 due to more stringent constraints. While not necessary in principle, these more stringent constraints are more amendable to analysis.

1) Long-Term and Short-Term Power Constraints: If the input power is subject to long- and short-term power constraints, we remove the constraint in (2d). Then, using the SVD equivalent channel model and the above definitions, the optimization problem in (2) becomes

$$\max_{P_k(\underline{\lambda}'),k=1,2,...,m} C = \mathbb{E}_{\underline{\lambda}'}\left[\sum_{k=1}^{m}\log\left(1+\frac{P_k(\underline{\lambda}')\lambda'_k}{\bar{P}/m}\right)\right], \tag{12}$$
$$\text{subject to} \qquad \mathbb{E}_{\underline{\lambda}'}\left[\sum_{k=1}^{m}P_k(\underline{\lambda}')\right] \le \bar{P},$$
$$\forall\underline{\lambda}' : \sum_{k=1}^{m}P_k(\underline{\lambda}') \le P_{max}.$$

The optimal power allocation structure can be found by examining the KKT conditions. Formally, we state and prove the following theorem.

Theorem 1. *The solution to the optimization problem* (12) *for* $\bar{P} \le P_{max}$ *is*

$$P_k(\underline{\lambda}') = \begin{cases} \left(\frac{1}{v} - \frac{\bar{P}}{m\lambda'_k}\right)^+, & \text{if } \sum_{k=1}^{m}\left(\frac{1}{v} - \frac{\bar{P}}{m\lambda'_k}\right)^+ \le P_{max} \\ \left(\frac{1}{\beta+v} - \frac{\bar{P}}{m\lambda'_k}\right)^+, & \text{otherwise} \end{cases}, \tag{13}$$

where $(x)^+ := \max(0, x)$. *The Lagrange multipliers* v *and* β *are obtaining by solving*

$$\sum_{k=1}^{m}\left(\frac{1}{\beta+v} - \frac{\bar{P}}{m\lambda'_k}\right)^+ = P_{max}, \tag{14}$$

$$\mathbb{E}_{\underline{\lambda}'}\left[\sum_{k=1}^{m}P_k(\underline{\lambda}')\right] = \bar{P}. \tag{15}$$

Proof: See Appendix A. ■

In the power allocation above, v is a constant that is fixed for all fading coefficients (all $\underline{\lambda}'$). We refer to v as "global constant", since it is fixed for all values of $\underline{\lambda}'$. We refer to β as the "local constant", since it depends on the current channel fading coefficients $\underline{\lambda}'$. From the power allocation (13), note that the local constant β is required only if $\sum_{k=1}^{m}\left(\frac{1}{v}-\frac{\bar{P}}{m\lambda'_k}\right)^+ > P_{max}$.

Despite the intricate structure of power allocation, the complex joint pdf of the random vector $\underline{\lambda}'$, and the multiple integral for computing the expectation, finding the threshold v is practical even if computationally involved. In fact, the threshold v of the system needs to be computed only once, and it can be obtained off-line through numerical methods. Once v is available, the local threshold β can be determined from (14) for each fading realization.

In Section III-A, finding the optimal power allocation was separated into three cases depending on the value of α and a constant α_{max}. An analogous situation arises for MIMO systems, as we now briefly discuss.

Case 1, $\alpha \leq 1$: In this case, the short-term power constraint is satisfied with equality, and the long-term power constraint can be removed. The optimal power allocation is the well-known water-filling across antennas, but not across time, as obtained in [5].

Case 2, $\alpha_{max} \leq \alpha$: In this case, the power allocation in Theorem 1 is valid, but it simplifies to

$$P_k = \left(\frac{1}{v'} - \frac{\bar{P}}{m\lambda'_k}\right)^+, \tag{16}$$

where the constant v' is determined by substituting the above power policy into (15). In fact, we can eliminate the short-term power constraint, and the problem reduces to finding the optimal power allocation for MIMO channels subject to only a long-term power constraint, which has been examined in [4].

Case 3, $1 < \alpha \leq \alpha_{max}$: Both power constraints paly a role and the optimal power allocation is given by Theorem 1, which was discussed earlier.

The only remaining task is to obtain the value of α_{max}. Note that for this case, we must ensure that the power allocation in (16) does not violate the short-term power constraint. Therefore, we have

$$\sum_{k=1}^{m}\left(\frac{1}{v'} - \frac{\bar{P}}{m\lambda'_k}\right)^+ \leq P_{max}, \forall \underline{\lambda}' \quad \Rightarrow \quad \sum_{k=1}^{m}\frac{1}{v'} \leq P_{max}$$
$$\Rightarrow \quad \frac{m}{v'\bar{P}} \leq \frac{P_{max}}{\bar{P}}.$$

Therefore, we have

$$\alpha_{max} = \frac{m}{v'\bar{P}}. \tag{17}$$

2) Long-Term and Per-Antenna Power Constraints: If the input power is subject to long-term and per-antenna power constraints, we remove the constraint in (2c). First, note that for a Hermitian matrix $\mathbf{Q}$ we have [15]

$$\max_{1\le k\le N_T} q_{kk} \le \max_{1\le k\le N_T} \text{eigenvalue}_k(\mathbf{Q}), \tag{18}$$

where q_{kk} is the k^{th} diagonal entry of the matrix $\mathbf{Q}$, and $\text{eigenvalue}_k(\mathbf{Q})$ is the k^{th} eigenvalue of the matrix $\mathbf{Q}$. If we consider the SVD method discussed earlier, then under the transformation $\tilde{\mathbf{x}} = \mathbf{V}^H\mathbf{x}$, the eigenvalues of the input covariance matrix do not change. On the other hand, the eigenvalues of the matrix $\text{tr}(\mathbf{Q}_{\tilde{\mathbf{x}}})$ are equal to the diagonal entries $P_k(\underline{\lambda}'), 1 \le k \le m$. Therefore, the constraint

$$P_k(\underline{\lambda}') \le \hat{P}, k = 1, ..., m, \tag{19}$$

is sufficient to satisfy (2d). Because of inequality (18), this condition is only sufficient and not necessary. Therefore, we can find a suboptimal solution by assuming a more stringent constraint. Specifically, we consider the optimization problem

$$\max_{P_k(\underline{\lambda}'),k=1,2,...,m} C = \mathbb{E}_{\underline{\lambda}'}\left[\sum_{k=1}^{m} \log\left(1 + \frac{P_k(\underline{\lambda}')\lambda'_k}{\bar{P}/m}\right)\right], \tag{20}$$
$$\text{subject to} \quad \mathbb{E}_{\underline{\lambda}'}\left[\sum_{k=1}^{m} P_k(\underline{\lambda}')\right] \le \bar{P},$$
$$\forall \underline{\lambda}' : P_k(\underline{\lambda}') \le \hat{P}, k = 1, ..., m.$$

Theorem 2. *The solution to the optimization problem* (20) *for* $\bar{P} \le m\hat{P}$ *is*

$$P_k(\underline{\lambda}') = \begin{cases} 0, & \text{if } \frac{\bar{P}}{m\lambda'_k} \ge \frac{1}{v} \\ \frac{1}{v} - \frac{\bar{P}}{m\lambda'_k}, & \text{if } \frac{1}{v} > \frac{\bar{P}}{m\lambda'_k} \ge \frac{1}{v} - \hat{P} \\ \hat{P}, & \text{otherwise} \end{cases}. \tag{21}$$

Proof: See Appendix B. ■

Note that in the power allocation above, v is a constant that is fixed for all fading coefficients, and is determined by substituting the power allocation (21) into (15). Let $\alpha := m\hat{P}/\bar{P}$. Note that the definition of α is slightly different from Sections III-A and III-B1, since there are m per-antenna power constraints here. Again, the general power allocation can be determined by considering three cases.

Case 1, $\alpha \le 1$: In this case, the long-term power constraint can be removed and the power policy is given by $P_k(\underline{\lambda}') = \hat{P}$.

Case 2, $\alpha_{max} \leq \alpha$: In this case, the per-antenna power constraint can be removed. The power allocation in Theorem 2 is valid and simplifies to (16), where v' is determined from (15). By the same procedure as before, we can find that $\alpha_{max} = m/(v'\bar{P})$.

