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Some remarks on G_2-structures

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Dedicated to the memory of Raoul Bott.

Abstract. This article consists of loosely related remarks about the geometry of G_2-structures on 7-manifolds, some of which are based on unpublished joint work with two other people: F. Reese Harvey and Steven Altschuler.

After some preliminary background information about the group G_2 and its representation theory, a set of techniques is introduced for calculating the differential invariants of G_2-structures and the rest of the article is applications of these results. Some of the results that may be of interest are as follows:

First, a formula is derived for the scalar curvature and Ricci curvature of a G_2-structure in terms of its torsion and covariant derivatives with respect to the 'natural connection' (as opposed to the Levi-Civita connection) associated to a G_2-structure. When the fundamental 3-form of the G_2-structure is closed, this formula implies, in particular, that the scalar curvature of the underlying metric is nonpositive and vanishes if and only if the structure is torsion-free. These formulae are also used to generalize a recent result of Cleyton and Ivanov [3] about the nonexistence of closed Einstein G_2-structures (other than the Ricci-flat ones) on compact 7-manifolds to a nonexistence result for closed G_2-structures whose Ricci tensor is too tightly pinched.

Second, some discussion is given of the geometry of the first and second order invariants of G_2-structures in terms of the representation theory of G_2.

Third, some formulae are derived for closed solutions of the Laplacian flow that specify how various related quantities, such as the torsion and the metric, evolve with the flow. These may be useful in studying convergence or long-time existence for given initial data.

Some of this work was subsumed in the work of Hitchin [12] and Joyce [14]. I am making it available now mainly because of interest expressed by others in seeing these results written up since they do not seem to have all made it into the literature.

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Contents

1. Introduction

This brief article consists of a collection of remarks on the geometry of G_2-structures on 7-manifolds, some of which are based on old unpublished joint work carried out on separate occasions with two other people: F. Reese Harvey and Steven Altschuler.

The work with Reese Harvey (recounted in §5) concerned techniques for calculating various quantities associated to a G_2-structure, possibly with torsion, and was carried out intermittently during the period 1988 through 1994.

The work with Steven Altschuler (recounted in §6) concerned the geometry of a natural Laplacian flow for G_2-structures and was carried out in 1992.

The main reason for making these remarks available now is that some of these formulae and results do not seem to have appeared yet in the literature and some people have expressed an interest in learning about them.

2. Algebra

This section will collect the main results about the group G_2 that will be needed. The reader may consult [2], [14], or [15] for details concerning the properties of the group G_2 that are not proved here. In general, the notation is chosen to agree with the notation in [2].

2.1. The group G_2

Let $e_1, e_2, \ldots, e_7$ denote the standard basis of $\mathbb{R}^7$ (whose elements will be referred to as column vectors of height 7) and let $e^1, e^2, \ldots, e^7 : \mathbb{R}^7 \to \mathbb{R}$ denote the corresponding dual basis.

For notational simplicity, write e^{ijk} for the wedge product $e^i{\wedge}e^j{\wedge}e^k$ in $\Lambda^3\bigl((\mathbb{R}^7)^*\bigr)$. Define

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{2.1}$$

By a theorem of Schouten [16] (see [2] for a proof), the subgroup of $\mathrm{GL}(7,\mathbb{R})$ that fixes ϕ is a compact, connected, simple Lie group of type G_2. In this article, this result will be used to justify the following definition:

Definition 1 (The group G_2).

$$G_2 = \{\ g \in GL(7,\mathbb{R}) \mid g^*(\phi) = \phi\ \}\,. \tag{2.2}$$

2.2. Associated structures

A few properties of G_2 will be needed in this article. The reader may consult [2] for proofs.

The group G_2 acts irreducibly on $\mathbb{R}^7$ and preserves the metric and orientation for which the basis $e_1, e_2, \dots, e_7$ is an oriented orthonormal basis. The notations g_ϕ and $\langle , \rangle_\phi$ will be used to refer to the metric. The Hodge star operator determined by this metric and orientation will be denoted $*_\phi$. Note, in particular, that G_2 also fixes the 4-form

$$*_\phi\phi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \tag{2.3}$$

2.3. Some G_2 actions

The group G_2 acts transitively on the unit sphere $S^6 \subset \mathbb{R}^7$. The stabilizer subgroup of any non-zero vector in $\mathbb{R}^7$ is isomorphic to $SU(3) \subset SO(6)$, so that $S^6 = G_2/\,SU(3)$. Since $SU(3)$ acts transitively on $S^5 \subset \mathbb{R}^6$, it follows that G_2 acts transitively on the set of orthonormal pairs of vectors in $\mathbb{R}^7$.

However, G_2 does not act transitively on the set of orthonormal triples of vectors in $\mathbb{R}^7$ since it preserves the 3-form ϕ.

2.4. The ε-notation

It will be convenient to use an ε-notation that will now be introduced. This is the unique symbol that is skew-symmetric in either three or four indices and satisfies

$$\phi = \tfrac{1}{6}\varepsilon_{ijk}\, e^i \wedge e^j \wedge e^k \tag{2.4}$$

$$*_\phi\phi = \tfrac{1}{24}\varepsilon_{ijkl}\, e^i \wedge e^j \wedge e^k \wedge e^l\,. \tag{2.5}$$

Thus, for example, $\varepsilon_{123} = 1$ and $\varepsilon_{4567} = 1$, while $\varepsilon_{124} = \varepsilon_{3456} = 0$. Another way to think of this symbol is via the cross product: $e_i \times e_j = \varepsilon_{ijk} e_k$.

The symbol ε satisfies various useful identities. For example (using the summation convention),

$$\varepsilon_{ijk}\,\varepsilon_{ijl} = 6\delta_{kl} \tag{2.6}$$

$$\varepsilon_{ijq}\,\varepsilon_{ijkl} = 4\varepsilon_{qkl} \tag{2.7}$$

$$\varepsilon_{ipq}\,\varepsilon_{ijk} = \varepsilon_{pqjk} + \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj} \tag{2.8}$$

$$\varepsilon_{ipq}\,\varepsilon_{ijkl} = \delta_{pj}\varepsilon_{qkl} - \delta_{jq}\varepsilon_{pkl} + \delta_{pk}\varepsilon_{jql} - \delta_{kq}\varepsilon_{jpl} + \delta_{pl}\varepsilon_{jkq} - \delta_{lq}\varepsilon_{jkp}\,. \tag{2.9}$$

These identities are actually quite easy to prove using the fact that G_2 acts transitively on orthonormal pairs. For example, identity (2.8) can be reduced to the case where $p = 1$ and $q = 2$. Then the only non-zero term on the left hand side is $\varepsilon_{312}\varepsilon_{3jk}$. By the definitions of ϕ and $*_\phi\phi$, both sides of the equation vanish unless $\{j, k\}$ is one of the subsets $\{1,2\}$, $\{4,7\}$, or $\{5,6\}$, and the identity clearly holds in those cases. The other identities can be proved similarly.

2.5. Matrix and vector representations

The ε-symbol can be used to describe the algebra $\mathfrak{g}_2$ as a subalgebra of $\mathfrak{so}(7)$, the space of skew-symmetric 7-by-7 matrices. A skew-symmetric matrix $a = (a_{ij})$ lies in $\mathfrak{g}_2$ if and only if $\varepsilon_{ijk}a_{jk} = 0$ for all i.

For any vector $v = v_i e_i \in \mathbb{R}^7$, define $[v] = (v_{ij}) \in \mathfrak{so}(7)$ by the formula $v_{ij} = \varepsilon_{ijk}v_k$. It then follows that

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \left[\mathbb{R}^7\right],$$

which is the G_2-invariant irreducible decomposition of $\mathfrak{so}(7)$. Note that $[v]$ is the matrix that represents the linear transformation of $\mathbb{R}^7$ induced by cross-product with $v \in \mathbb{R}^7$.

Conversely, define the map $\langle\cdot\rangle\colon \mathfrak{gl}(7) \to \mathbb{R}^7$ by $\langle\,(a_{ij})\,\rangle = (\varepsilon_{ijk}a_{jk})$. The kernel of this mapping intersected with $\mathfrak{so}(7)$ is $\mathfrak{g}_2$ and the ε-identities imply that, for all $a, b \in \mathbb{R}^7$,

$$\langle\,[a]\,\rangle = 6a \tag{2.10}$$

$$\langle\,[a][b]\,\rangle = 3[b]a = -3[a]b. \tag{2.11}$$

2.6. The G_2-type decomposition of exterior forms

To avoid writing $(\mathbb{R}^7)^*$ many times, I will, for the rest of this section, use V as an abbreviation for the vector space $\mathbb{R}^7$.

Although G_2 acts irreducibly on V and hence on $\Lambda^1(V^*)$ and $\Lambda^6(V^*)$, it does not act irreducibly on $\Lambda^p(V^*)$ for $2 \le p \le 5$. In order to understand the irreducible decomposition of $\Lambda^p(V^*)$ for p in this range, it suffices to understand the cases $p = 2$ and $p = 3$, since the operator $*_\phi$ induces an isomorphism of G_2-modules $\Lambda^p(V^*) = \Lambda^{7-p}(V^*)$.

In [2], it is shown that there are irreducible G_2-module decompositions

$$\Lambda^2(V^*) = \Lambda^2_{14}(V^*) \oplus \Lambda^2_7(V^*) \tag{2.12}$$

$$\Lambda^3(V^*) = \Lambda^3_{27}(V^*) \oplus \Lambda^3_7(V^*) \oplus \Lambda^3_1(V^*) \tag{2.13}$$

where $\Lambda^p_d(V^*)$ denotes an irreducible G_2-module of dimension d. For $p = 4$ or 5, adopt the convention that $\Lambda^p_d(V^*) = *_\phi(\Lambda^{7-p}_d(V^*))$.

These summands can be characterized as follows:

$$\begin{aligned}
\Lambda^2_7(V^*) &= \left\{\, *_\phi(\alpha \wedge *_\phi\phi) \mid \alpha \in \Lambda^1(V^*) \,\right\}\\
&= \left\{\, \alpha \in \Lambda^2(V^*) \mid \alpha \wedge \phi = 2*_\phi\alpha \,\right\}\\
\Lambda^2_{14}(V^*) &= \left\{\, \alpha \in \Lambda^2(V^*) \mid \alpha \wedge \phi = -*_\phi\alpha \,\right\} = \mathfrak{g}_2^\flat\\
\Lambda^3_1(V^*) &= \{\, r\phi \mid r \in \mathbb{R} \,\}\\
\Lambda^3_7(V^*) &= \left\{\, *_\phi(\alpha \wedge \phi) \mid \alpha \in \Lambda^1(V^*) \,\right\}\\
\Lambda^3_{27}(V^*) &= \left\{\, \alpha \in \Lambda^3(V^*) \mid \alpha \wedge \phi = 0 \text{ and } \alpha \wedge *_\phi\phi = 0 \,\right\} = \mathsf{i}_\phi\big(S^2_0(V^*)\big).
\end{aligned} \tag{2.14}$$

The notations $\mathfrak{g}_2^\flat$ and $\mathsf{i}_\phi\big(S^2_0(V^*)\big)$ used in (2.14) need some explanation.

First, $\mathfrak{g}_2^\flat$: Under the "musical isomorphism" $\flat\colon V \to V^*$ induced by the G_2-invariant inner product $\langle , \rangle_\phi$, the Lie algebra of G_2, namely $\mathfrak{g}_2 \subset V \otimes V^*$, is identified with $\mathfrak{g}_2^\flat = (\flat \otimes 1)(\mathfrak{g}_2) \subset \Lambda^2(V^*) \subset V^* \otimes V^*$. This subspace is an irreducible G_2-module since G_2 is simple.

Second, $\mathsf{i}_\phi\big(S^2_0(V^*)\big)$: Consider the linear mapping $\mathsf{i}_\phi\colon S^2(V^*) \to \Lambda^3(V^*)$, defined on decomposable elements by

$$\mathsf{i}_\phi(\alpha \circ \beta) = \alpha \wedge *_\phi(\beta \wedge *_\phi \phi) + \beta \wedge *_\phi(\alpha \wedge *_\phi \phi). \tag{2.15}$$

The mapping i_ϕ is G_2-invariant and one can show that $S^2(V^*) = \mathbb{R} g_\phi \oplus S^2_0(V^*)$ is a decomposition of $S^2(V^*)$ into G_2-irreducible summands. Evidently, i_ϕ is nonzero on each summand and is therefore injective. Hence, the image $\mathsf{i}_\phi\left(S^2_0(V^*)\right) \subset \Lambda^3(V^*)$ is 27-dimensional and irreducible. The equation

$$\Lambda^3_{27}(V^*) = \left\{\ \alpha \in \Lambda^3(V^*) \mid \alpha \wedge \phi = 0 \text{ and } \alpha \wedge *_\phi \phi = 0\ \right\} \tag{2.16}$$

defines $\Lambda^3_{27}(V^*)$ as a G_2-invariant, 27-dimensional subspace of $\Lambda^3(V^*)$. By dimension count, it must intersect $\mathsf{i}_\phi\left(S^2_0(V^*)\right)$ nontrivially. Since this intersection is also G_2-invariant and since $\mathsf{i}_\phi\left(S^2_0(V^*)\right)$ is G_2-irreducible, $\mathsf{i}_\phi\left(S^2_0(V^*)\right) = \Lambda^3_{27}(V^*)$.