Case 3, $1 < \alpha \leq \alpha_{max}$: In this case, both power constraints play a role and the power policy is given by (21).

3) Long-Term, Short-Term, and Per-Antenna Power Constraints: If the input power is subject to long-term, short-term, and per-antenna power constraints, we obtain a suboptimal power allocation. The corresponding optimization problem is given in (2). In order to make this optimization problem mathematically tractable, we will combine the two power constraints (2c) and (2d) into a more stringer power constraint. Using [9, Lemma 1] and inequality (18), the following lemma applies.

Lemma 1. *If* $\frac{P_{max}}{m} \leq \hat{P} \leq P_{max}$*, and the following* p*-norm power constraint is satisfied*

$$\forall \underline{\lambda}' : \left(\sum_{k=1}^{m} \left(P_k(\underline{\lambda}') \right)^p \right)^{\frac{1}{p}} \leq \hat{P}, \tag{22}$$

where $p = \ln(m)/\ln(m\hat{P}/P_{max})$*, then the two power constraints* (2c) *and* (2d) *would be satisfied.*

Proof: The proof follows from inequality (18), the described SVD method and its properties, and [9, Lemma 1]. ■

Note that the p-norm power constraint (22) is sufficient, not necessary. Therefore, the power allocation in the sequel is suboptimal. Considering Lemma 1, the optimization problem in (2) can be stated as following.

$$\begin{aligned} \max_{P_k(\underline{\lambda}'), k=1,2,\ldots,m} \quad & C = \mathbb{E}_{\underline{\lambda}'} \left[\sum_{k=1}^{m} \log \left(1 + \frac{P_k(\underline{\lambda}')\lambda'_k}{\bar{P}/m} \right) \right], \\ \text{subject to} \quad & \mathbb{E}_{\underline{\lambda}'} \left[\sum_{k=1}^{m} P_k(\underline{\lambda}') \right] \leq \bar{P}, \\ & \forall \underline{\lambda}' : \left(\sum_{k=1}^{m} \left(P_k(\underline{\lambda}') \right)^p \right)^{\frac{1}{p}} \leq \hat{P}. \end{aligned} \tag{23}$$

Theorem 3. *The solution to the optimization problem* (23) *for* $\bar{P} \leq m^{\frac{p-1}{p}} \hat{P}$*, is the optimal power allocation policy as*

$$P_k(\underline{\lambda}') = \begin{cases} 0, & \text{if} \quad \frac{\bar{P}}{m\lambda'_k} \geq \frac{1}{v} \\ \frac{1}{v} - \frac{\bar{P}}{m\lambda'_k}, & \text{if} \frac{1}{v} < \frac{\bar{P}}{m\lambda'_k} \text{and} \left(\sum_{k=1}^{m}\left(\left(\frac{1}{v'} - \frac{\bar{P}}{m\lambda'_k}\right)^+\right)^p\right)^{\frac{1}{p}} \leq \hat{P} \\ \text{solution of} - v + (P_k(\underline{\lambda}') + \frac{\bar{P}}{m\lambda'_k})^{-1} = \left(P_k(\underline{\lambda}')\right)^{p-1}\beta, & \text{otherwise} \end{cases} \tag{24}$$

where β is a constant depending on the fading coefficients and is chosen such that the inequality (22) *is satisfied with equality for the corresponding fading coefficient. The constant v like before, is a global constant, which is chosen such that the long-term power constraint is satisfied with equality, i.e., is determined from substituting the power allocation* (24) *into* (15).

Proof: The proof is based on the KKT conditions, and is very similar to the proof of Theorem 1. In fact, the KKT conditions are the same as in Appendix A, but instead of the condition in (44d), (44f), and (44g), we have the following conditions

$$\theta'\left((\sum_{k=1}^{m} P_k^p)^{\frac{1}{p}} - \hat{P})\right) = 0,$$

$$(\sum_{k=1}^{m} P_k^p)^{\frac{1}{p}} \leq \hat{P},$$

$$-\frac{f(\underline{\lambda}')}{P_k + \frac{\bar{P}}{m\lambda'_k}} - \theta_k + \theta'(pP_k^{p-1}\hat{P}^{1-p}) + $$

$$vf(\underline{\lambda}') = 0, \quad \forall 1 \leq k \leq m.$$

Then, applying the same procedure as in Appendix A, the power policy in (24) results. ■

As before, we are interested in finding the conditions for which one (or more) of the power constrains can be eliminated without being violated. We have more options than before, because there are three power constraints. First, consider the short-term and per-antenna power constraints. Clearly, if $\hat{P} \geq P_{max}$, then the per-antenna power constraint can be removed and the optimal power policy is given by (13), because we need to consider long- and short-term power constraints only. On the other hand, if $\frac{P_{max}}{m} \geq \hat{P}$, then the short-term power constraint can be removed and the suboptimal power policy is given by (21), because we can consider long-term and per-antenna power constraints only. However, if $\frac{P_{max}}{m} \leq \hat{P} \leq P_{max}$, according to Lemma 1, we can combine the short-term and per-antenna power constraints to one p-norm power constraint as in (22). Before considering the conditions for which we can eliminate one of the long-term and p-norm power constraints, we state and prove the following lemma, and then we will divide the problem into three cases.

Lemma 2. *Let* $p \geq 1$ *and* $x = [x_1, ..., x_m]^T \in \mathbb{R}^m$. *Then*

$$\sum_{k=1}^{m} x_k \leq m^{\frac{p-1}{p}} \left(\sum_{k=1}^{m} x_k^p \right)^{\frac{1}{p}} = m^{\frac{p-1}{p}} \|x\|_p. \tag{25}$$

Proof: According to Hölder's inequality, if $x, y \in \mathbb{R}^m$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$< x, y >= \sum_{k=1}^{m} x_k y_k \leq \|x\|_p \|y\|_q.$$

If we let $y = [1, 1, ..., 1]^T$, then the inequality (25) follows. ∎

Now, let $\alpha := m^{\frac{p-1}{p}} \hat{P}/\bar{P}$ and consider the following three cases.

Case 1, $\alpha \leq 1$: In this case, the p-norm power constraint is satisfied with equality, and the long-term power constraint can be eliminated, because

$$\sum_{k=1}^{m} P_k(\underline{\lambda}') \leq m^{\frac{p-1}{p}} \left(\sum_{k=1}^{m} \left(P_k(\underline{\lambda}')\right)^p \right)^{\frac{1}{p}} \tag{26}$$

$$\leq m^{\frac{p-1}{p}} \hat{P} \tag{27}$$

$$\leq \bar{P}, \tag{28}$$

where (26) holds according to Lemma 2, (27) holds because of the p-norm power constraint, and (28) holds because $\alpha \leq 1$. Therefore, the average of the left term of (26) is always less than $\bar{P}$. Thus, the long-term power constraint is satisfied and can be removed in this case and the problem reduces to maximizing the ergodic capacity subject to the p-norm power constraint only. This problem has been considered in [9] and the optimal power policy for the p-norm power constraint is given by

$$P_k(\underline{\lambda}') = \begin{cases} 0, & \text{if} \quad \lambda'_k = 0 \\ \text{solution of } \left(P_k(\underline{\lambda}')\right)^p + \frac{1}{\lambda'_k} \left(P_k(\underline{\lambda}')\right)^{p-1} = \beta', & \text{otherwise} \end{cases}, \tag{29}$$

where the constant β' is chosen such that the p-norm power constraint is satisfied with equality.

Case 2, $\alpha_{max} \leq \alpha$: In this case, we can eliminate the p-norm power constraint (and therefore, short-term and per-antenna power constraints) and maximizing the ergodic capacity is subject to the long-term power constraint only. The power allocation in Theorem 3 is valid and simplifies to (16), where v' is determined from (15).

Case 3, $1 < \alpha \leq \alpha_{max}$: In this case, both power constraints play a role and the power policy is given by (24).

For obtaining the value of α_{max}, we find the conditions for which the power policy (16) does not violate the p-norm power constraint. The worst case occurs as $\lambda'_k \to \infty, k = 1, ..., m$. Then we should have

$$\left(\sum_{k=1}^{m}\left(\frac{1}{v'}\right)^p\right)^{\frac{1}{p}} \leq \hat{P} \quad \Rightarrow \quad \frac{m}{v'\bar{P}} \leq \frac{m^{\frac{p-1}{p}}\hat{P}}{\bar{P}} = \alpha,$$

which means that $\alpha_{max} = m/(v'\bar{P})$ like before.