Using the ε-notation, one can express the map i_ϕ in indices as

$$\mathsf{i}_\phi(h_{ij} e^i e^j) = \varepsilon_{ikl}\, h_{ij}\, e^j \wedge e^k \wedge e^l, \tag{2.17}$$

making it evident that $\mathsf{i}_\phi(g_\phi) = 6\phi$.

It will be useful to have a way to invert the map i_ϕ. Define $\mathsf{j}_\phi : \Lambda^3(V^*) \to S^2(V^*)$ by the formula

$$\mathsf{j}_\phi(\gamma)(v,w) = *_\phi\big((v \lrcorner \phi) \wedge (w \lrcorner \phi) \wedge \gamma\big). \tag{2.18}$$

for $\gamma \in \Lambda^3(V^*)$ and $v, w \in V$. It is not difficult to verify that

$$\mathsf{j}_\phi\big(\mathsf{i}_\phi(h)\big) = 8h + 4\big(\mathrm{tr}_{g_\phi}(h)\big)\, g_\phi \tag{2.19}$$

for all $h \in S^2(V^*)$. Note also that $\mathsf{j}_\phi(\phi) = 6 g_\phi$, while $\mathsf{j}_\phi\big(\Lambda^3_7(V^*)\big) = 0$.

Note that i_ϕ and j_ϕ are not isometries when $S^2_0(V^*)$ and $\Lambda^3_{27}(V^*)$ are given their natural metrics.[1] Instead, $\gamma \in \Lambda^3_{27}(V^*)$ satisfies $|\mathsf{j}_\phi(\gamma)|^2 = 8\,|\gamma|^2$ while $h \in S^2_0(V^*)$ satisfies $|\mathsf{i}_\phi(h)|^2 = 8\,|h|^2$.

2.7. More G_2 representation theory

It will, from time to time, be useful to have some deeper knowledge of the representation theory of G_2, so some of these facts will be collected here. For details, consult [13].

Since G_2 is a simple Lie group of rank 2, its irreducible representations can be indexed by a pair of integers (p,q) that represent the highest weight of the representation with respect to a fixed maximal torus in G_2 endowed with fixed base for its root system. The irreducible representation of highest weight (p,q) will be denoted $\mathsf{V}_{p,q}$.

[1] The usual inner product on exterior forms is meant here, while, when $h = h_{ij}\, e^i e^j$ with (e^i) being a g-orthonormal coframe of V, one sets $|h|^2 = h_{ij} h_{ij}$.

2.7.1. The standard representation

The fundamental representation $\mathsf{V}_{1,0} \simeq \mathbb{R}^7$ is the 'standard' representation in which G_2 has been defined in this article.

The representation $\mathsf{V}_{p,0}$ for $p \geq 0$ is isomorphic to $\mathsf{S}^p_0(\mathbb{R}^7)$, i.e., the symmetric, trace-free polynomials of degree p in seven variables. (It is somewhat remarkable that these irreducible representations of $\mathrm{SO}(7)$ remain irreducible when thought of as representations of G_2.) In this article, the only representations $\mathsf{V}_{p,0}$ in this series that will be important are those for $p = 0, 1, 2$.

2.7.2. The adjoint representation

The other fundamental representation, $\mathsf{V}_{0,1} \simeq \mathbb{R}^{14}$ is isomorphic to $\mathfrak{g}_2$, i.e., is the adjoint representation of G_2. The representation $\mathsf{V}_{0,p}$ for $p \geq 0$ is then the irreducible constituent of highest weight in $\mathsf{S}^p(\mathfrak{g}_2)$.

In this article, only $\mathsf{V}_{0,1} \simeq \mathfrak{g}_2$ and $\mathsf{V}_{0,2} \simeq \mathbb{R}^{77}$ from this series will be important. (This latter one will be important because it is the space of curvature tensors of G_2-metrics.) The reader must be careful not to confuse the representation $\mathsf{V}_{0,2}$ with $\mathsf{V}_{3,0}$, which also happens to have dimension 77.

A few more facts about this representation will be needed: The group G_2 has rank 2 and a maximal torus for G_2 can be obtained by simply taking a maximal torus in the subgroup $\mathrm{SU}(3)$. Moreover, every element in $\mathfrak{g}_2$ is $\mathrm{Ad}(\mathrm{G}_2)$-conjugate to an element in such a maximal torus. Consequently, every element in $\Lambda^2_{14}(\mathbb{R}^7) = \mathfrak{g}_2^\flat$ is conjugate to an element of the form

$$\alpha = \lambda_1\, e^{23} + \lambda_2\, e^{45} - (\lambda_1{+}\lambda_2)\, e^{67} \tag{2.20}$$

since these span $\mathfrak{t}^\flat \subset \mathfrak{g}_2^\flat$, where $\mathfrak{t} \subset \mathfrak{g}_2$ is a Cartan subalgebra. Moreover, it is well-known that the ring of $\mathrm{Ad}(\mathrm{G}_2)$-invariant polynomials on $\mathfrak{g}_2$ is a free polynomial ring on two generators, one of degree 2 and one of degree 6. One sees from the above normal form that these two generators can be taken to be $|\alpha|^2$ and $|\alpha^3|^2$. Thus, two elements α and β in $\Lambda^2_{14}(\mathbb{R}^7)$ are conjugate under the action of G_2 if and only if they satisfy $|\alpha|^2 = |\beta|^2$ and $|\alpha^3|^2 = |\beta^3|^2$. In particular, the normal form (2.20) can be made unique by requiring that $0 \leq \lambda_1 \leq \lambda_2$.

In particular, one obtains, for all $\alpha \in \Lambda^2_{14}(V^*)$, the useful identity

$$|\, \alpha^2 \,|^2 = |\alpha|^4 \tag{2.21}$$

and inequality

$$|\, \alpha^3 \,|^2 \leq \tfrac{2}{3}\, |\alpha|^6, \tag{2.22}$$

which are easily verified by checking them on elements of the form (2.20).

In fact, using the normal form (2.20), one can prove other useful exterior algebra identities. One that will be needed later is

$$\alpha \wedge *_\phi(\alpha \wedge \alpha) = |\alpha|^2 *_\phi \alpha - \tfrac{1}{3} *_\phi(\alpha^3) \wedge *_\phi \phi \qquad \text{for } \alpha \in \Lambda^2_{14}(V^*). \tag{2.23}$$

2.7.3. Other representations

Of the representations $\mathsf{V}_{p,q}$ with p and q positive, only $\mathsf{V}_{1,1} \simeq \mathbb{R}^{64}$ will play any significant role in this article (and mainly as a nuisance at that). In fact, each of the other representations $\mathsf{V}_{p,q}$ with both p and q positive has dimension at least 189, so these can easily be ruled out for dimension reasons in the calculations to follow.

The following tensor product and Schur functor decompositions will be useful:

$$\begin{aligned}
\mathsf{S}^2\big(V_{1,0}\big) &\simeq \mathsf{V}_{0,0} \oplus \mathsf{V}_{2,0}\\
\Lambda^2\big(V_{1,0}\big) &\simeq \mathsf{V}_{1,0} \oplus \mathsf{V}_{0,1}\\
\mathsf{V}_{1,0} \otimes \mathsf{V}_{0,1} &\simeq \mathsf{V}_{1,0} \oplus \mathsf{V}_{2,0} \oplus \mathsf{V}_{1,1}\\
\mathsf{S}^2\big(V_{0,1}\big) &\simeq \mathsf{V}_{0,0} \oplus \mathsf{V}_{2,0} \oplus \mathsf{V}_{0,2}\\
\Lambda^2\big(V_{0,1}\big) &\simeq \mathsf{V}_{0,1} \oplus \mathsf{V}_{3,0}
\end{aligned} \tag{2.24}$$

2.7.4. An example of G_2-type decomposition

As an application of these formulae that will be used below, consider the problem of decomposing $\beta\wedge\beta \in \Lambda^4(V^*)$ into its G_2-types where β lies in $\Lambda^2_{14}(V^*) \simeq \mathsf{V}_{0,1}$. Since

$$\Lambda^4(V^*) \simeq \Lambda^4_1(V^*) \oplus \Lambda^4_7(V^*) \oplus \Lambda^4_{27}(V^*) \simeq \mathsf{V}_{0,0} \oplus \mathsf{V}_{1,0} \oplus \mathsf{V}_{0,2} \tag{2.25}$$

and since, by (2.24), we have $\mathsf{S}^2\big(\mathsf{V}_{0,1}\big) \simeq \mathsf{V}_{0,0} \oplus \mathsf{V}_{2,0} \oplus \mathsf{V}_{0,2}$, it follows that $\beta\wedge\beta$ can have no component in $\Lambda^4_7(V^*) \simeq \mathsf{V}_{1,0}$. Moreover, since there is, up to multiples, only one G_2-invariant quadratic form on $\mathsf{V}_{0,1}$ and since $*_\phi\phi$ spans $\Lambda^4_1(V^*) \simeq \mathsf{V}_{0,0}$, it follows that there is a constant λ such that

$$\beta\wedge\beta = \lambda\,|\beta|^2 *_\phi\phi + \big(\beta\wedge\beta - \lambda\,|\beta|^2 *_\phi\phi\big) \tag{2.26}$$

where the first term on the right lies in $\Lambda^4_1(V^*)$ while the second term (in parentheses) lies in $\Lambda^4_{27}(V^*)$.

The constant λ is determined as follows: Wedging both sides with ϕ and using the fact that $\beta\wedge\phi = -*_\phi\beta$ while $\gamma\wedge\phi = 0$ for $\gamma \in \Lambda^4_{27}(V^*)$ yields

$$-\,|\beta|^2 *_\phi 1 = \beta\wedge\beta\wedge\phi = \big(\lambda\,|\beta|^2 *_\phi\phi\big)\wedge\phi = 7\lambda\,|\beta|^2 *_\phi 1, \tag{2.27}$$

showing that $\lambda = -\frac{1}{7}$. Thus, the G_2-type decomposition is given by

$$\beta\wedge\beta = -\tfrac{1}{7}\,|\beta|^2 *_\phi\phi + \big(\beta\wedge\beta + \tfrac{1}{7}\,|\beta|^2 *_\phi\phi\big). \tag{2.28}$$

for $\beta \in \Lambda^2_{14}(V^*)$. Of course, this decomposition is orthogonal, so, using the identity (2.21), one can take the square norms of both sides, yielding

$$|\beta|^4 = |\beta\wedge\beta|^2 = \tfrac{1}{7}|\beta|^4 + \big|\beta\wedge\beta + \tfrac{1}{7}\,|\beta|^2 *_\phi\phi\big|^2. \tag{2.29}$$

Consequently, for $\beta \in \Lambda^2_{14}(V^*)$, one has

$$\big|\,\beta\wedge\beta + \tfrac{1}{7}\,|\beta|^2 *_\phi\phi\,\big|^2 = \tfrac{6}{7}|\beta|^4, \tag{2.30}$$

an identity that will be used below. (Note that (2.30) implies, in particular, that the $\Lambda^4_{27}(V^*)$-piece of $\beta{\wedge}\beta$ cannot vanish unless β itself vanishes, a result equivalent to Lemma 5.8 of [3].)

Similar sorts of calculations can be used to establish the (sharp) inequalities for quadratic forms

$$-\,2\,|\beta|^2\,g \;\leq\; \mathsf{j}\big(*_\phi(\beta\wedge\beta)\big) \;\leq\; \tfrac{2}{3}\,|\beta|^2\,g. \tag{2.31}$$

Details are left to the reader.

2.8. Definite forms

The dimension of G_2 is 14 and so, by dimension count, the $\mathrm{GL}(V)$-orbit of ϕ in $\Lambda^3(V^*)$ is open. Denote this orbit by $\Lambda^3_+(V^*)$ and speak of the elements of $\Lambda^3_+(V^*)$ as *definite* 3-forms on V. Note that $\Lambda^3_+(V)$ has two components, since $\mathrm{GL}(V)$ does and since G_2 is connected. Each component is the negative of the other. It is known [10] that $\mathrm{SO}(7)/\mathrm{G}_2 \simeq \mathbb{RP}^7$, so that each component of $\Lambda^3_+(V)$ is diffeomorphic to $\mathbb{RP}^7\times\mathbb{R}^{28}$.

2.8.1. On general 7-dimensional vector spaces

If W is any 7-dimensional vector space, an isomorphism $u\colon W \xrightarrow{\sim} V$ induces an isomorphism $u^*\colon \Lambda^3(V^*) \xrightarrow{\sim} \Lambda^3(W^*)$. Denote by $\Lambda^3_+(W^*)$ the open subset $u^*\left(\Lambda^3_+(V^*)\right) \subset \Lambda^3(W^*)$. Since $\Lambda^3_+(V^*)$ consists of a single $\mathrm{GL}(V)$-orbit, this set does not depend on the choice of u.

2.8.2. Associated algebraic structures

Each $\varphi\in\Lambda^3_+(W^*)$ has a stabilizer in $\mathrm{GL}(W)$ that is isomorphic to G_2 and hence defines a canonical inner product $\langle,\rangle_\varphi$ (with associated quadratic form g_φ) and orientation (Hodge star) $*_\varphi : \Lambda^p(W^*)\to\Lambda^{7-p}(W^*)$.

Similarly, using φ in the place of ϕ in the formulae (2.15) and (2.18), one defines mappings $\mathsf{i}_\varphi : S^2(W^*)\to\Lambda^3(W^*)$ and $\mathsf{j}_\varphi : \Lambda^3(W^*)\to S^2(W^*)$. These maps are frequently useful in formulae.

For example, let $\mathsf{G} : \Lambda^3_+(W^*)\to S^2_+(W^*)$ be the nonlinear $\mathrm{GL}(W)$-equivariant mapping that satisfies $\mathsf{G}(\varphi)=g_\varphi$. It is not difficult to show that G is smooth and satisfies

$$\mathsf{G}'(\varphi)(\psi) = \tfrac12\,\mathsf{j}_\varphi(\psi) - \tfrac13\,*_\varphi(\psi\wedge *_\varphi\varphi)\,g_\varphi\,. \tag{2.32}$$

There is also an associated *vector cross product* $\times_\varphi : W\times W\to W$ defined by the condition

$$\langle w_1\times_\varphi w_2\,,w_3\rangle_\varphi = \varphi(w_1,w_2,w_3). \tag{2.33}$$

Remark 1 (The vector cross product definition of G_2). Given a vector space V over $\mathbb{R}$ endowed with a positive definite inner product $\langle,\rangle : V\times V\to\mathbb{R}$, a (2-fold) *vector cross product* on $\big(V,\langle,\rangle\big)$ is a skew-symmetric bilinear pairing $\times : V\times V\to V$ that satisfies

$$\langle v_1\times v_2, v_1\rangle = 0 \qquad\text{and}\qquad |v_1\times v_2|^2 = |v_1|^2\,|v_2|^2 - \langle v_1,v_2\rangle^2 \tag{2.34}$$

for all $v_1,v_2\in V$.

It can be shown that the $GL(7, \mathbb{R})$-stabilizer of the vector cross product $\times_\phi$ is equal to G_2. Hence one could take $\times_\phi : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ as the algebraic structure defining G_2. In fact, this is what Gray did in his work on G_2-structures. However, I find that the 3-form formulation is more congenial for computations, so vector cross products will not play any significant role in this article.

2.8.3. Definite 4-forms

The canonical mapping $\mathsf{S} : \Lambda^3_+(W^*) \to \Lambda^4(W^*)$ defined by $\mathsf{S}(\varphi) = *_\varphi\varphi$ is a double covering onto an open set $\Lambda^4_+(W^*)$ in $\Lambda^4(W^*)$, which will be referred to as the space of 'definite' 4-forms on W.

The $GL(W)$-stabilizer of an element $\psi \in \Lambda^4_+(W^*)$ is then isomorphic to $\pm G_2 = G_2 \cup \big(G_2 \cdot (-\mathrm{id}_W)\big)$. Thus, a definite 4-form on W defines an inner product on W, but not an orientation.

3. G_2-structures

3.1. Definite forms on manifolds

Let M be a smooth manifold of dimension 7. The union of the subspaces $\Lambda^3_+(T^*_xM)$ is an open subbundle $\Lambda^3_+(T^*M) \subset \Lambda^3(T^*M)$ of the bundle of 3-forms on M.

Definition 2 (Definite 3-forms on manifolds). A 3-form σ on M that takes values in $\Lambda^3_+(T^*M)$ will be said to be a *definite* 3-form on M. The set of definite 3-forms on M will be denoted $\Omega^3_+(M)$.

3.1.1. G_2-structures and definite 3-forms

Each definite 3-form on M defines a G_2-structure on M in the following way:

Let $\mathcal{F}$ denote the principal right $GL(V)$-bundle over M consisting of V-coframes $u\colon T_xM \xrightarrow{\sim} V$. Given any $\sigma \in \Omega^3_+(M)$, define a G_2-bundle

$$F_\sigma = \{u \in \mathrm{Hom}(T_xM, V) \ \mid \ x \in M \text{ and } u^*(\phi) = \sigma_x\}\,. \tag{3.1}$$

Every G_2-reduction of $\mathcal{F}$ (i.e., G_2-structure on M in the usual sense) is of the form F_σ for some unique $\sigma \in \Omega^3_+(M)$. For this reason, a 3-form $\sigma \in \Omega^3_+(M)$ will usually, by abuse of language, be called a G_2-structure in this article.

Remark 2 (Alternative terminologies). Some authors use 'almost G_2-structure' to refer to what I am calling a G_2-structure in this article. Apparently, this practice stems from an imagined analogy with the distinction between 'almost complex structure' and 'complex structure'.

However, for a subgroup $G \subset GL(n, \mathbb{R})$, the use of '$G$-structure' on an n-manifold M to mean a G-subbundle of the $GL(n, \mathbb{R})$-bundle of frames (or coframes) on M is well established. It seems unwise to tamper with this usage, especially since 'almost G-structure' suggests a structure that lacks some property of actual G-structures. Making an exception for the case $G = G_2$ merely invites confusion.

This use of 'G-structure' does not conflict with the 'almost complex structure' vs. 'complex structure' usage since a complex structure on a $2n$-manifold is not simply a $\mathrm{GL}(n,\mathbb{C})$-structure, but is (by the Newlander-Nirenberg theorem, equivalent to) a $\mathrm{GL}(n,\mathbb{C})$-structure with an assumed integrability property, whereas an 'almost complex structure' actually is (equivalent to) a $\mathrm{GL}(n,\mathbb{C})$-structure, not an 'almost $\mathrm{GL}(n,\mathbb{C})$-structure'.

Some authors speak of an 'integrable G_2-structure', meaning a G_2-structure $\sigma \in \Omega^3_+(M)$ satisfying some differential equations, such as $\mathrm{d}\sigma = 0$ (the exact differential equation intended varies with the author). Again, this usage appears to stem from an imagined analogy with a symplectic structure, which is defined by a nondegenerate 2-form ω that is closed, i.e., $\mathrm{d}\omega = 0$. In the symplectic case, Darboux' Theorem says that ω is, indeed, locally equivalent to the flat model, i.e., is 'integrable' in the standard terminology of the theory of Lie pseudo-groups. (In a similar way, one speaks of 'integrable almost complex structures'.) This usage of 'integrable' for G_2-structures also seems ill-advised to me since, as will be seen below, no first order condition on a G_2-structure implies that it is locally equivalent to the flat model (which is the only interpretation of 'integrable' in this context that would be consistent with the established usage in the theory of Lie pseudo-groups). Moreover, this encourages the confusing shift of terminology in which 'G_2-structure' is used to mean 'integrable G_2-structure' and 'almost G_2-structure' is used to mean an actual G_2-structure.

For this reason, none of the modifiers 'integrable', 'almost', 'nearly', or their ilk will be used in this article when referring to G_2-structures.

However, since it seems to be harmless, the terminology 'G_2-manifold' will sometimes be used to denote a manifold endowed with a G_2-structure that is flat to first order (i.e., 'torsion-free' in the usual terminology).

Definition 3 (Associated metric, orientation, and vector cross product). For any $\sigma \in \Omega^3_+(M)$, denote by g_σ, $*_\sigma$, and $\times_\sigma$ the metric, Hodge star operator, and vector cross product on M that are canonically associated to σ. When it is needed, the oriented orthonormal frame bundle of g_σ with this orientation will be denoted $\mathsf{F}_\sigma = F_\sigma \cdot \mathrm{SO}(7)$.

Remark 3 (Existence of G_2-structures). Because G_2 is both connected and simply connected, a connected 7-manifold M can support a G_2-structure only if it is both orientable and spinnable, i.e., if the first two Stiefel-Whitney classes of M vanish.

Conversely, by an observation due to Gray [9], these two necessary conditions are also sufficient:

Since G_2 is simply connected, it is the image under the standard double covering map $\rho : \mathrm{Spin}(7) \to \mathrm{SO}(7)$ of a unique subgroup of $\mathrm{Spin}(7)$, which, by abuse of language, will also be called G_2. Now, $\mathrm{Spin}(7)$ has a faithful representation on $\mathbb{R}^8$ and hence can be regarded as a subgroup of $\mathrm{SO}(8)$. The restriction of this representation to G_2 must also be faithful and hence, for dimension reasons, it must be isomorphic to $\mathsf{V}_{0,0} \oplus \mathsf{V}_{1,0}$. In particular, G_2 fixes a vector in $\mathbb{R}^8$ and acts transitively on the unit 6-sphere orthogonal to this vector. Consequently, $\mathrm{Spin}(7)$ must act transitively on the unit 7-sphere in $\mathbb{R}^8$ with stabilizer subgroup G_2.

Now, suppose M^7 to be orientable and spinnable. Choose a Riemannian metric g, an orientation, and a spin structure $\tilde{\mathsf{F}} \to M$, i.e., a spin double cover of the SO(7)-bundle $\mathsf{F} \to M$ consisting of oriented, g-orthonormal coframes on M. The associated spinor bundle $\mathbb{S} = \tilde{\mathsf{F}} \times_{\mathrm{Spin}(7)} \mathbb{R}^8$ is a vector bundle of rank 8 over the 7-manifold M and therefore has a nonvanishing unit section $s : M \to \mathbb{S}$. This allows one to reduce the structure group of $\tilde{\mathsf{F}}$ (and hence F) from Spin(7) to G_2 (since, by the previous paragraph, this is, up to conjugacy, the Spin(7)-stablizer of any nonzero vector in $\mathbb{R}^8$). Thus, M admits a G_2-structure whose associated metric and orientation are the chosen ones.

3.2. Type decomposition

Since G_2 acts reducibly on $\Lambda^p(V^*)$ for $2 \le p \le 5$, one can associate to any G_2-structure σ on M natural splittings of the p-form bundles $\Lambda^p(T^*M)$ into direct summands. These will be labeled as $\Lambda^p_d(T^*M, \sigma)$, or more simply, $\Lambda^p_d(T^*M)$ when the structure σ is clear from context. Denote the space of sections of $\Lambda^p_d(T^*M, \sigma)$ by $\Omega^p_d(M, \sigma)$.

Thus, for example, in view of (2.14), one has

$$\Omega^2_7(M,\sigma) = \big\{\, \beta \in \Omega^2(M) \mid \beta \wedge \sigma = 2{*_\sigma}\beta \,\big\} \tag{3.2}$$

$$\Omega^2_{14}(M,\sigma) = \big\{\, \beta \in \Omega^2(M) \mid \beta \wedge \sigma = -{*_\sigma}\beta \,\big\}\,. \tag{3.3}$$

Fortunately, the irreducible modules of dimensions 14 and 27 only occur in one dual pair of dimensions each. Meanwhile, the irreducible module of dimension 7 occurs in each degree $1 \le p \le 6$. From time to time, it is useful to be able to recognize the scale factors that can be introduced by the various different isomorphisms between these different modules. For example, for $\alpha \in \Omega^1_7(M)$ one has

$$\begin{aligned} {*_\sigma}\big({*_\sigma}(\alpha \wedge \sigma) \wedge \sigma\big) &= -4\,\alpha \\ {*_\sigma}\big({*_\sigma}(\alpha \wedge {*_\sigma}\sigma) \wedge {*_\sigma}\sigma\big) &= \quad 3\,\alpha, \end{aligned} \tag{3.4}$$

and these identities can sometimes be useful in simplifying various expressions. One should also keep in mind that, using the metric, each 1-form α has a corresponding dual vector field $\alpha^\sharp$ and there are useful identities of the form

$$\begin{aligned} {*_\sigma}(\alpha \wedge \sigma) &= -\alpha^\sharp \lrcorner {*_\sigma}\sigma \\ {*_\sigma}(\alpha \wedge {*_\sigma}\sigma) &= \quad \alpha^\sharp \lrcorner \sigma\,. \end{aligned} \tag{3.5}$$

Remark 4 (G_2-structures with the same associated metric and orientation). These type of decompositions have many uses. For example, they furnish a description of all of the G_2-structures that have the same associated metric and orientation as a given $\sigma \in \Omega^3_+(M)$:

Let a and α be a function and a 1-form, respectively, on M with $a^2 + |\alpha|^2_\sigma = 1$. Then the 3-form

$$\tilde{\sigma} = \big(a^2 - |\alpha|^2_\sigma\big)\,\sigma + 2a\,{*_\sigma}\big(\alpha \wedge \sigma\big) + \mathsf{i}\big(\alpha \circ \alpha\big) \tag{3.6}$$

is definite and has the same associated metric and orientation as σ. (This pointwise fact is most easily proved by checking it in the case $\sigma = \phi$ and $(a, \alpha) = (c, s\,e^1)$ where $c^2 + s^2 = 1$

and then using the fact that G_2 acts transitively on the unit 6-sphere in $\mathbb{R}^7$ to reduce to this case.)