IV. Rayleigh Fading Channels

In Section III, we obtained the optimal power allocation policies for general channel models subject to various combinations of power constraints. In this section, we consider the iid Rayleigh fading channel model, and simplify the described power policies to the extent possible, and provide some numerical results.

A. Analysis

We are interested in studying the optimal power policy for Rayleigh fading channel model more precisely, because this model is usually assumed in practical systems. In particular, we obtain some of the constants and thresholds introduced in the power policies before. In fact, most of these constants are a function of the distribution of the fading coefficients. We will simplify the calculations needed to obtain the constants for the Rayleigh fading case. However, it is not always easy to find a closed form for the thresholds. We will simplify the equations for all the thresholds in the SISO and MISO Rayleigh fading case and for some of the thresholds in the MIMO Rayleigh fading case.

1) SISO and MISO: Consider the parameter γ defined in Section III-A. For Rayleigh fading γ is an exponential random variable with expected value $\bar{P}/N_0$. Let $\bar{\gamma}$ be the average SNR of the system, which in the case of SISO and MISO systems is $\bar{\gamma} = \bar{P}/N_0$. We denote the probability density function of γ by $f(\gamma)$, which for the SISO Rayleigh fading channel model is

$$f(\gamma) = \begin{cases} \frac{e^{-\frac{\gamma}{\bar{\gamma}}}}{\bar{\gamma}}, & \gamma \geq 0 \\ 0, & \text{otherwise} \end{cases} . \tag{30}$$

Beginning with the SISO with distribution (30), we derive the thresholds γ'_0, α_{max}, and γ_0 in the optimal power allocation from Section III-A. In [16], the value of γ'_0 is derived for Rayleigh fading channel as the solution to the equation

$$\frac{e^{-\frac{\gamma'_0}{\bar{\gamma}}}}{\frac{\gamma'_0}{\bar{\gamma}}} - E_1\left(\frac{\gamma'_0}{\bar{\gamma}}\right) = \bar{\gamma}, \tag{31}$$

where $E_n(x)$ is the exponential integral defined by

$$E_n(x) := \int_1^{+\infty} t^{-n} e^{-xt}, dt, x \geq 0. \tag{32}$$

From [16], we know that (31) has a unique solution and γ_0' always lies in the interval $[0, 1]$. From (9), $\alpha_{max} = 1/\gamma_0'$ is unique and is always larger than one. According to (31), note that the value of α_{max} is only a function of $\bar{\gamma}$. For the threshold γ_0, we have the following theorem.

Theorem 4. *If* $1 < \alpha \leq \alpha_{max}$ *(Case 3), then the threshold* γ_0 *in* (8) *for the Rayleigh fading SISO channel can be determined from*

$$\begin{aligned} \frac{e^{-x}}{x} - \frac{e^{-\frac{x}{1-\alpha x\bar{\gamma}}}}{x} - E_1(x) + E_1\left(\frac{x}{1-\alpha x\bar{\gamma}}\right) \\ + \alpha\bar{\gamma}e^{-\frac{x}{1-\alpha x\bar{\gamma}}} = \bar{\gamma}, \end{aligned} \tag{33}$$

where $x := \gamma_0/\bar{\gamma}$, *and* $E_1(.)$ *is the exponential integral defined in* (32). *Furthermore,* γ_0 *is unique and* $\gamma_0 \in [0, \frac{1}{\alpha}]$.

Proof: See Appendix C. ∎

Now, consider a MISO system with $N_T = n$ antennas at the transmitter and $N_R = 1$ antenna at the receiver. The discrete-time input-output relationship is a special case of (1), where the output and noise are scalars. We consider two different transmission schemes, one based upon optimal beamforming and referred to as the SVD method, and another based upon antenna selection.

Since the rank of channel matrix $\mathbf{H}$ is one, after singular value decomposition, we can see that the channel in (1) is equivalent to the following scalar channel

$$\tilde{y} = \sqrt{\lambda_1}\tilde{x} + \tilde{n}, \tag{34}$$

where $\lambda_1 = \sum_{i=1}^{n} |h_i|^2$. The instantaneous received SNR without power adaptation in this context is

$$\gamma = \frac{\lambda_1 \bar{P}}{N_0}. \tag{35}$$

Therefore, we observe that the MISO problem reduces to the SISO problem with a different distribution. For Rayleigh fading, γ in (35) is a n-Erlang random variable with the distribution function

$$f(\gamma) = \begin{cases} \frac{1}{(n-1)!\bar{\gamma}} \left(\frac{\gamma}{\bar{\gamma}}\right)^{n-1} \exp\left(-\frac{\gamma}{\bar{\gamma}}\right), & \gamma \geq 0 \\ 0, & \text{otherwise} \end{cases}, \tag{36}$$

where $\bar{\gamma} = \bar{P}/N_0$ as before. The optimal power policy is as in Section III-A with three regions as before (Case 1, Case 2, and Case 3) corresponding to the power allocation policy in (5), (6), and (8), respectively. However, the value of α_{max}, γ_0', and γ_0 are different, since they depend on the distribution of γ. According to [16] and [4], with the distribution in (36), the value of γ_0' is given by

$$\frac{\Gamma\left(n, \frac{\gamma_0'}{\bar{\gamma}}\right)}{\frac{\gamma_0'}{\bar{\gamma}}} - \Gamma\left(n-1, \frac{\gamma_0'}{\bar{\gamma}}\right) = (n-1)!\bar{\gamma}, \tag{37}$$

where $\Gamma(.,.)$ is the complementary incomplete gamma function

$$\Gamma(a,x) := \int_x^\infty \exp{(-t)} x^{a-1}\, dt.$$

As proved in [16], the above equation has a unique solution and $\gamma_0' \in [0,1]$. Therefore, α_{max} is uniquely determined by (9), i.e., $\alpha_{max} = 1/\gamma_0'$. As we can see in (37), the value of α_{max} for a fixed n is only a function of $\bar{\gamma}$.

The value of γ_0 in Case 3, i.e., $1 < \alpha \leq \alpha_{max}$, is a constant that determines the power allocation in (8), and is given by (7). Similarly, we have the following theorem.

Theorem 5. *If $1 < \alpha \leq \alpha_{max}$ (Case 3), then the threshold γ_0 in* (8) *for the MISO Rayleigh fading channel using the SVD method can be determined from*

$$\begin{aligned} \Gamma(n-1, \frac{x}{1-\alpha x\bar{\gamma}}) - \Gamma(n-1,x) + \frac{1}{x}\Gamma(n,x) + \\ (\bar{\gamma}\alpha - \frac{1}{x})\Gamma(n, \frac{x}{1-\alpha x\bar{\gamma}}) = \bar{\gamma}(n-1)!, \end{aligned} \tag{38}$$

where $x := \gamma_0/\bar{\gamma}$. Furthermore, γ_0 is unique and $\gamma_0 \in [0, \frac{1}{\alpha}]$.

Proof: The steps of the proof are exactly like those in the proof of Theorem 4 with the pdf (36) for γ. ■

For MISO with antenna selection, by choosing the transmitter antenna corresponding to the largest channel gain, we have another scalar channel with

$$\gamma = \frac{\bar{P}|h|^2}{N_0}, \tag{39}$$

where $|h|^2 = \max{(|h_1|^2, |h_2|^2, ..., |h_n|^2)}$. Therefore, the pdf of γ is given by

$$f(\gamma) = \begin{cases} \frac{n}{\bar{\gamma}} \exp\left(-\frac{\gamma}{\bar{\gamma}}\right)\left(1-\exp\left(-\frac{\gamma}{\bar{\gamma}}\right)\right)^{n-1}, & \gamma \geq 0, \\ 0, & \text{otherwise} \end{cases}, \tag{40}$$

where $\bar{\gamma} = \bar{P}/N_0$ as before. Again, the optimal power policy is as in Section III-A. According to [16] and [4], with the distribution in (40), the value of γ_0' is given by the following equation

$$\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left(\frac{e^{-(k+1)\gamma_0'/\bar{\gamma}}}{(k+1)\gamma_0'/\bar{\gamma}}\right.$$
$$\left.-E_1((k+1)\gamma_0'/\bar{\gamma})\right)=\frac{\bar{\gamma}}{n}. \tag{41}$$

As proved in [16], the above equation has a unique solution and $\gamma_0' \in [0,1]$. Therefore, α_{max} is uniquely determined by (9). As for the constant γ_0, which specifies the power allocation in Case 3 (8), we have the following theorem.