Moreover, any definite 3-form on M that has g_σ and $*_\sigma$ as associated metric and orientation is of the form (3.6) for some pair (a, α) satisfying $a^2 + |\alpha|^2_\sigma = 1$, unique up to replacement by $(-a, -\alpha)$. (If $H^1(M, \mathbb{Z}_2) \neq 0$, the pair (a, α) might only be defined up to sign.)

Of course, some such formula was expected, since $\mathrm{SO}(7)/\mathrm{G}_2 \simeq \mathbb{RP}^7$ (a consequence of the result $\mathrm{Spin}(7)/\mathrm{G}_2 \simeq S^7$ discussed in Remark 3). What (3.6) displays is a concrete isomorphism between the bundle $\mathsf{F}_\sigma/\mathrm{G}_2$ and the $\mathbb{RP}^7$-bundle $\mathbb{P}\big(\mathbb{R} \oplus T^*M\big)$ over M.

3.3. Exterior derivative formulae

The decomposition of the p-forms on M allows one to express the exterior derivatives of both σ and $*_\sigma\sigma$ in fairly simple terms:

Proposition 1 (The torsion forms)**.** *For any* G_2*-structure* $\sigma \in \Omega^3_+(M)$*, there exist unique differential forms* $\tau_0 \in \Omega^0(M)$*,* $\tau_1 \in \Omega^1(M)$*,* $\tau_2 \in \Omega^2_{14}(M, \sigma)$*, and* $\tau_3 \in \Omega^3_{27}(M, \sigma)$ *so that the following equations hold:*

$$
\begin{aligned}
\mathrm{d}\,\sigma &= \tau_0 *_\sigma \sigma + 3\,\tau_1 \wedge \sigma + *_\sigma \tau_3\,,\\
\mathrm{d} *_\sigma \sigma &= \qquad\; 4\,\tau_1 \wedge *_\sigma \sigma + \tau_2 \wedge \sigma\,.
\end{aligned}
\tag{3.7}
$$

Proof. In view of the decomposition (2.14), the only part of this proposition that is not simply the definition of the τ_i is the occurrence of τ_1 in two places. In fact, by (2.14), there exist unique forms $\tau_0 \in \Omega^0(M)$, $\tau_1, \tilde\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M, \sigma)$, and $\tau_3 \in \Omega^3_{27}(M, \sigma)$ so that the above equation for $d\,\sigma$ holds while $\mathrm{d} *_\sigma \sigma = 4\,\tilde\tau_1 \wedge *_\sigma \sigma + \tau_2 \wedge \sigma$.

However, as is shown in [2] (see Remark 5 below for a sketch of the proof), there is an identity

$$
*_\sigma \sigma \wedge *_\sigma \big(\mathrm{d}(*_\sigma \sigma)\big) + (*_\sigma \mathrm{d}\sigma) \wedge \sigma = 0 \tag{3.8}
$$

valid for all $\sigma \in \Omega^3_+(M)$, and, in view of (3.4), this is equivalent to $\tilde\tau_1 = \tau_1$. $\square$

Definition 4 (The torsion forms). For a definite 3-form $\sigma \in \Omega^3_+(M)$, the quadruple of forms $(\tau_0, \tau_1, \tau_2, \tau_3)$ defined by (3.7) will be referred to as the *intrinsic torsion forms* of σ.

Remark 5 (General intrinsic torsion). The existence of the identity (3.8) may seem surprising at first, but the existence of such an identity can be understood by general considerations.

For any subgroup $G \subset \mathrm{SO}(n)$, the first order invariants (usually called the 'intrinsic torsion') of a G-structure F on an n-manifold M take values in a bundle over M associated to the natural G-representation on $(\mathfrak{so}(n)/\,\mathfrak{g}) \otimes \mathbb{R}^n$. (See §4.2 below for a further explication of this fact.) When the first order invariants of a given G-structure vanish, it is said to be '1-flat' or 'flat to first order'. For more discussion of this notion, see [2].

In the case of $\mathrm{G}_2 \subset \mathrm{SO}(7)$, this torsion representation space is

$$
(\mathfrak{so}(7)/\,\mathfrak{g}_2) \otimes \mathbb{R}^7 \simeq \mathsf{V}_{1,0} \otimes \mathsf{V}_{1,0} \simeq \mathsf{V}_{0,0} \oplus \mathsf{V}_{1,0} \oplus \mathsf{V}_{0,1} \oplus \mathsf{V}_{2,0}. \tag{3.9}
$$

and, as has already been remarked, these four summands are isomorphic, respectively, to $\Lambda^0(V^*)$, $\Lambda^1(V^*)$, $\Lambda^2_{14}(V^*)$, and $\Lambda^3_{27}(V^*)$. Since the exterior derivatives of the defining forms σ and $*_\sigma\sigma$ can be expressed linearly in terms of the first order invariants of F_σ and since there is only one $\Lambda^1(V^*)$ in the above representation list, it follows that the two 1-forms τ_1 and $\tilde{\tau}_1$ alluded to in the above proof must satisfy some universal linear relation.

Consideration of the fact that replacing σ by $\lambda^3\sigma$ for some positive function λ will replace $*_\sigma\sigma$ by $\lambda^4 *_\sigma\sigma$ shows that this relation must be the one given in Proposition 1.

Proposition 2 (1-flatness of G_2-structures)**.** *A G_2-structure $\sigma \in \Omega^3_+(M)$ is flat to first order if and only if its torsion forms all vanish, i.e., if and only if $d\sigma = d{*_\sigma}\sigma = 0$.*

Proof. A G_2-structure $\sigma \in \Omega^3_+(M)$ is flat to first order at $p \in M$ if there exists a p-centered coordinate chart $x : U \to \mathbb{R}^7$ such that the 3-form $\sigma - x^*(\phi)$ on U vanishes to order at least 2 at p.

Recall that the map $\mathsf{S} : \Lambda^3_+(W^*) \to \Lambda^4_+(W^*)$ defined in §2.8 is a smooth double covering. This implies that if $\sigma - x^*(\phi)$ vanishes to order 2 at p, then $*_\sigma\sigma - x^*(*_\phi\phi)$ vanishes to order 2 at p as well.

Since $d\phi = d{*_\phi}\phi = 0$, if σ is flat to first order at p, then $d\sigma$ and $d{*_\sigma}\sigma$ must vanish to at least first order at p. Thus, the claim in one direction is established.

To demonstrate the claim in the converse direction, it suffices to show that any definite 3-form σ defined on a neighborhood of $0 \in \mathbb{R}^7$ that satisfies $\sigma_0 = \phi$ and $d\sigma = d{*_\sigma}\sigma = 0$ is flat to first order at $0 \in \mathbb{R}^7$.

Now, if ψ is any 3-form on $\mathbb{R}^7$ that vanishes at the origin $0 \in \mathbb{R}^7$, then, because $\Lambda^3_+(V^*)$ is an open set in $\Lambda^3(V^*)$, the 3-form $\sigma = \phi + \psi$ is a definite 3-form on some open neighborhood of $0 \in \mathbb{R}^7$. Since $d\sigma = d\psi$, and since, for any 4-form $\Psi \in \Lambda^4(V^*)$, there exists a 3-form ψ on $\mathbb{R}^7$ that vanishes at $0 \in \mathbb{R}^7$ and that satisfies $(d\psi)_0 = \Psi$, it follows that the condition $d\sigma = 0$, i.e., $\tau_0 = \tau_1 = \tau_3 = 0$, imposes 35 independent linear conditions on the intrinsic torsion of σ. Since these conditions must define some G_2-invariant subspace of the torsion representation $\mathsf{V}_{0,0}\oplus\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}\oplus\mathsf{V}_{2,0}$, it follows by dimension count that it is the subspace $\mathsf{V}_{0,0}\oplus\mathsf{V}_{1,0}\oplus\mathsf{V}_{2,0}$.

Similarly, since $\Lambda^4_+(W^*)$ is an open subset of $\Lambda^4(W^*)$ and since $\mathsf{S} : \Lambda^3_+(W^*) \to \Lambda^4_+(W^*)$ is a smooth double covering, it follows that if ψ is any smooth 4-form vanishing at the origin $0 \in \mathbb{R}^7$, then there is an open neighborhood U of $0 \in \mathbb{R}^7$ on which there exists a definite form σ such that $\sigma_0 = \phi$ and $*_\sigma\sigma = *_\phi\phi + \psi$. Moreover, if Ψ is any 5-form in $\Lambda^5(V^*)$, then there exists a smooth 4-form ψ vanishing at $0 \in \mathbb{R}^7$ such that $(d\psi)_0 = \Psi$. The corresponding definite 3-form σ will then satisfy $d{*_\sigma}\sigma = d\psi$, so that $(d{*_\sigma}\sigma)_0 = \Psi$. It follows that the condition $d{*_\sigma}\sigma = 0$, i.e., $\tau_1 = \tau_2 = 0$, must be 21 independent linear equations on the intrinsic torsion of σ. Since these conditions must define some G_2-invariant subspace of the torsion representation $\mathsf{V}_{0,0}\oplus\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}\oplus\mathsf{V}_{2,0}$, it follows by dimension count that it is the subspace $\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}$.

Thus, the conditions $d\sigma = 0$ and $d{*_\sigma}\sigma = 0$ together imply that all of the intrinsic torsion of σ vanishes, i.e., that σ is flat to first order at each point. $\square$

Remark 6 (Fernández and Gray's theorem on vector cross products). Proposition 2 implies the 1982 result of Fernández and Gray [6] that a vector cross product $\times : TM \times TM \to TM$ that is compatible with a Riemannian metric g on M is g-parallel if and only if the corresponding 3-form is closed and coclosed (with respect to g).

The essential difference between Proposition 2 and their result is that they assume a specific metric g and vector cross product to be given, whereas Proposition 2 starts with a definite 3-form σ and constructs a specific metric associated to σ.

4. Frame Bundle Calculations

4.1. The associated Levi-Civita connection

Let $\sigma \in \Omega^3_+(M)$ be a G_2-structure with associated G_2-bundle $F_\sigma \subset \mathcal{F}$. This bundle can be canonically enlarged to an oriented orthonormal frame bundle $\mathsf{F}_\sigma = F_\sigma \cdot SO(7) \subset \mathcal{F}$ and this larger bundle will be referred to as the associated metric frame bundle of σ.

Now $\pi\colon \mathsf{F}_\sigma \to M$ has a tautological V-valued 1-form ω defined by requiring that $\omega(v) = u(\pi_*(v))$ for all $v \in T_u\mathsf{F}$. It may help the reader to think of ω as expanded in the basis e_i in the form $\omega = \omega_1\, e_1 + \cdots + \omega_7\, e_7$ and then think of ω as a column of height 7, i.e., $\omega = (\omega_i)$.

The Levi-Civita connection is then represented on F_σ as a 1-form ψ on F_σ taking values in $\mathfrak{so}(7)$, i.e., the 7-by-7 skew-symmetric matrices. As such, $\psi = (\psi_{ij})$ where $\psi_{ij} = -\psi_{ji}$.

The defining property of ψ is that it satisfies the *first structure equation* of Cartan:

$$\mathrm{d}\,\omega = -\psi \wedge \omega. \tag{4.1}$$

In indices (i.e., components) this matrix equation becomes the system of equations $\mathrm{d}\omega_i = -\psi_{ij}\wedge\omega_j$.

The curvature of this connection is represented by the 2-form $\Psi = \mathrm{d}\psi + \psi\wedge\psi$. It satisfies the *first Bianchi identity*

$$\Psi \wedge \omega = 0 \tag{4.2}$$

and has the indicial expression

$$\Psi_{ij} = \mathrm{d}\psi_{ij} + \psi_{ik} \wedge \psi_{kj} = \tfrac{1}{2}\mathsf{R}_{ijkl}\, \omega_k \wedge \omega_l\,. \tag{4.3}$$

4.1.1. The natural connection and intrinsic torsion on F_σ

To save writing, I will denote the pullbacks of ω and ψ to F_σ by the same letters, trusting the reader to keep in mind where various equations are taking place.

The pullback of ψ to F_σ will not generally have values in $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. However, keeping in mind the canonical decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus [V]$, there is a unique decomposition of the form

$$\psi = \theta + 2[\tau] \tag{4.4}$$

where θ takes values in $\mathfrak{g}_2$ and τ takes values in V. (The coefficient 2 simplifies subsequent formulas.)

Then θ is a connection 1-form on F_σ and defines what will be referred to as the *natural* connection associated to the G_2-structure σ. This connection will not be torsion-free (and hence is not the Levi-Civita connection) unless τ vanishes identically.

4.2. General G-structure torsion

This construction of a natural connection for a G_2-structure σ is an instance of a general construction valid for any $G \subset \mathrm{O}(n)$.

Letting $\mathfrak{g} \subset \mathfrak{so}(n)$ denote the Lie algebra of G, there is a unique G-equivariant splitting $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ obtained by using the standard $\mathrm{O}(n)$-invariant inner product on $\mathfrak{so}(n)$.

For any G-structure $\pi : F \to M$, one has the associated orthonormal frame bundle $\mathsf{F} = F \cdot \mathrm{O}(n)$. One can then pull back the Levi-Civita connection ψ on F to F and decompose it uniquely in the form $\psi = \theta + \tau$ where θ takes values in $\mathfrak{g}$ and τ takes values in $\mathfrak{g}^\perp \simeq \mathfrak{so}(n)/\,\mathfrak{g}$. The 1-form θ defines a natural connection on F (one that is the pullback to F of a metric-compatible connection, generally with torsion, on F). The 1-form τ represents a section T of the associated *torsion bundle* $F \times_\rho (\mathfrak{g}^\perp \otimes \mathbb{R}^n)$, where $\rho : G \to \mathrm{End}\big(\mathfrak{g}^\perp \otimes \mathbb{R}^n\big)$ is the tensor product of the two obvious representations.

It is a general result (essentially due to É. Cartan) that all of the pointwise first-order diffeomorphism invariants of a G-structure $F \subset \mathcal{F}$ that are polynomial in the derivatives of the corresponding defining section σ of the bundle $\mathcal{F}/G$ are expressible as polynomials in the section T.

Moreover, for $k \geq 2$, all of the pointwise k-th order diffeomorphism invariants of a G-structure $F \subset \mathcal{F}$ that are polynomial in the first k derivatives of the corresponding defining section σ of the bundle $\mathcal{F}/G$ are expressible as polynomials in the section T, its first $k{-}1$ covariant derivatives with respect to the connection θ, the curvature of θ, and its first $k{-}2$ covariant derivatives (with respect to θ).

Consequently, for each $k \geq 1$, the polynomial pointwise invariants of order k are polynomials in a canonically defined section of a vector bundle of the form

$$F \times_{\rho_1 \times \cdots \times \rho_k} \big(V_1(\mathfrak{g}) \oplus \cdots \oplus V_k(\mathfrak{g})\big)$$

where $V_k(\mathfrak{g})$ is the unique G-representation that satisfies

$$\big(\mathfrak{gl}(n,\mathbb{R})/\,\mathfrak{g}\big) \otimes \mathsf{S}^k(\mathbb{R}^n) = V_k(\mathfrak{g}) \oplus \big(\mathbb{R}^n \otimes \mathsf{S}^{k+1}(\mathbb{R}^n)\big). \tag{4.5}$$

In the familiar case in which $\mathfrak{g} = \mathfrak{so}(n)$, the first torsion space $V_1\big(\mathfrak{so}(n)\big)$ vanishes (this is simply the fundamental lemma of Riemannian geometry) and one has the result (due to Cartan and Weyl) that all of the pointwise invariants of a metric can be expressed in terms of the Riemann curvature tensor and its covariant derivatives with respect to the Levi-Civita connection.

Remark 7 (Canonical connections). The use of the term 'natural' with regard to the connection θ on the G-structure F should not be construed to mean that this is the only 'canonical' connection on M that is compatible with F. In many cases, this is only one of a family of possible 'canonical' connections that can be defined in terms of the first-order invariants of the G-structure F and that are preserved under equivalence of G-structures.

For example, if the G-modules $V_1(\mathfrak{g})$ and $\mathfrak{g}\otimes\mathbb{R}^n$ have common constituents, so that the space $\mathrm{Hom}^G\big(V_1(\mathfrak{g}),\mathfrak{g}\otimes\mathbb{R}^n\big)$ of G-equivariant homomorphisms between the two spaces has dimension $r>0$, there will be an r-parameter family of ways of modifying θ, by adding a $\mathfrak{g}$-valued 1-form whose coefficients are linear in the torsion functions, in such a way that the resulting modification defines a connection on M compatible with the G-structure F. Each element in this r-parameter family of connections can be regarded as canonical in the sense that equivalence of G-structures will induce isomorphisms between the corresponding connections in the r-parameter family.

Of course, there is no *a priori* reason to consider only connection modifications that are linear in the torsion functions; for example, any G-equivariant polynomial mapping $V_1(\mathfrak{g})\to\mathfrak{g}\otimes\mathbb{R}^n$ could be used to define such a modification of θ. However, these 'higher' modifications do not often arise in practice.

Depending on the intended use, it could well be that one of these other connections (rather than the one being called 'natural' in the present article) is better suited for expressing identities of one kind or another.

4.3. G_2-specific calculations

In the specific case of $\mathrm{G}_2\subset\mathrm{SO}(7)$, one finds, as has already been remarked,

$$V_1(\mathfrak{g}_2)\simeq\mathsf{V}_{0,0}\oplus\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}\oplus\mathsf{V}_{2,0}, \tag{4.6}$$

while $V_2(\mathfrak{g}_2)$, which has dimension 392, has the decomposition

$$V_2(\mathfrak{g}_2)\simeq\mathsf{V}_{0,0}\oplus2\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}\oplus3\mathsf{V}_{2,0}\oplus2\mathsf{V}_{1,1}\oplus\mathsf{V}_{0,2}\oplus\mathsf{V}_{3,0}\,. \tag{4.7}$$

Naturally, this latter space has $V_2\big(\mathfrak{so}(7)\big)$, i.e., the curvature tensors of metrics in dimension 7, as a quotient. For comparison, note that, as G_2-modules:

$$V_2\big(\mathfrak{so}(7)\big)\simeq\mathsf{V}_{0,0}\oplus2\mathsf{V}_{2,0}\oplus\mathsf{V}_{1,1}\oplus\mathsf{V}_{0,2}\,. \tag{4.8}$$

The Ricci tensor takes values in a subspace isomorphic to $\mathsf{V}_{0,0}\oplus\mathsf{V}_{2,0}$ while the remainder represents the Weyl tensor.

Remark 8 (Canonical G_2-connections). Since $\mathfrak{g}_2\otimes\mathsf{V}_{1,0}=\mathsf{V}_{1,0}\oplus\mathsf{V}_{2,0}\oplus\mathsf{V}_{1,1}$ shares two G_2-irreducible modules with $V_1(\mathfrak{g})$, it follows from Remark 7 that there is actually a 2-parameter family of canonical connections associated to any G_2-structure σ. Each element in this family is compatible with σ (in the sense that σ is parallel under the corresponding parallel translation). Since the common constituents $\mathsf{V}_{1,0}$ and $\mathsf{V}_{2,0}$ correspond to the torsion forms τ_1 and τ_3, respectively, it follows that the entire two-parameter family of canonical connections collapses to a single connection if and only if the G_2-structure σ satisfies $\tau_1=\tau_3=0$. In this case, differentiating the equations (3.7) shows that $\tau_0\tau_2=0$ and $\mathrm{d}\tau_0=0$. In particular, when M is connected, it follows that τ_0 is constant. If $\tau_0=0$, then the G_2-structure is closed. If $\tau_0\neq0$, then $\tau_2=0$ and one has the equation $\mathrm{d}\sigma=\tau_0\,{*_\sigma}\sigma$, which is the defining equation for the so-called 'nearly G_2-manifolds'.

Thus, the family of canonical G_2-connections associated to a G_2-structure σ collapses to a single G_2-connection if and only if either σ is closed or it defines a nearly G_2-manifold.

4.4. The second structure equations

It is helpful to make the following observation: The identities (2.10) imply that the 2-form $2[\tau]\wedge[\tau] + [\,[\tau]\wedge\tau\,]$ takes values in $\mathfrak{g}_2$. This motivates the definitions

$$D\tau = \mathrm{d}\tau + \theta\wedge\tau - [\tau]\wedge\tau \tag{4.9}$$

$$D\theta = \mathrm{d}\theta + \theta\wedge\theta + 4[\tau]\wedge[\tau] + 2[\,[\tau]\wedge\tau\,], \tag{4.10}$$

for, with these definitions, $D\theta$ takes values in $\mathfrak{g}_2$. Moreover

$$\Psi = d(\theta + 2[\tau]) + (\theta + 2[\tau])\wedge(\theta + 2[\tau]) = D\theta + 2[D\tau] \tag{4.11}$$

so that the first Bianchi identity takes the form

$$(D\theta + 2[D\tau])\wedge\omega = 0. \tag{4.12}$$

Remark 9 (Covariant differentials). The decisive advantage of using the forms $D\tau$ and $D\theta$ to express the curvature tensor is that these forms do *not* contain all of the information about the second order invariants of the underlying G_2-structure σ although they do contain enough information to recover the Riemann curvature tensor of the underlying metric.

4.5. Indicial calculations

The indicial expression of (4.1) in terms of (4.4) is

$$\mathrm{d}\omega_i = -\theta_{ij}\wedge\omega_j - 2\varepsilon_{ijk}\tau_k\wedge\omega_j\,. \tag{4.13}$$

Denote $\pi^*(\sigma)$ by σ and, with a slight abuse of notation, denote $\pi^*(*_\sigma\sigma)$ by $\star\sigma$. Then

$$\sigma = \tfrac{1}{6}\varepsilon_{ijk}\,\omega_i\wedge\omega_j\wedge\omega_k \tag{4.14}$$

$$\star\sigma = \tfrac{1}{24}\varepsilon_{ijkl}\,\omega_i\wedge\omega_j\wedge\omega_k\wedge\omega_l\,. \tag{4.15}$$

These give rise, via (4.13) and the ε-identities, to the formulae

$$\begin{aligned}\mathrm{d}\,\sigma &= \varepsilon_{ijkl}\,\tau_i\wedge\omega_j\wedge\omega_k\wedge\omega_l\\ \mathrm{d}\star\sigma &= -(\tau_p\wedge\omega_p)\ \wedge\ (\varepsilon_{ijk}\,\omega_i\wedge\omega_j\wedge\omega_k)\\ &= -6\,(\tau_p\wedge\omega_p)\wedge\sigma\end{aligned} \tag{4.16}$$

4.5.1. Torsion decomposition.

There are unique functions T_{ij} on F_σ so that

$$\tau_i = \mathsf{T}_{ij}\,\omega_j\,. \tag{4.17}$$

These functions can be used to express the intrinsic torsion forms in indicial form:

$$\begin{aligned}\pi^*(\tau_0) &= \tfrac{24}{7}\,\mathsf{T}_{ii}\,,\\ \pi^*(\tau_1) &= \varepsilon_{ijk}\mathsf{T}_{ij}\ \omega_k\,,\\ \pi^*(\tau_2) &= 4\,\mathsf{T}_{ij}\,\omega_i\wedge\omega_j - \varepsilon_{ijkl}\mathsf{T}_{ij}\,\omega_k\wedge\omega_l\,,\\ \pi^*(\tau_3) &= -\tfrac{3}{2}\varepsilon_{ikl}\,(\mathsf{T}_{ij} + \mathsf{T}_{ji})\,\omega_j\wedge\omega_k\wedge\omega_l + \tfrac{18}{7}\,\mathsf{T}_{ii}\,\sigma\,.\end{aligned} \tag{4.18}$$

(In these formulae, one sums over repeated indices in any term.)

4.5.2. Curvature identities

The covariant differentials can be expressed in indices as

$$D\tau = (D\tau_i) = \left(\tfrac{1}{2}\mathsf{T}_{ijk}\,\omega_j\wedge\omega_k\right) \tag{4.19}$$
$$D\theta = (D\theta_{ij}) = \left(\tfrac{1}{2}\mathsf{S}_{ijkl}\,\omega_j\wedge\omega_k\right) \tag{4.20}$$

where each of T and S are skew-symmetric in their last two indices, S is skew-symmetric in its first two indices and, since $D\theta$ takes values in $\mathfrak{g}_2$, the functions S also satisfy

$$\varepsilon_{ijm}\mathsf{S}_{ijkl} = 0$$

for all m, k, and l.