Theorem 6. *If $1 < \alpha \leq \alpha_{max}$ (Case 3), then the threshold γ_0 in* (8) *for the MISO Rayleigh fading using the antenna selection method can be determined from*

$$\sum_{k=0}^{n-1}(-1)^K\binom{n-1}{k}\left(\frac{e^{-(k+1)x}}{(k+1)x}-E_1((k+1)x)\right.$$
$$\left.+E_1\left(\frac{(k+1)x}{1-\alpha x\bar{\gamma}}\right)+\frac{1}{k+1}(\alpha\bar{\gamma}-\frac{1}{x})e^{-\frac{(k+1)x}{1-\alpha x\bar{\gamma}}}\right)=\frac{\bar{\gamma}}{n}, \tag{42}$$

where $x := \gamma_0/\bar{\gamma}$. Furthermore, γ_0 is unique and $\gamma_0 \in [0, \frac{1}{\alpha}]$.

Proof: The steps of the proof are exactly like those in the proof of Theorem 4 with the pdf (40) for γ. ■

2) MIMO: In the MIMO case, because of more intricate structure of the power policies, finding a simplified equation for expressing some of the thresholds is not easy. In particular, when the power allocation contains a local constant, which depends on the instantaneous fading coefficients (like β in Sections III-B1 and III-B3), there is not a closed form for this constant, and therefore, other thresholds in that power policy that depend on this local constant cannot be expressed through more simplified equations. Furthermore, the complex joint pdf of the random vector $\underline{\lambda}'$ and the multiple integral for computing the expectations in the MIMO channel model, make it less convenient to find the thresholds analytically, and numerical computation is more appropriate in this case.

First, we find α_{max} (the threshold that determines Case 2) in the Rayleigh fading channel model. In all the Sections III-B1, III-B2, and III-B3, we have $\alpha_{max} = m/(v'\bar{P})$, where v' is the constant in the optimal power policy (16). Therefore, for a fixed m and $\bar{P}$, α_{max} is the same in Sections III-B1, III-B2, and III-B3. If we define $\gamma_0 := v'\bar{P}/m$, then $\alpha_{max} = 1/\gamma_0$. From (15), γ_0 becomes the threshold as in [4, Eq. (36)]

$$\int_{\gamma_0}^{\infty}\left(\frac{1}{\gamma_0}-\frac{1}{x}\right)f_{\lambda'}(x)\,dx = 1, \tag{43}$$

where $f_{\lambda'}(x)$ is the probability distribution function of a normalized unordered eigenvalue of the Wishart matrix [5]. More simplified equations for determining γ_0 and the proof of its uniqueness can be found in [4]. Once γ_0 is found, $\alpha_{max} = 1/\gamma_0$ can be determined.

Finally, consider the threshold v in (21) in Section III-B2. Because this power policy does not contain any local constant, it is possible to obtain equations for v and prove its uniqueness. However, we do not go through the lengthy calculations, because the procedure is very similar to the one in [4], in which the authors have analyzed the case in which the input power is subject to a long-term power constraint only.

B. Numerical Results and Discussion

As with the analysis in Section IV-A, all the numerical results in this section are for Rayleigh fading channels. For the ease of presentation, we split this section into two subsections.

1) SISO and MISO: In Figure 1, we plot α_{max} versus $\bar{\gamma}$ for SISO systems, and MISO systems with $n = 2$ and $n = 10$ for the two introduced schemes, SVD and antenna selection.

Note that in practical wireless communication systems, it is advantageous to have a small α_{max}, since it means that we can ignore the short-term power constraint for a wider range of α (remember this is the Case 2 if $\alpha_{max} \leq \alpha$, and the short-term power constraint is eliminated). In general, according to Figure 1, as we increase the average SNR, the value of α_{max} gets smaller and finally approaches 1 for high average SNRs. Intuitively, power allocation (water-filling) is most important at lower SNR, which might require the short-term power to fluctuate significantly with the fading coefficients. As a result, for optimality at low SNR, we need to allow higher short-term powers relative to long-term powers, and correspondingly α_{max} is larger at low SNR. That is, the short-term power constraint is somehow more relevant at low SNR compared to high SNR. Also, note that in the MISO systems, the value of α_{max} is always smaller than in the SISO system. As we can see in Figure 1, the more antennas at the transmitter, the smaller α_{max}. In addition to the above observations, we can see that the value of α_{max} in the optimal SVD scheme is less than that of the suboptimal antenna selection scheme for a fixed number of antennas at the transmitter.

In Figure 2, we plot the capacity for the two schemes in the MISO systems with different values of n, but with a fixed P_{max} and N_0. For comparison, we have also plotted the capacity for the case, in which there is channel state information at the receiver only (labeled "Without CSIT") [5]. The capacity is given by (3) and the power allocation policies described earlier, and depends on the value of average SNR, and indirectly on α and α_{max} that determine which one of the three cases in Section III-A occurs. As an example, for the optimal SVD scheme with $n = 2$, the three regions $\bar{\gamma} \in [-10\,\text{dB}, 12.79\,\text{dB}]$, $\bar{\gamma} \in$

$[12.79\,\text{dB}, 13.01\,\text{dB}]$, and $\bar{\gamma} \in [13.01\,\text{dB}, 20\,\text{dB}]$ correspond to Case 2, Case 3, and Case 1, respectively. As another example, for the suboptimal antenna selection scheme with $n = 2$, the three regions $\bar{\gamma} \in [-10\,\text{dB}, 12.71\,\text{dB}]$, $\bar{\gamma} \in [12.71\,\text{dB}, 13.01\,\text{dB}]$, and $\bar{\gamma} \in [13.01\,\text{dB}, 20\,\text{dB}]$ correspond to Case 2, Case 3, and Case 1, respectively. In the third region (Case 1), we observe saturation of the ergodic capacity because of the short-term power constraint. In that region, the horizontal axes shows the value of available average SNR, not the actual one. As we can see, there is always a small range in which Case 3 occurs, and this range becomes smaller if we increase the number of transmit antennas n.

Note that as the number of antennas at the transmitter increases, the capacity per channel use also increases for the case of complete CSI. However, there is not that much of improvement in the capacity by increasing the number of antennas if there is no CSIT. It is known that the capacity in MIMO systems increases linearly by $m := \min(N_T, N_R)$ [6], which for MISO channels, it is equal to 1 because $N_R = 1$. However, when CSI is also available at the transmitter, the capacity increases logarithmically as the number of antennas at the transmitter increases [17].

Next, we examine the effect of α on the capacity. Figure 3 shows the capacity versus average SNR for each of the two schemes in the MISO channel with the two extreme possibilities for α: $\alpha = \infty$ so that $\alpha_{max} \leq \alpha, \forall \bar{\gamma}$, and $\alpha = 1, \forall \bar{\gamma}$. For other values of $1 < \alpha < \infty$ between the two extremes, the capacity versus average SNR curve lies between the two curves for $\alpha = 1$ and $\alpha = \infty$. Note that because α is fixed and $\bar{P}$ is varying, P_{max} is varying unlike previous plot.

As we can see from Figure 3, the difference between the capacities for the two extreme values of α is small, and becomes smaller for large $\bar{\gamma}$ for both SVD and antenna selection. In fact, if $\alpha = \infty$, the problem reduces to the case in which there is no short-term power constraint (it is also the case when $\alpha_{max} \leq \alpha, \forall \bar{\gamma}$), and if $\alpha = 1$, the problem reduces to constant power allocation since the power policy is $P(\gamma) = \bar{P}$. Note that even in this case of constant power allocation, we use the CSIT because we perform singular value decomposition or antenna selection. If we do not use the CSIT, there would be a large gape between the capacities as we observed in Figure 2.