Since $\Psi = D\theta + 2\big[D\tau\big]$, the Riemann curvature functions are expressed as

$$\mathsf{R}_{ijkl} = \mathsf{S}_{ijkl} + 2\,\varepsilon_{ijp}\mathsf{T}_{pkl}\,, \tag{4.21}$$

so that the first Bianchi identity becomes

$$\mathsf{S}_{ijkl} + \mathsf{S}_{iljk} + \mathsf{S}_{iklj} + 2\,\varepsilon_{ijp}\mathsf{T}_{pkl} + 2\,\varepsilon_{ilp}\mathsf{T}_{pjk} + 2\,\varepsilon_{ikp}\mathsf{T}_{plj} = 0. \tag{4.22}$$

The identities (4.22) impose 28 linear conditions on the T_{ijk} alone. Perhaps the easiest way to derive these 28 conditions is to expand the identities

$$\mathrm{d}(\mathrm{d}(\sigma)) = \mathrm{d}(\mathrm{d}(\star\sigma)) = 0 \tag{4.23}$$

and use the structure equations (4.13) together with the definitions (4.9), (4.19), and (4.20). This will be left as an exercise for the reader. The result is that the conditions (4.23) are equivalent to the following equations (some of which are redundant):

$$\begin{aligned} 0 &= \mathsf{T}_{iij}\,,\\ 0 &= \varepsilon_{ipq}\mathsf{T}_{jpq} - \varepsilon_{jpq}\mathsf{T}_{ipq}\,,\\ 0 &= \varepsilon_{ipq}\mathsf{T}_{pqj} - \varepsilon_{jpq}\mathsf{T}_{pqi}\,. \end{aligned} \tag{4.24}$$

This implies that the function (T_{ijk}), which nominally takes values in a G_2-module of the form

$$\begin{aligned} \mathsf{V}_{1,0}\otimes\Lambda^2\big(\mathsf{V}_{1,0}\big) &= \mathsf{V}_{1,0}\otimes\big(\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}\big)\\ &= \mathsf{V}_{0,0}\oplus 2\mathsf{V}_{1,0}\oplus\mathsf{V}_{0,1}\oplus 2\mathsf{V}_{2,0}\oplus\mathsf{V}_{1,1}\,, \end{aligned} \tag{4.25}$$

actually takes values in a submodule of the form

$$\mathsf{V}_{0,0}\oplus 2\mathsf{V}_{2,0}\oplus\mathsf{V}_{1,1}\,. \tag{4.26}$$

4.5.3. The Ricci identity

It was Bonan [1] who first observed that the Bianchi identities imply that a G_2-structure with vanishing torsion must necessarily have vanishing Ricci tensor. On general abstract grounds, it then follows that the Bianchi identities (4.22) must allow one to express the Ricci curvature in terms of the T_{ijk}. Indeed, by combining the first Bianchi identities via the ε-identities (another exercise for the reader), one derives the following expression for the Ricci curvature components $\mathsf{R}_{ij} = \mathsf{R}_{kikj}$:

$$\mathsf{R}_{ij} = 6\,\varepsilon_{pqi}\mathsf{T}_{pqj}\,. \tag{4.27}$$

This allows one to express the Ricci curvature directly in terms of the four torsion forms and their exterior derivatives. The resulting formula for the scalar curvature of the underlying metric g_σ is

$$\mathrm{Scal}(g_\sigma) = 12\,\delta\tau_1 + \tfrac{21}{8}\,{\tau_0}^2 + 30\,|\tau_1|^2 - \tfrac{1}{2}\,|\tau_2|^2 - \tfrac{1}{2}\,|\tau_3|^2. \tag{4.28}$$

The full Ricci tensor is somewhat more complicated, but can be expressed as follows:

First, define a G_2-invariant quadratic pairing $\mathsf{Q} : \Lambda^3(T^*) \times \Lambda^3(T^*) \to \Lambda^3(T^*)$ by the following recipe: Choose a local basis $e_1, \ldots, e_7$ of orthonormal vector fields such that $\sigma(e_i, e_j, e_k) = \varepsilon_{ijk}$ (such a basis is often called a G_2-frame field). Then, for $\alpha, \beta \in \Omega^3(M)$ set

$$\mathsf{Q}(\alpha,\beta) = *_\sigma\left[\varepsilon_{ijkl}\,\big((e_i\wedge e_j) \lrcorner *_\sigma\alpha\big)\wedge\big((e_k\wedge e_l)\lrcorner *_\sigma\beta\big)\right]. \tag{4.29}$$

The resulting mapping Q does not depend on the choice of local G_2-frame field. With this definition (and keeping in mind the definition (2.18) of j) one finds

$$\begin{aligned}
\mathrm{Ric}(g_\sigma) = &-\Big(\tfrac{3}{2}\,\delta\tau_1 - \tfrac{3}{8}\,{\tau_0}^2 + 15\,|\tau_1|^2 - \tfrac{1}{4}\,|\tau_2|^2 + \tfrac{1}{2}\,|\tau_3|^2\Big)\,g_\sigma \\
&+\mathsf{j}\Big(-\tfrac{5}{4}\,\mathrm{d}\big(*_\sigma(\tau_1\wedge *_\sigma\sigma)\big) - \tfrac{1}{4}\,\mathrm{d}\tau_2 + \tfrac{1}{4}\,*_\sigma\mathrm{d}\tau_3 \\
&\qquad + \tfrac{5}{2}\,\tau_1\wedge *_\sigma(\tau_1\wedge *_\sigma\sigma) - \tfrac{1}{8}\,\tau_0\tau_3 + \tfrac{1}{4}\,\tau_1\wedge\tau_2 \\
&\qquad + \tfrac{3}{4}\,*_\sigma(\tau_1\wedge\tau_3) + \tfrac{1}{8}\,*_\sigma(\tau_2\wedge\tau_2) + \tfrac{1}{64}\,\mathsf{Q}(\tau_3,\tau_3)\Big).
\end{aligned} \tag{4.30}$$

While a formula in this generality is not of much practical use, when one goes to investigate special classes of G_2-structures, this formula can simplify considerably, as will be seen.

Formulae essentially equivalent to a special case of the formulae (4.28) and (4.30) were found in [7, 8], where those authors considered what they called 'integrable G_2-structures', which, in the terminology of this article, means G_2-structures σ satisfying $\tau_2 = 0$.

Remark 10 (General identities). It is perhaps worth remarking on why the identities (4.28) and (4.30) could be expected to have the form that they do.

In the first place, one knows that the scalar curvature must be expressible in a G_2-invariant manner as a sum of a linear expression in the second order invariants, i.e., a section of a vector bundle modeled on $V_2(\mathfrak{g}_2)$, and an expression in the first order invariants, i.e., the torsion forms, that is at most quadratic. A glance at (4.7) shows that there is only one trivial summand in the representation $V_2(\mathfrak{g}_2)$ and hence there is essentially only one possible second order term up to a universal constant multiple. Since $\delta\tau_1$ is a scalar second order invariant, it must represent this copy of $\mathsf{V}_{0,0}$ in $V_2(\mathfrak{g}_2)$. As for the first order terms, since $V_1(\mathfrak{g}_2)$ consists of four mutually inequivalent G_2-modules, the space of G_2-invariant quadratic forms on this space has dimension 4 and must be represented by the square norms of the four torsion forms. Thus, a formula of the form (4.28) was inevitable; it was just a matter of determining the numerical coefficients, which was done with the aid of MAPLE.

The argument for the form of (4.30) is quite similar. Since the scalar curvature has already been determined, it is a question of writing down a formula for the trace-free part of the Ricci tensor, i.e., finding linear terms in $V_2(\mathfrak{g}_2)$ and quadratic terms in $V_1(\mathfrak{g}_2)$ that take values in the G_2-module $\mathsf{V}_{2,0}$. Again, a glance at (4.7) shows that there are at most three possible second order terms and it is not difficult to see that the three second order terms that take values in $\mathsf{V}_{2,0}$ found by taking derivatives of τ_1, τ_2, and τ_3 and projecting into a suitable $\mathsf{V}_{2,0}$ representation are, in fact, independent and generate the three copies of $\mathsf{V}_{2,0}$ that appear in $V_2(\mathfrak{g}_2)$. On the other hand, using representation theory to compute the second symmetric power of $V_1(\mathfrak{g}_2)$ shows that there exist eight copies of $\mathsf{V}_{2,0}$ in this symmetric power. Of those eight copies, five are computable via wedge product and appear in the formula for Ricci. Of the remaining three, one bilinear in τ_2 and τ_3 and the other two quadratic in τ_3, only one of the terms quadratic in τ_3 actually makes an appearance. The rest is just a matter of determining constants.

4.6. Closed G_2-structures

Now, consider the case of a *closed* $\sigma \in \Omega^3_+(M)$, i.e., $\mathrm{d}\sigma = 0$. In this case, by Proposition 1, it follows that

$$\mathrm{d}{*_\sigma}\sigma = \tau_2 \wedge \sigma, \tag{4.31}$$

where τ_2 lies in $\Omega^2_{14}(M,\sigma)$. In particular,

$$\tau_2 \wedge {*_\sigma}\sigma = 0. \tag{4.32}$$

Taking the exterior derivative of (4.31) yields

$$0 = \mathrm{d}\tau_2 \wedge \sigma\,, \tag{4.33}$$

implying that $\mathrm{d}\tau_2$ has no component in $\Omega^3_7(M,\sigma)$. Differentiating (4.32) yields

$$\begin{aligned} 0 = \mathrm{d}\big(\tau_2 \wedge {*_\sigma}\sigma\big) &= \mathrm{d}\tau_2 \wedge {*_\sigma}\sigma + \tau_2 \wedge \mathrm{d}{*_\sigma}\sigma \\ &= \mathrm{d}\tau_2 \wedge {*_\sigma}\sigma + \tau_2 \wedge \tau_2 \wedge \sigma = \mathrm{d}\tau_2 \wedge {*_\sigma}\sigma - |\tau_2|^2 *_\sigma 1\,. \end{aligned} \tag{4.34}$$

Thus, from (4.33) and (4.34) it follows that there exists a $\gamma \in \Omega^3_{27}(M,\sigma)$ so that

$$\mathrm{d}\tau_2 = \tfrac{1}{7}|\tau_2|^2\,\sigma + \gamma. \tag{4.35}$$

In summary, formulae (4.28) and (4.30) can be simplified, in this case, to

$$\mathrm{Scal}(g_\sigma) = -\tfrac{1}{2}\,|\tau_2|^2. \tag{4.36}$$

and

$$\mathrm{Ric}(g_\sigma) = \tfrac{1}{4}\,|\tau_2|^2\,g_\sigma - \tfrac{1}{4}\,\mathsf{j}\left(\mathrm{d}\tau_2 - \tfrac{1}{2} *_\sigma(\tau_2 \wedge \tau_2)\right). \tag{4.37}$$

Remark 11 (Differential invariants of closed G_2-structures). Just as one can compute the dimension of the space of k-jets of G-structures as in §4.2, one can compute the dimension of the space of k-jets of G-structures satisfying some set of differential equations. In the case of closed G_2-structures, denote the module of k-th order differential invariants by $V_k'(\mathfrak{g}_2) \subset V_k(\mathfrak{g}_2)$. One finds, for example, that

$$V_1'(\mathfrak{g}_2) \simeq \mathsf{V}_{0,1} \qquad\qquad V_2'(\mathfrak{g}_2) \simeq \mathsf{V}_{2,0} \oplus \mathsf{V}_{1,1} \oplus \mathsf{V}_{0,2}. \tag{4.38}$$

This implies, on abstract grounds, that the scalar curvature of the underlying metric of a closed G_2-structure must be expressed in terms of the first order invariants (since there is no $\mathsf{V}_{0,0}$ component in $V_2'(\mathfrak{g}_2)$) and that the full Ricci tensor can be expressed in terms of τ_2 and $\mathrm{d}\tau_2$. Thus, the form of (4.36) and (4.37) could have been anticipated, if not the numerical coefficients.

Of course, it is easy to 'write down' the general closed G_2-structure locally: If $\beta \in \Omega^2(\mathbb{R}^7)$ is a (smooth) 2-form that vanishes to second order at $0 \in \mathbb{R}^7$, then the 3-form $\sigma = \phi + \mathrm{d}\beta$ will equal ϕ at 0 and hence will be a closed, definite 3-form on some open neighborhood of $0 \in \mathbb{R}^7$. Conversely, if σ is a closed G_2-structure on a manifold M^7, then any point $p \in M$ has an open neighborhood U on which there exists a p-centered coordinate chart $x : U \to \mathbb{R}^7$ such that $\sigma_U = x^*(\phi + \mathrm{d}\beta)$ where $\beta \in \Omega^2(\mathbb{R}^7)$ is a 2-form that vanishes to second order at $0 \in \mathbb{R}^7$.

In a sense that it is possible to make precise using Cartan's notion of the generality of the space of solutions of a system of PDE, one can develop this discussion further to show that the general closed G_2-structure modulo diffeomorphism depends on 8 functions of seven variables.

An immediate consequence of (4.36) is the following:

Corollary 1. *For any closed* G_2*-structure* $\sigma \in \Omega^3_+(M)$*, the scalar curvature of the underlying metric is non-positive and vanishes identically if and only if the entire Ricci tensor of the underlying metric vanishes. Equivalently, the scalar curvature vanishes identically if and only if* σ *satisfies* $\mathrm{d}\sigma = \mathrm{d}{*_\sigma}\sigma = 0$*.*

Using the formulae for i and j, the formula (4.37) can be rewritten as

$$\mathrm{d}\tau_2 = \tfrac{3}{14}\,|\tau_2|^2\,\sigma + \tfrac{1}{2}{*_\sigma}(\tau_2 \wedge \tau_2) - \tfrac{1}{2}\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big), \tag{4.39}$$

where $\mathrm{Ric}^0(g_\sigma)$ is the traceless Ricci tensor of g_σ.

Corollary 2. *A closed* G_2*-structure* $\sigma \in \Omega^3_+(M)$ *has an Einstein underlying metric if and only if it satisfies* $\mathrm{d}{*_\sigma}\sigma = \tau_2{\wedge}\sigma$ *where* $\mathrm{d}\tau_2 = \frac{3}{14}\,|\tau_2|^2\,\sigma + \frac{1}{2}{*_\sigma}(\tau_2{\wedge}\tau_2)$*.*

Remark 12 (Nonexistence of compact Einstein examples). I do not know whether there exist any closed G_2-structures that are Einstein but not Ricci-flat, even local (i.e., incomplete) ones.

After Version 1.0 of the present article was posted to the arXiv, Cleyton and Ivanov [3] gave an argument (based on a comparison of the Ricci curvatures of the Levi-Civita connection and the canonical connection of the underlying G_2-structure) showing that no *compact* 7-manifold can support a closed G_2-structure σ whose underlying metric g_σ is Einstein unless σ is also coclosed, i.e., $\mathrm{d}{*_\sigma}\sigma = 0$. Their argument is rather involved, but Corollary 2 yields a simple proof:

Suppose that $\sigma \in \Omega^3_+(M)$ is a closed G_2-structure whose underlying metric g_σ is Einstein. Then, by Corollary 2, it follows that $\mathrm{d}{*_\sigma}\sigma = \tau_2{\wedge}\sigma$ where $\mathrm{d}\tau_2 = \frac{3}{14}\,|\tau_2|^2\,\sigma + \frac12{*_\sigma}(\tau_2{\wedge}\tau_2)$. Now, using this formula together with the formula (2.21), one finds

$$\begin{aligned} \mathrm{d}\big(\tfrac13{\tau_2}^3\big) &= {\tau_2}^2\wedge\mathrm{d}\tau_2 = {\tau_2}^2\wedge\big(\tfrac{3}{14}\,|\tau_2|^2\,\sigma + \tfrac12{*_\sigma}(\tau_2\wedge\tau_2)\big)\\ &= -\tfrac{3}{14}\,|\tau_2|^4{*_\sigma}1 + \tfrac12|\tau_2\wedge\tau_2|^2\,{*_\sigma}1 = \tfrac27|\tau_2|^4{*_\sigma}1. \end{aligned} \tag{4.40}$$

Now, suppose that M were compact. Integrating both ends of (4.40) over M and applying Stokes' theorem yields

$$0 = \int_M \mathrm{d}\big(\tfrac13{\tau_2}^3\big) = \int_M \tfrac27|\tau_2|^4{*_\sigma}1\,, \tag{4.41}$$

implying that τ_2 must vanish identically, as was to be shown.

In view of (4.39), this nonexistence can be seen as a special case of a general result about pinching of Ricci curvature:

Corollary 3. *Suppose that $\sigma \in \Omega^3_+(M)$ is a closed G_2-structure on a compact 7-manifold M that satisfies the pinching condition*

$$\big|\mathrm{Ric}^0(g_\sigma)\big|^2 \le \tfrac{4}{21}\,C\,\mathrm{Scal}(g_\sigma)^2. \tag{4.42}$$

for some constant $C \le 1$. If $C < 1$, then σ is also coclosed. If $C = 1$, then equality must hold in (4.42) *everywhere on M. Moreover, in this case, the identity*

$$\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big) = \tfrac23\big({*_\sigma}(\tau_2\wedge\tau_2) + \tfrac17|\tau_2|^2\,\sigma\big) \tag{4.43}$$

or, equivalently,

$$\mathrm{d}\tau_2 = \tfrac16\,\big(|\tau_2|^2\,\sigma + {*_\sigma}(\tau_2\wedge\tau_2)\big) \tag{4.44}$$

must hold everywhere on M.

Proof. Using (4.39), one obtains, after using (2.21), the orthogonality of $\Omega^3_1(M,\sigma)$ and $\Omega^3_{27}(M,\sigma)$, the identity (2.30), and the Cauchy-Schwartz inequality,

$$\begin{aligned} \mathrm{d}\big(\tfrac13{\tau_2}^3\big) &= {\tau_2}^2\wedge\mathrm{d}\tau_2 = {\tau_2}^2\wedge\big(\tfrac{3}{14}\,|\tau_2|^2\,\sigma + \tfrac12{*_\sigma}(\tau_2\wedge\tau_2) - \tfrac12\,\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big)\big)\\ &= \tfrac27|\tau_2|^4{*_\sigma}1 - \tfrac12\,{\tau_2}^2\wedge\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big)\\ &= \tfrac27|\tau_2|^4{*_\sigma}1 - \tfrac12\,\big({\tau_2}^2 + \tfrac17\,|\tau_2|^2{*_\sigma}\sigma\big)\wedge\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big)\\ &\ge \left(\tfrac27|\tau_2|^4 - \tfrac12\sqrt{\tfrac67}\,|\tau_2|^2\,\big|\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big|\right)\,{*_\sigma}1. \end{aligned} \tag{4.45}$$

Now, the expression at the end of (4.45) will be a nonnegative multiple of the volume form ${*_\sigma}1$ as long as

$$\big|\mathrm{Ric}^0(g_\sigma)\big| = \sqrt{\tfrac18}\,\big|\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big| \le \sqrt{\tfrac{1}{21}}|\tau_2|^2 = -\sqrt{\tfrac{4}{21}}\,\mathrm{Scal}(g_\sigma). \tag{4.46}$$

Since $-\,\mathrm{Scal}(g_\sigma) \geq 0$, the inequality (4.42) with $C < 1$ will evidently imply that the expression at the end of (4.45) is a positive multiple of $|\tau_2|^2 *_\sigma 1$. By Stokes' theorem, this will imply that τ_2 vanishes identically, as desired.

Suppose now that (4.42) holds with $C = 1$. Then the expression at the end of (4.45) is still a nonnegative multiple of $|\tau_2|^2 *_\sigma 1$ and hence, by Stokes' theorem, must vanish identically. However, by the strong form of the Cauchy-Schwartz inequality, this can only happen if the relation

$$\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big) = \tfrac{2}{3}\big(*_\sigma(\tau_2 \wedge \tau_2) + \tfrac{1}{7}|\tau_2|^2\,\sigma\big) \tag{4.47}$$

holds identically on the open set where $|\tau_2| > 0$. Now, if the locus $|\tau_2| = 0$ has any interior, then $\mathrm{Ric}(g_\sigma)$ vanishes on this interior since σ is both closed and coclosed there. Thus, (4.47) holds on both the open set where $|\tau_2| > 0$ and the interior of the locus where $|\tau_2| = 0$. Consequently, it must hold on all of M, as desired. $\square$

Remark 13 (Extremally Ricci-pinched closed G_2-structures). Note that another way of phrasing Corollary 3 is to use (4.45) to show that the inequality

$$\int_M \big|\mathrm{Ric}^0(g_\sigma)\big|^2 *_\sigma 1 \geq \frac{4}{21}\int_M \mathrm{Scal}(g_\sigma)^2 *_\sigma 1 \tag{4.48}$$

holds for any closed G_2-structure σ on a compact manifold M^7 and that equality holds in (4.48) if and only if σ satisfies

$$\mathrm{d}\sigma = 0, \qquad \mathrm{d}{*_\sigma}\sigma = \tau \wedge \sigma, \qquad \mathrm{d}\tau = \tfrac{1}{6}\,\big(|\tau|^2\,\sigma + *_\sigma(\tau \wedge \tau)\big). \tag{4.49}$$

Indeed, Corollary 3 suggests that the G_2-structures σ that satisfy (4.49) might be of particular interest, since these are, in some sense, the most 'extremally Ricci-pinched' that a closed G_2-structure can be on a compact 7-manifold.

One can see that there are some rather subtle restrictions on such structures on compact manifolds by developing these equations a bit further: Note that (4.49) implies

$$\begin{aligned}\mathrm{d}\big(\tau^3\big) &= 3\,\tau^2 \wedge \mathrm{d}\tau = \tau^2 \wedge \big(\tfrac{1}{2}\,|\tau|^2\,\sigma + \tfrac{1}{2}{*_\sigma}(\tau \wedge \tau)\big)\\ &= -\tfrac{1}{2}\,|\tau|^4 *_\sigma 1 + \tfrac{1}{2}|\tau \wedge \tau|^2 *_\sigma 1 = 0.\end{aligned} \tag{4.50}$$

On the other hand, computation using the structure equations and (4.49) yields

$$0 = \mathrm{d}\big(\mathrm{d}\tau\big) = \mathrm{d}\,\big(\tfrac{1}{6}\,\big(|\tau|^2\,\sigma + *_\sigma(\tau \wedge \tau)\big)\big) = \alpha \wedge \sigma + *_\sigma\gamma \tag{4.51}$$

where γ lies in $\Omega^3_{27}(M,\sigma)$ and

$$\alpha = \tfrac{1}{8}\,\big(\mathrm{d}\big(|\tau|^2\big) - \tfrac{2}{9}{*_\sigma}(\tau^3)\big). \tag{4.52}$$

Consequently, any solution of (4.49) must satisfy[2]

$$\mathrm{d}\big(|\tau|^2\big) = \tfrac{2}{9}{*_\sigma}(\tau^3). \tag{4.53}$$

[2]The vanishing of γ as defined in (4.51) imposes 27 more equations on the covariant derivative of τ, but these are not as easily stated as (4.53).

Combining this with (4.50) yields

$$\Delta_\sigma\big(|\tau|^2\big) = 0. \tag{4.54}$$

Assume now that M is compact and connected. It then follows from (4.54) that $|\tau|^2$ must be a constant.

Of course, if $|\tau|^2 = 0$, then $\tau = 0$ and σ is coclosed and hence g_σ-parallel. Thus, assume from now on that $|\tau|^2 > 0$.

Then (4.53) implies that $\tau^3 = 0$. However, $|\tau{\wedge}\tau|^2 = |\tau|^4 \neq 0$, implying that τ has constant rank 4 (instead of the *a priori* maximum of 6) and hence that $\tau{\wedge}\tau$ is a nonzero simple 4-form of constant norm. Using (2.23) and the fact that $\tau^3 = 0$ then yields

$$\begin{aligned} \mathrm{d}\big(\tau^2\big) &= 2\,\tau\wedge\mathrm{d}\tau = \tfrac13\tau\wedge\big(\,|\tau|^2\,\sigma + *_\sigma(\tau\wedge\tau)\big)\\ &= -\tfrac13\,|\tau|^2 *_\sigma\tau + \tfrac13\,|\tau|^2 *_\sigma\tau = 0, \end{aligned} \tag{4.55}$$

So that the simple 4-form $\tau{\wedge}\tau$ is closed.

Since $\tau{\wedge}\tau$ is simple with constant norm, the 3-form $*_\sigma(\tau{\wedge}\tau)$ is also nonzero and simple, with constant norm. Moreover, in view of the constancy of $|\tau|^2$, expanding $\mathrm{d}\big(\mathrm{d}\tau\big) = 0$ and using (4.49) shows that $*_\sigma(\tau{\wedge}\tau)$ is also closed.

Consequently, the tangent bundle of M splits as an orthogonal direct sum of two integrable subbundles

$$TM = P \oplus Q \tag{4.56}$$

with $P = \{\, v \in TM \mid v \lrcorner (\tau{\wedge}\tau) = 0\,\}$ of rank 3 and $Q = \{\, v \in TM \mid v \lrcorner *_\sigma(\tau{\wedge}\tau) = 0\,\}$ of rank 4. The P-leaves are calibrated by $-|\tau|^{-2} *_\sigma(\tau{\wedge}\tau)$ while the Q-leaves are calibrated by $-|\tau|^{-2}\,(\tau{\wedge}\tau)$. (The reason for the minus signs is that they correctly orient the P-leaves as associative submanifolds and the Q-leaves as coassociative submanifolds.)

The Ricci curvature in this case simplifies to

$$\operatorname{Ric}(g_\sigma) = \tfrac{1}{12}\,\mathsf{j}\big(*_\sigma(\tau\wedge\tau)\big) = -\tfrac16\,|\tau|^2\,\big(g_\sigma\big)\big|_P\,, \tag{4.57}$$

so that, in particular, the Ricci curvature is nonpositive, with one eigenvalue $-\frac16\,|\tau|^2$ of multiplicity 3 and the other eigenvalue 0 of multiplicity 4.

Example 1 (A homogeneous example). Just how general the G_2-structures σ satisfying (4.49) with $\tau \neq 0$ are, even locally, is an interesting question. I will now show that these equations do have a nontrivial solution, by producing a (homogeneous) example.