We observe that the value of the short-term power constraint (P_{max}) does not significantly impact the ergodic capacity of the channel for large average SNRs, as long as it is larger than the long-term power constraint, i.e., $\alpha \geq 1$. In other words, if $\bar{P} \leq P_{max}$, the difference between the capacity obtained from the optimal power allocation policy subject to the long-term power constraint only and the one obtained from the optimal power allocation subject to both long- and short-term power constraints is relatively small (about 3 dB at low SNR and 1 dB at higher SNR for the MISO systems with $n = 2$).

2) MIMO: we present some numerical results for MIMO systems with Rayleigh fading. Figure 4 shows the value of α_{max} versus the average SNR per parallel channel ($\bar{\lambda} = \frac{\bar{P}}{mN_0}$) for 2×2 and 4×4 MIMO systems. This plot suggests similar observations as for MISO systems: for large average SNRs, α_{max} approaches unity, and as the number of antennas increases, the value of α_{max} decreases, which is desirable in practice. Intuitively, water-filling has more short-term fluctuations at low SNR, which makes α_{max} larger. In fact, the short-term power constraint is more relevant at low SNR compared to high SNR.

Figure 5 examines the effect of P_{max} on the capacity of MIMO systems with Rayleigh fading. We compare the capacity of two extreme cases: $P_{max} = \bar{P}$ and $P_{max} \to \infty$ (or $\alpha = 1$ and $\alpha \to \infty$ with α defined in Section III-B1). The figure shows the capacity versus average SNR for these two cases in a 2×2 MIMO system along with the case of having CSI only at the receiver but not at the transmitter. Note that this figure also examines the effect of $\hat{P}$ on the capacity if we consider two extreme cases: $\hat{P} = \bar{P}/m$ and $\hat{P} \to \infty$ (or $\alpha = 1$ and $\alpha \to \infty$ with α defined in Section III-B2). For other values of $1 < \alpha < \infty$ between the two extremes, the capacity versus average SNR curve lies between the first two curves in Figure 5.

As we can see, the difference between the first two curves appears negligible. The same result is reported in [18], in which the curves for the two cases of space-time water-filling and spatial water-filling are equivalent to the case with $\alpha = \infty$ and $\alpha = 1$ in the figure, respectively. To interpret this, note that at each fixed average SNR, the small difference between the first two curves in Figure 5 is the difference between the ergodic capacities with a same average SNR, but with two extreme possibilities when P_{max} is equal to $\bar{P}$ and when the short-term power constraint is removed ($P_{max} = \infty$). The ergodic capacity of the channel with the same average SNR but with a short-term power constraint $\bar{P} < P_{max} < \infty$ has a value between the two extreme cases. Therefore, for a fixed average SNR, the value of the short-term power constraint has a very small effect on the ergodic capacity as long as it is equal to or larger than the long-term power constraint. Note that in the case of MISO systems in Figure 3, the P_{max} has a considerable effect on the ergodic capacity for low SNR regimes. However, for MIMO systems, this effect is negligible for all plotted range of SNRs, since the performance of the spatial water-filling (it happens when $P_{max} = \bar{P}$) is very close to the one of the space-time water-filling (it happens when we remove the short-term power constraint or $\alpha = \infty$).

According to the above discussion, one might argue that if we use a constant power allocation across the time, i.e. replacing the two long- and short-term power constraints in (12) with a stronger power constraint $\sum_{k=1}^{m} P_k \leq \bar{P}$ when $\bar{P} \leq P_{max}$ (or $1 \leq \alpha$), then we would have almost the same ergodic capacity as in the optimal power allocation in (13). This reasoning is in fact true, but surprisingly, the complexity of computing the constant power allocation across time is often more than that of the optimal one. In the optimal power allocation (13), except for the threshold v, which needs to be pre-computed offline, we need to compute the threshold β online only for some of the channel realizations (when $\sum_{k=1}^{m}\left(\frac{1}{v}-\frac{\bar{P}}{m\lambda'_k}\right)^+ > P_{max}$). However, in the constant power allocation across the time, we need to compute a threshold online (the threshold for the spatial water-filling) for each channel realization.

When there is no CSI at the transmitter, the capacity for a 2×2 Rayleigh fading MIMO system is plotted in Figure 5 and is labeled as “Without CSIT”. As we can see, we have approximately $3\,\mathrm{dB}$ gain if we have full CSI comparing to the case of having CSI only at the receiver in the low SNR regime. However, for high SNRs, the ergodic capacity is the same for the two cases. This always happens for a $m\times m$ i.i.d. Rayleigh fading MIMO system as discussed in [17], since at high SNR, the water-filling strategy allocates an equal amount of power to all the spatial modes, as well as an equal amount of power over time, and that is what the transmitter does when it does not have CSI [17]. This reasoning remains true for a general $m\times n$ i.i.d. Rayleigh fading MIMO system with $m\leq n$, which is not the case in a MISO system.

We stress that the results and observations in this section are only applicable to iid Rayleigh fading. By contrast, the effect of P_{max} on the ergodic capacity is considerable if the model is Rayleigh fading with log-normal shadowing. We anticipate such a result based upon our observations in conjunction with results in [18], in which the authors demonstrate that with log-normal shadowing, space-time water-filling achieves significantly higher ergodic capacity than spatial water-filling at low to moderate SNR regimes.

V. Summary and Extensions

In this paper, we derived the optimal power allocation in a fading environment to maximize the ergodic capacity if the input power is subject to long- and short-term power constraints for SISO and MIMO systems. Furthermore, we determined a suboptimal power allocation if the input power is subject to long-term and per-antenna power constraints and if the input power is subject to long-term, short-term, and per-antenna power constraints in MIMO systems. We studied the conditions for which one or more of the power constraints can essentially be eliminated. For the Rayleigh fading channel model, we obtained the value of α_{max}, the threshold ratio between short- and long-term power constraints, for the SISO, MISO, and MIMO channels, and discussed its importance in practical systems. We evaluated the effect of the short-term power constraint on the ergodic capacity in MISO and MIMO systems with Rayleigh fading through our numerical results.

This work can be extended to multiple access channels as well as broadcast channels. Another trend of future work is to focus on a limited feedback scenario in which only partial CSI is available at the transmitter. Considering fairness criteria in the power allocation in multiple-antenna channels subject to both long- and short-term power constraints is another potential extension to this work. Finally, instead of power constraints, we could consider energy constraints, which are relevant in sensor networks if there is no possibility of recharging the battery and the energy would be consumed completely after some period of time.

Appendix A
Proof of Theorem 1

Since (12) is a convex optimization problem, We prove the theorem using the KKT conditions [14]. For simplicity, we drop $\underline{\lambda}'$ from $P_k(\underline{\lambda}')$ and simply denote it by P_k in the proof of Theorems 1, 2 and 3. To maximize the capacity, the long-term power constraint should be satisfied with equality, which is possible since $\alpha \geq 1$. Let $\theta_1, \theta_2, ..., \theta_m$ denote the Lagrange multipliers corresponding to the constraints that force the powers to be positive ($P_1 \geq 0, P_2 \geq 0, ..., P_m \geq 0$, respectively), θ' be the Lagrange multiplier corresponding to the short-term power constraint, and v be the Lagrange multiplier corresponding to the long-term power constraint (we consider the long-term power constraint with equality). Then, the KKT conditions can be written as

$$\theta_k P_k = 0, k = 1, 2, ..., m, \tag{44a}$$

$$\theta_k \geq 0, k = 1, 2, ..., m, \tag{44b}$$

$$P_k \geq 0, k = 1, 2, ..., m, \tag{44c}$$

$$\theta' (\sum_{k=1}^{m} P_k - P_{max}) = 0, \tag{44d}$$

$$\theta' \geq 0, \tag{44e}$$

$$\sum_{k=1}^{m} P_k \leq P_{max}, \tag{44f}$$

$$-\frac{m f(\underline{\lambda}')}{m P_k + \frac{\bar{P}}{\lambda'_k}} - \theta_k + \theta' + v f(\underline{\lambda}') = 0, k = 1, 2, ..., m, \tag{44g}$$

$$\mathbb{E}_{\underline{\lambda}'} \left[\sum_{k=1}^{m} P_k \right] = \bar{P}. \tag{44h}$$

Now, based on the above conditions, we obtain some restrictions on the solution:

Restriction 1: if $P_k \neq 0$, then from (44a), $\theta_k = 0$, so $\theta' = (\frac{1}{P_k + \bar{P}/(m\lambda'_k)} - v) f(\underline{\lambda}') \geq 0$ (from (44g) and (44e)).

Restriction 2: if $\frac{\bar{P}}{m\lambda'_k} \geq \frac{1}{v}$, then $P_k = 0$ (from Restriction 1).

Now, consider two different situations:

Situation 1, $\sum_{k=1}^{m} \left(\frac{1}{v} - \frac{\bar{P}}{m\lambda'_k} \right)^+ \leq P_{max}$: In this case, the power allocation $P_k = \left(\frac{1}{v} - \frac{\bar{P}}{m\lambda'_k} \right)^+, 1 \leq k \leq m$ is a valid solution and satisfies all the KKT conditions (note that in this case $\theta' = 0$, and from Restriction 1 and 2, above power allocation results).

Situation 2, $\sum_{k=1}^{m} \left(\frac{1}{v} - \frac{\bar{P}}{m\lambda'_k} \right)^+ > P_{max}$: In this case, $\theta' \neq 0$ and from (44d), we have $\sum_{k=1}^{m} P_k - P_{max} = 0$. Now, consider two cases:

2.1: If $\frac{\bar{P}}{m\lambda'_k} \geq \frac{1}{\theta'/f(\underline{\lambda}') + v}$, then $P_k = 0$ (because if $P_k > 0$, then from (44a) $\theta_k = 0$, and from (44g) $P_k = \frac{1}{\theta'/f(\underline{\lambda}') + v} - \frac{\bar{P}}{m\lambda'_k} > 0$, which is a contradiction).

2.2: If $\frac{\bar{P}}{m\lambda'_k} < \frac{1}{\theta'/f(\underline{\lambda}') + v}$, then $\theta_k = 0$ (because if $\theta_k > 0$, then from (44a) $P_k = 0$, and from (44g) $\theta_k = \theta' + v f(\underline{\lambda}') - \frac{f(\underline{\lambda}')}{\bar{P}/(m\lambda'_k)} > 0$, so $\frac{\bar{P}}{m\lambda'_k} > \frac{1}{\theta'/f(\underline{\lambda}') + v}$, which is a contradiction).

With the explanation in 2.1 and 2.2 cases, and defining $\beta := \theta'/f(\underline{\lambda}')$, we now can determine the power allocation in situation 2. That is $P_k = \left(\frac{1}{\beta + v} - \frac{\bar{P}}{m\lambda'_k} \right)^+, 1 \leq k \leq m$, where β is the answer to the equation $\sum_{k=1}^{m} \left(\frac{1}{\beta + v} - \frac{\bar{P}}{m\lambda'_k} \right)^+ = P_{max}$.

The power allocation described in Situation 1 and Situation 2 completes the proof.

Appendix B
Proof of Theorem 2

Let $\theta_1, \theta_2, ..., \theta_m$ denote the Lagrange multipliers corresponding to the constraints that force the powers to be positive ($P_1 \geq 0, P_2 \geq 0, ..., P_m \geq 0$, respectively), $\theta'_1, \theta'_2, ..., \theta'_m$ be the Lagrange multipliers corresponding to the per-antenna power constraints ($P_1 \leq \hat{P}, P_2 \leq \hat{P}, ..., P_m \leq \hat{P}$, respectively), and v be the Lagrange multiplier corresponding to the long-term power constraint (we consider the long-term power constraint with equality). Then, the KKT conditions can be written as [14]:

$$\theta_k P_k = 0, k = 1, 2, ..., m,$$

$$\theta_k \geq 0, k = 1, 2, ..., m,$$

$$P_k \geq 0, k = 1, 2, ..., m,$$

$$\theta'_k (P_k - \hat{P}) = 0,$$

$$\theta'_k \geq 0,$$

$$P_k \leq \hat{P},$$

$$-\frac{f(\underline{\lambda}')}{P_k + \frac{\bar{P}}{m\lambda'_k}} - \theta_k + \theta'_k + vf(\underline{\lambda}') = 0, k = 1, 2, ..., m,$$

$$\mathbb{E}_{\underline{\lambda}'}\left[\sum_{k=1}^{m} P_k\right] = \bar{P}.$$

Then, with the same procedure as in Appendix A, we can see that the power policy in (21) results from above conditions.

APPENDIX C
PROOF OF THEOREM 4

If we put the power allocation (8) into (7), we have

$$\int_{\gamma_0}^{\frac{\gamma_0}{1-\alpha\gamma_0}} \left(\frac{\bar{P}}{\gamma_0} - \frac{\bar{P}}{\gamma}\right) f(\gamma)\, d\gamma + \int_{\frac{\gamma_0}{1-\alpha\gamma_0}}^{\infty} P_{max} f(\gamma)\, d\gamma = \bar{P}.$$

Putting the distribution $f(\gamma)$ from (30) into the above equation, (33) results. Note that in the power allocation (8), the inequality $\gamma_0 \leq \frac{\gamma_0}{1-\alpha\gamma_0}$ should always be satisfied for the power allocation to be valid. Therefore, the threshold γ_0 should lie in the region $[0, 1/\alpha]$ ($0 \leq \gamma_0 \leq 1/\alpha$, therefore, $0 \leq x \leq 1/\bar{\gamma}\alpha$). Now, define the function $g(x)$ as

$$g(x) := \frac{e^{-x}}{x} - \frac{e^{-\frac{x}{1-\alpha x\bar{\gamma}}}}{x} - E_1(x) + E_1\left(\frac{x}{1-\alpha x\bar{\gamma}}\right) + \alpha\bar{\gamma}e^{-\frac{x}{1-\alpha x\bar{\gamma}}} - \bar{\gamma}.$$

Then, we have:

$$\frac{\partial g(x)}{\partial x} = \frac{e^{-\frac{x}{1-\alpha x\bar{\gamma}}} - e^{-x}}{x^2} \leq 0, \forall x \in [0, \frac{1}{\bar{\gamma}\alpha}], \tag{45}$$

$$\lim_{x\to 0^+} g(x) = (\alpha - 1)\bar{\gamma} > 0 \text{ (since } 1 < \alpha), \tag{46}$$

$$\lim_{x\to\left(\frac{1}{\bar{\gamma}\alpha}\right)^-} g(x) = \frac{e^{-x_1}}{x_1} - E_1(x_1) - \bar{\gamma}, x_1 = \frac{1}{\bar{\gamma}\alpha}. \tag{47}$$

Note that $1 < \alpha \leq \alpha_{max}$ for the power allocation (8), and from (9) $\alpha_{max} = \frac{1}{\gamma_0'}$, where γ_0' is given by (31). Therefore, $x_1 = \frac{1}{\bar{\gamma}\alpha} \geq \frac{\gamma_0'}{\bar{\gamma}}$, and from [16], we have $\frac{e^{-x_1}}{x_1} - E_1(x_1) - \bar{\gamma} \leq 0$ for $x_1 = \frac{1}{\bar{\gamma}\alpha}$. Therefore,

$$\lim_{x\to\left(\frac{1}{\bar{\gamma}\alpha}\right)^-} g(x) \leq 0. \tag{48}$$

From (45), (46), and (48), we observe that γ_0 is uniquely determined and $\gamma_0 \in [0, \frac{1}{\alpha}]$ and the proof is complete.

References

[1] "Federal Communictions Commission: Rules and Regulations" available online at http://www.fcc.gov/oet/info/rules/

[2] D. P. Palomar, "Unified Framework for Linear MIMO Transceivers with Shaping Constraints," *IEEE Communication Letters*, vol. 8, no. 12, pp. 697-699, 2004.

[3] A. J. Goldsmith and P. P. Varaiya, "Capacity of Fading Channels with Channel Side Information," *IEEE Transactions on Information Theory*, vol. 43, pp. 1986–1992, Nov. 1997.

[4] S. K. Jayaweera and H. V. Poor, "Capacity of Multiple-Antenna Systems with Both Receiver and Transmitter Channel State Information," *IEEE Transactions on Information Theory*, vol. 49, no. 10, pp. 2697–2709, Oct. 2003.

[5] E. Telatar, "Capacity of Multi-Antenna Gaussian Channels," *Eur. Trans. Telecomm. ETT*, vol. 10, no. 6, pp. 585–596, Nov. 1999.

[6] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity Limits of MIMO Channels," *IEEE Journal on Selected Area in Communications*, vol. 21, no. 5, p. 684–702, Jun. 2003.

[7] G. Caire, G. Taricco, and E. Biglieri, "Optimum Power Control over Fading Channels," *IEEE Transactions on Information Theory*, vol. 45, pp. 1468-1489, July 1999.

[8] B. M. Hochwald, L. M. A. Jalloul, "Statistical Distributions that Arise in Constrained Beamforming with Incomplete Channel Information," *IEEE Information Theory and Applications Workshop (ITA)*, 2010.

[9] A. Feiten, R. Mathar, S. Hanly, "Eigenvalue-Based Optimum-Power Allocation for Gaussian Vector Channels," *IEEE Transactions on Information Theory*, vol. 53, pp. 2304-2309, June 2007.

[10] D. Tse and S. Hanly,"Multiaccess Fading Channels-Part I: Polymatroid Structure, Optimal Resource Allocation and Throughput Capacities," *IEEE Transactions on Information Theory*, vol. 44, pp. 2796-2815, Nov. 1998.

[11] R. Knopp and P. Humblet, "Information Capacity and Power Control in Single-Cell Multiuser Communications," *IEEE International Conference on Communications*, 1995, pp. 331-335.

[12] R. R. Chen, Y. Lin, "Optimal Power Control for Multiple Access Channel with Peak and Average Power Constraints," *IEEE International Conference on Wireless Networks, Communications and Mobile Computing*, 2005, pp. 1408-1411.

[13] M. A. Khojastepour and B. Aazhang, "The Capacity of Average and Peak Power Constrainted Fading Channels with Channel Side Information," *IEEE Wireless Cotnitnnications and Networking Conference (WCNC)*, vol. 1, pp. 77–82, Mar. 2004.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge, U.K.: Cambridge Univ. Press, 2004.

[15] A. W. Marshall and I. Olkin, *Theory of Majorization and Its Applications.* New York: Academic, 1979.

[16] M. S. Alouini and A. J. Goldsmith, "Capacity of Rayleigh Fading Channels under Different Adaptive Transmission and Diversity-Combining Techniques," *IEEE Transactions on Vehicular Technol*, vol. 48, pp. 1165–1181, July 1999.

[17] D. N. C. Tse and P. Viswanath, *Fundamentals of Wireless Communications.* Cambridge, U.K.: Cambridge Univ. Press, 2005.

[18] Zukang Shen, Robert W. Heath, Jr., Jeffrey G. Andrews, and Brian L. Evans, "Comparison of Space-Time Water-filling and Spatial Water-filling for MIMO Fading Channels," *IEEE Global Telecommunications Conference*, Dec. 2004, vol. 1, pp. 431–435.

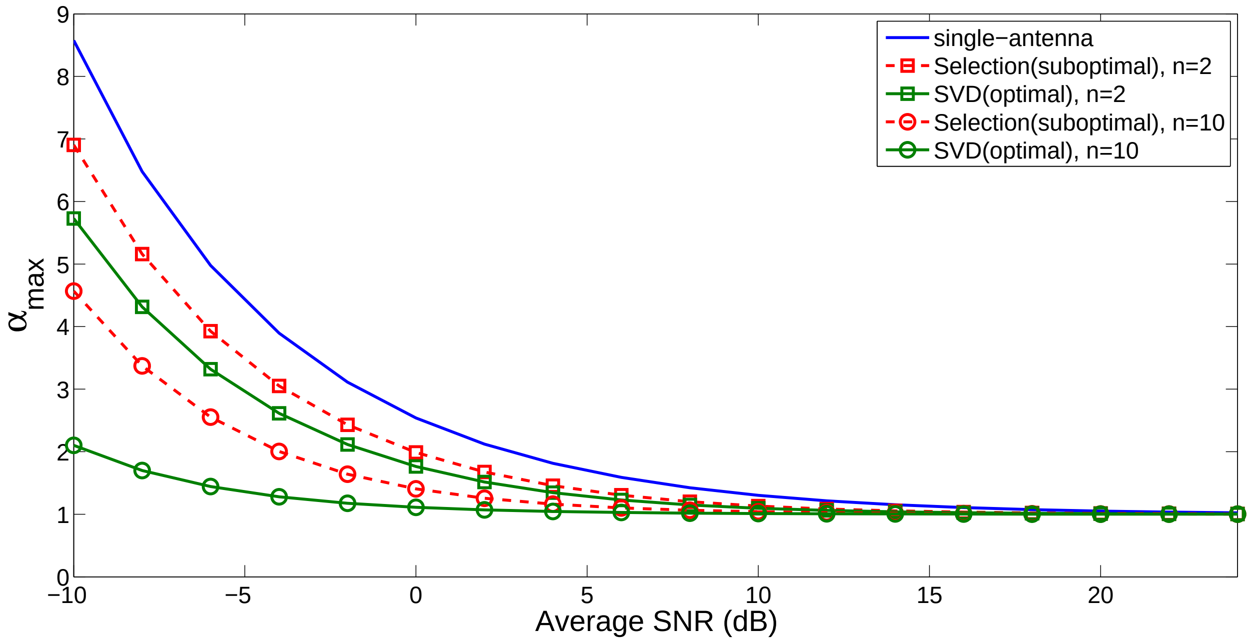

Fig. 1. The value of α_{max}, the threshold separating Cases 2 and 3, versus average SNR $\bar{\gamma}$ for SISO and MISO systems with Rayleigh fading.

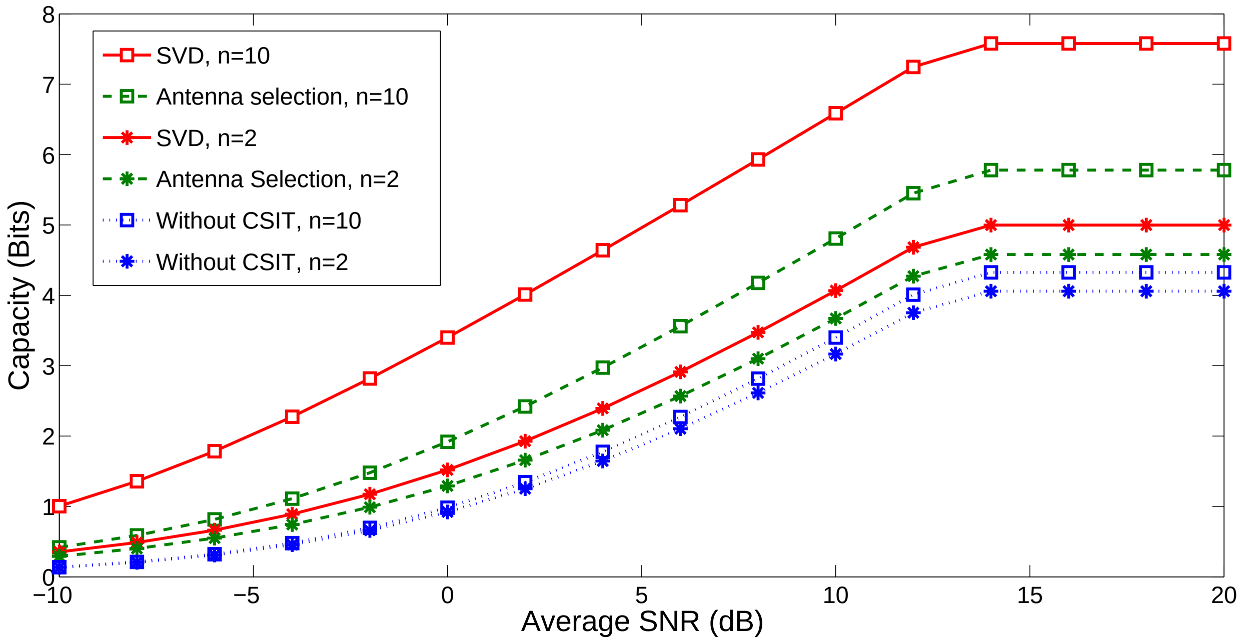

Fig. 2. Ergodic capacity versus average SNR for MISO systems with Rayleigh fading, $P_{max} = 20(13dB)$ and $N_0 = 1$, and $n = 2, 10$.

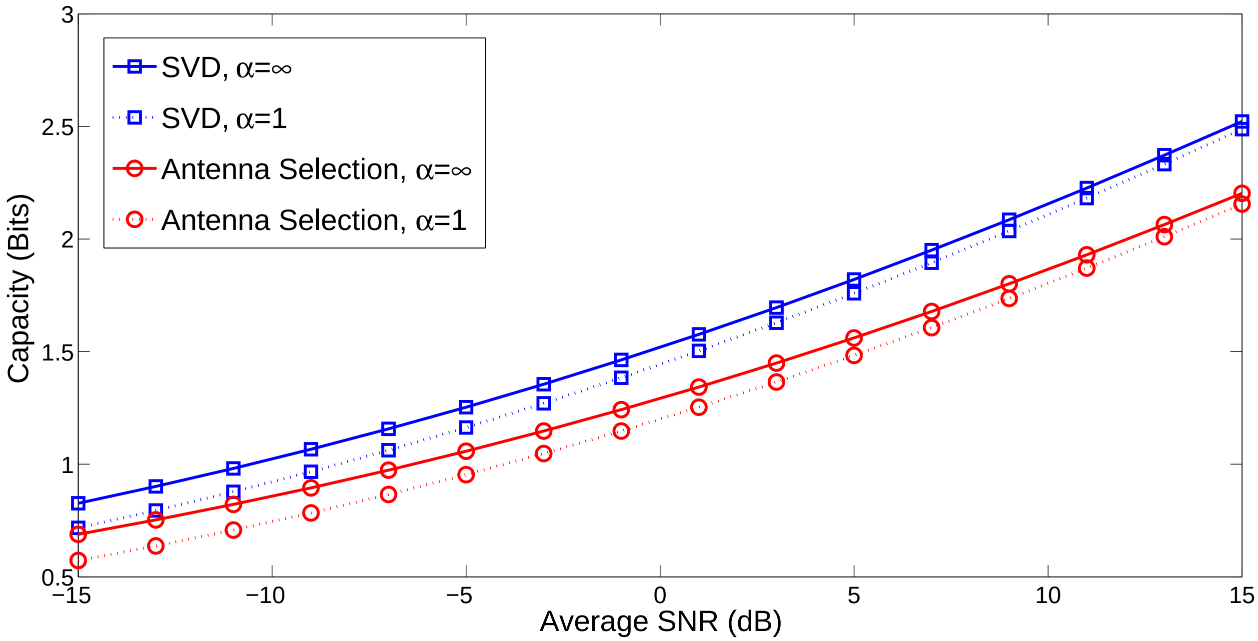

Fig. 3. Ergodic capacity versus average SNR for MISO systems with Rayleigh fading, $n = 2$ and $\alpha = \infty$ and $\alpha = 1$.

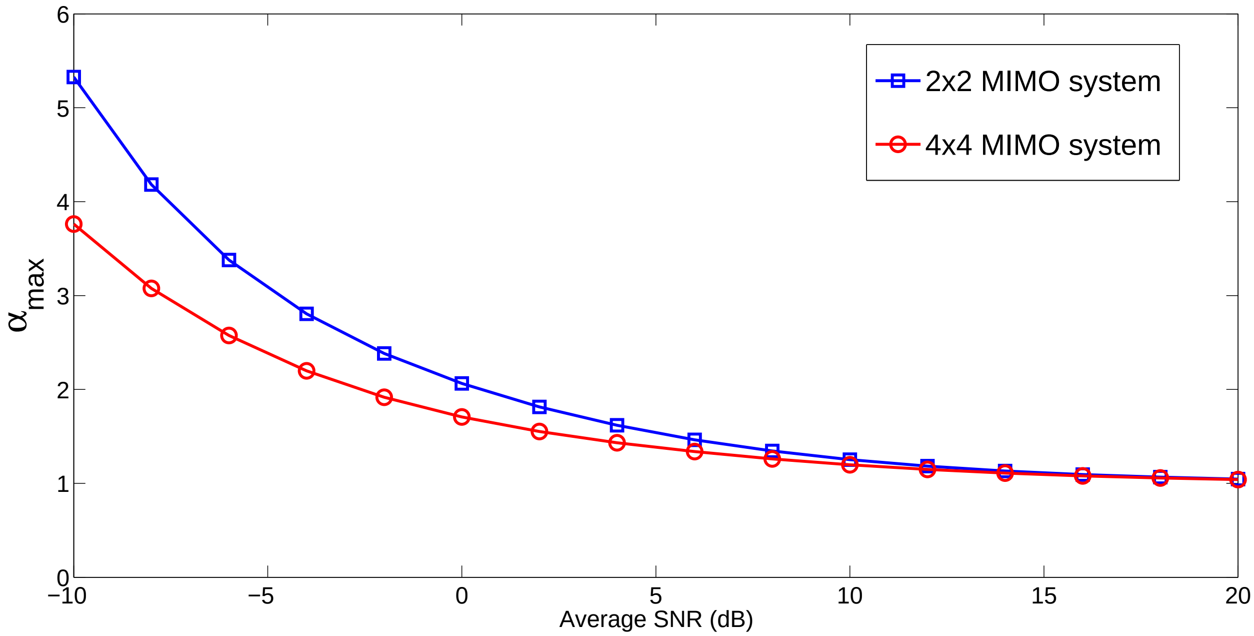

Fig. 4. The value of α_{max}, the threshold separating Cases 2 and 3, versus Average SNR per parallel channel ($\frac{\bar{P}}{mN_0}$) for MIMO systems with Rayleigh fading.

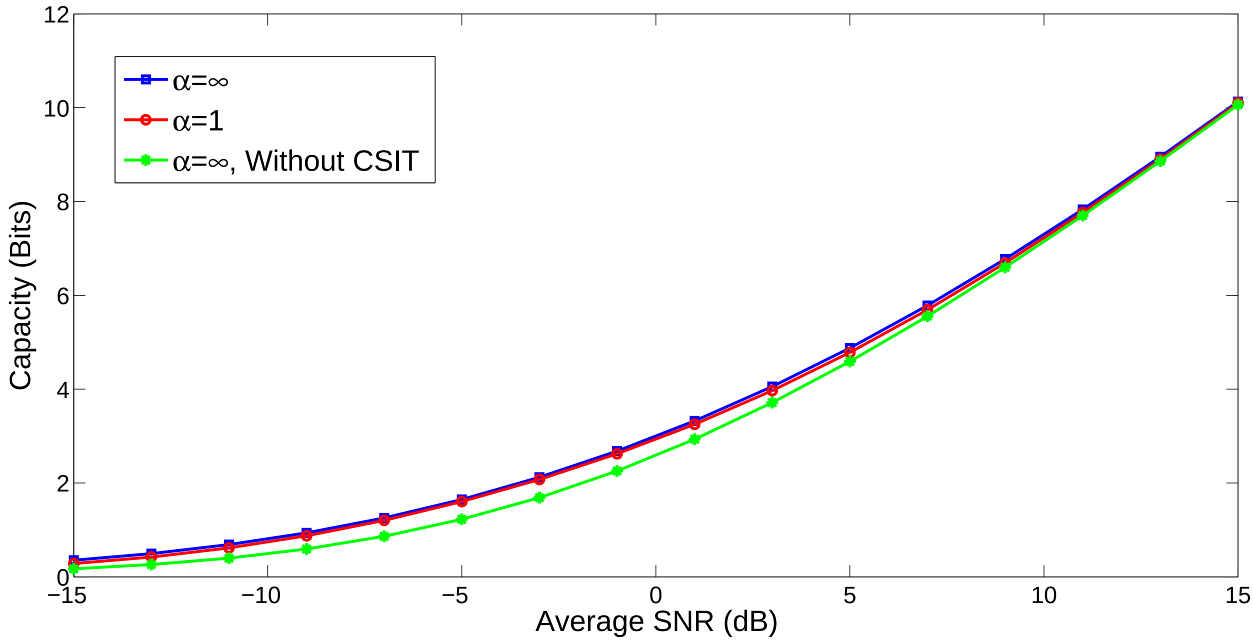

Fig. 5. Ergodic capacity versus average SNR per parallel channel ($\frac{\bar{P}}{mN_0}$) for 2×2 MIMO systems with Rayleigh fading.