Let G be the group of volume-preserving affine transformations of $\mathbb{C}^2$. Thus G can be regarded as the matrix group consisting of the 3-by-3 matrices with complex entries of the form

$$g = \begin{pmatrix} a & b & x\\ c & d & y\\ 0 & 0 & 1\end{pmatrix} \tag{4.58}$$

where $ad - bc = 1$. Write the canonical left-invariant form on G as

$$\alpha = g^{-1}\,\mathrm{d}g = \begin{pmatrix} -\omega^1 + \mathrm{i}\,\eta^1 & -\omega^3 - \eta^3 + \mathrm{i}\,(\eta^2 - \omega^2) & \omega^4 + \mathrm{i}\,\omega^5\\ -\omega^3 + \eta^3 + \mathrm{i}\,(\eta^2 + \omega^2) & \omega^1 - \mathrm{i}\,\eta^1 & \omega^6 - \mathrm{i}\,\omega^7\\ 0 & 0 & 0 \end{pmatrix}. \tag{4.59}$$

Then $\mathrm{d}\alpha = -\alpha{\wedge}\alpha$ implies that the left-invariant 3-form $\tilde\sigma$ defined by

$$\tilde\sigma = \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{347} - \omega^{356} \tag{4.60}$$

(where ω^{ijk} stands for the wedge product $\omega^i{\wedge}\omega^j{\wedge}\omega^k$, etc.) satisfies $\mathrm{d}\tilde\sigma = 0$. Consequently, $\tilde\sigma$ is the pullback to G of a definite 3-form σ on the left coset space $M^7 = G/\mathrm{SU}(2)$. (Here, $\mathrm{SU}(2) \subset G$ is the subgroup whose left cosets are the integral leaves of the differentially closed system $\omega^i = 0$ on G.) Moreover, letting $\pi : G \to M$ denote the coset projection, one sees that

$$\pi^*\big({*_\sigma}\sigma\big) = \omega^{4567} + \omega^{2367} + \omega^{2345} + \omega^{1357} - \omega^{1346} - \omega^{1256} - \omega^{1247} \tag{4.61}$$

while

$$\pi^*\big(g_\sigma\big) = (\omega^1)^2 + \cdots + (\omega^7)^2. \tag{4.62}$$

Finally, one finds that there exists a 2-form τ on M so that

$$\pi^*(\tau) = 6\,\omega^{45} - 6\,\omega^{67}. \tag{4.63}$$

The equation $\mathrm{d}\alpha = -\alpha{\wedge}\alpha$ then implies that the pair (σ,τ) satisfy (4.49).

Note that M is diffeomorphic to $\mathbb{R}^7$ and that the P-leaves and Q-leaves are, respectively, the fibers of maps $M \to \mathbb{C}^2 = G/\mathrm{SL}(2,\mathbb{C})$ and $M \to \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. Although M is not compact, it has compact quotients on which σ is well-defined. To see this, let $\mathsf{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$ be a cocompact, discrete, torsion-free subgroup that preserves a lattice $\mathsf{L} \subset \mathbb{C}^2$. (Such $\mathsf{\Gamma}$ do exist. For example, let $q \in \mathbb{Z}[\mathrm{i}]$ be a prime in the Gaussian integers satisfying $|q|^2 = q\bar q > 4$ and let $\mathsf{\Gamma}_q \subset \mathrm{SL}\big(2,\mathbb{Z}[\mathrm{i}]\big)$ be the finite index subgroup consisting of the elements $\gamma \in \mathrm{SL}\big(2,\mathbb{Z}[\mathrm{i}]\big)$ that satisfy $\gamma \equiv \mathrm{I}_2 \mod (q)$. Then $\mathsf{\Gamma}_q$ has the required properties and preserves the lattice $\mathbb{Z}[\mathrm{i}]^2 \subset \mathbb{C}^2$.) Now consider the discrete subgroup $\Gamma \subset G$ consisting of elements of the form

$$g = \begin{pmatrix} \gamma & \ell \\ 0 & 1 \end{pmatrix} \tag{4.64}$$

where γ lies in $\mathsf{\Gamma}$ and ℓ lies in L. Then Γ acts on $M = G/\mathrm{SU}(2)$ on the left preserving σ and it is not difficult to see that this action is both free and properly discontinuous. The quotient $\bar M = \Gamma\backslash M$ is compact and supports a closed extremally Ricci-pinched G_2-structure $\bar\sigma$ that pulls back to M to equal σ.

Remark 14 (Natural equations for closed G_2-structures). Let λ be a constant and consider the system of equations

$$\mathrm{d}\sigma = 0, \qquad \mathrm{d}{*_\sigma}\sigma = \tau\wedge\sigma, \qquad \mathrm{d}\tau = \tfrac17\,|\tau|^2\,\sigma + \lambda\left(\tfrac17\,|\tau|^2\,\sigma + {*_\sigma}(\tau\wedge\tau)\right). \tag{4.65}$$

for a G_2-structure σ on a manifold M^7. This family includes both the Einstein condition ($\lambda = \frac12$) and the 'extremally pinched Ricci' condition ($\lambda = \frac16$). Indeed, in view of (4.35) and (2.28) and since $\mathsf{S}^2(\mathsf{V}_{0,1}) \simeq \mathsf{V}_{0,0}{\oplus}\mathsf{V}_{2,0}{\oplus}\mathsf{V}_{0,2}$ while $\Lambda^3(\mathsf{V}_{1,0}) \simeq \mathsf{V}_{0,0}{\oplus}\mathsf{V}_{0,1}{\oplus}\mathsf{V}_{2,0}$, the 1-parameter family of natural equations (4.65) for closed G_2-structures describes the most general way in which $\mathrm{d}\tau$ can be prescribed naturally and quadratically in terms of τ. In view of the fact that $\mathrm{d}\tau$ can have no component in $\Omega^3_7(M,\sigma)$ and that the component of $\mathrm{d}\tau$ in $\Omega^3_1(M,\sigma)$ is determined by (4.35), it follows that (4.65) is a system of 27 ($= \dim \mathsf{V}_{2,0}$) equations for a closed G_2-structure σ. In view of the discussion in Remark 11, one should regard (4.65) as an overdetermined system of PDE. This system is not involutive for any value of λ, as the following discussion will show.

First, the computation (4.50) can be redone for G_2-structures satisfying (4.65), yielding

$$\mathrm{d}\big(\tau^3\big) = \frac{3(6\lambda-1)}{7}\,|\tau|^4 *_\sigma 1\,. \tag{4.66}$$

In particular, on a compact 7-manifold, the only value of λ that is possible for such a structure with τ not identically zero is $\lambda = \frac{1}{6}$.

Redoing the computation (4.51) using the structure equations and (4.65) instead of (4.49) yields

$$0 = \mathrm{d}\big(\mathrm{d}\tau\big) = \alpha\wedge\sigma + *_\sigma\gamma \tag{4.67}$$

where γ lies in $\Omega^3_{27}(M,\sigma)$ and

$$\alpha = \frac{\lambda(2\lambda-1)}{4} *_\sigma(\tau^3) - \frac{(3\lambda-4)}{28}\,\mathrm{d}\big(|\tau|^2\big). \tag{4.68}$$

Consequently, any solution of (4.65) satisfies

$$(3\lambda-4)\,\mathrm{d}\big(|\tau|^2\big) = 7\lambda(2\lambda-1) *_\sigma(\tau^3). \tag{4.69}$$

When $\lambda = \frac{4}{3}$, this condition implies $\tau^3 = 0$, which, by (4.66), then implies $|\tau| = 0$, i.e., $\tau = 0$. Thus, there are no G_2-structures σ satisfying (4.65) with $\lambda = \frac{4}{3}$ except those that are closed and coclosed.

When $\lambda \neq \frac{4}{3}$, the system (4.69) represents 7 ‘new’ second order equations on σ that are not algebraic consequences of (4.65). The existence of these ‘new’ equations implies that the system (4.65) is not involutive.

Even beyond this, when $\lambda \neq 0$, the vanishing of the term γ in (4.67) yields 27 more equations of second order on σ that are not algebraic consequences of (4.65) and (4.69) combined. Whether further differentiation of these combined equations would yield more second (or even first) order equations remains to be seen. It is this phenomenon that makes the analysis of systems of type (4.65) troublesome.

5. The Torsion-free Case

A G_2-structure $\sigma \in \Omega^3_+(M)$ is said to be *torsion-free* if all of its four torsion forms vanish. There is an aspect of the geometry of torsion-free G_2-structures that is analogous to the Kähler identities in complex Riemannian geometry and that is the concern of this section.

The material in this section was the result of a joint project with F. Reese Harvey and was carried out between 1991 and 1994.

5.1. Reference modules

It will be convenient to chose a ‘reference’ representation for each of the irreducible G_2-modules that appear in the exterior algebra on $\mathsf{V}_{1,0}$.

Given any G_2-structure $\sigma \in \Omega^3_+(M)$, these will be chosen to correspond to the spaces of differential forms $\Omega^0(M)$, $\Omega^1(M)$, $\Omega^2_{14}(M,\sigma)$, and $\Omega^3_{27}(M,\sigma)$. For simplicity, these spaces will be referred to as Ω_1, Ω_7, Ω_{14}, and Ω_{27} when M and σ are clear from context.

5.2. Exterior derivative identities

When a G_2-structure σ has vanishing intrinsic torsion, the fundamental forms σ and $*_\sigma\sigma$ are parallel with respect to the natural connection (which is torsion-free) and so are all of the various natural isomorphisms between the different constituents of the bundle of exterior differential forms. Consequently, the various differential operators that one can define by decomposing the exterior derivative into its constituent components are really manifestations of first order differential operators between the abstract bundles. Thus, there will be identities (analogous to the identities one proves in Kähler geometry) between these different manifestations. In this subsection, these will be made explicit. Essentially, the proof of the following proposition is a matter of checking constants.

Proposition 3 (Exterior derivative identities)**.** *Suppose that σ is a torsion-free G_2-structure on M. Then, for all $p,q \in \{1,7,14,27\}$, there exists a first order differential operator $\mathrm{d}^p_q\colon \Omega_p \to \Omega_q$, so that the exterior derivative formulas given in Table 1 hold for all $f \in \Omega_1$, $\alpha \in \Omega_7$, $\beta \in \Omega_{14}$, and $\gamma \in \Omega_{27}$. These operators are non-zero except for d^1_{27}, d^{27}_1, d^1_{14}, d^{14}_1, d^1_1, and d^{14}_{14}. With respect to the natural metrics on the underlying bundles, $(\mathrm{d}^p_q)^* = \mathrm{d}^q_p$. The identity $\mathrm{d}^2 = 0$ is equivalent to the second order identities on the operators d^p_q listed in Table 2. Finally, the formulas for the Hodge Laplacians in terms of the operators d^p_q are as given in Table 3.*

Proof. The operators d^p_q are defined by decomposing the exterior derivative operator into types (much as ∂ and $\bar\partial$ are defined in Kähler geometry by the projection of the exterior derivative into types). For example, take the formula $\mathrm{d}^7_7\alpha = *_\sigma(\mathrm{d}(\alpha\wedge *_\sigma\sigma))$ as the *definition* of $\mathrm{d}^7_7\colon \Omega_7 \to \Omega_7$ and define $\mathrm{d}^7_{27}\alpha$ to be the $\Omega^3_{27}(M,\sigma)$-component of $\mathrm{d}\big(*_\sigma(\alpha\wedge *_\sigma\sigma)\big)$. Verifying the exterior derivative formulas is a routine matter that is best left to the reader. Once these have been established, the second order identities and the Laplacian formulas follow by routine computation. □

Remark 15 (Torsion perturbations). In the general case of a G_2-structure with torsion, all of the formulae in the tables listed above must be modified by lower order terms. For example, in Table 1 the second line would be modified to

$$\mathrm{d}(f\,\sigma) = \mathrm{d}^1_7 f\wedge\sigma + f\tau_0 *_\sigma\sigma + 3f\tau_1\wedge\sigma + f *_\sigma\tau_3. \tag{5.1}$$

The zero right hand sides in Table 2 have to be replaced by first order operators whose coefficients depend on the torsion terms and, in Table 3, one must take into account which particular part of the exterior algebra a given form occupies before writing down the appropriate formula for the Laplacian. For example, it is not true, in general, that $\Delta(f\,\sigma) = \Delta f\,\sigma$.

$$
\begin{array}{lclrrr}
\mathrm{d}\,f & = & & \mathrm{d}^1_7 f & & \\
\mathrm{d}\,(f\,\sigma) & = & & \mathrm{d}^1_7 f\wedge\sigma & & \\
\mathrm{d}\,(f\,{*_\sigma}\sigma) & = & & \mathrm{d}^1_7 f\wedge{*_\sigma}\sigma & & \\
\mathrm{d}\,\alpha & = & & \frac{1}{3}{*_\sigma}(\mathrm{d}^7_7\alpha\wedge{*_\sigma}\sigma) & +\mathrm{d}^7_{14}\alpha & \\
\mathrm{d}\,{*_\sigma}(\alpha\wedge{*_\sigma}\sigma) & = & -\frac{3}{7}\,\mathrm{d}^7_1\alpha\,\sigma & -\frac{1}{2}{*_\sigma}(\mathrm{d}^7_7\alpha\wedge\sigma) & & +\mathrm{d}^7_{27}\alpha \\
\mathrm{d}\,{*_\sigma}(\alpha\wedge\sigma) & = & \frac{4}{7}\,\mathrm{d}^7_1\alpha\,{*_\sigma}\sigma & +\frac{1}{2}\,\mathrm{d}^7_7\alpha\wedge\sigma & & +{*_\sigma}\,\mathrm{d}^7_{27}\alpha \\
\mathrm{d}\,(\alpha\wedge\sigma) & = & & \frac{2}{3}\,\mathrm{d}^7_7\alpha\wedge{*_\sigma}\sigma & -{*_\sigma}\,\mathrm{d}^7_{14}\alpha & \\
\mathrm{d}\,(\alpha\wedge{*_\sigma}\sigma) & = & & -{*_\sigma}\,\mathrm{d}^7_7\alpha & & \\
\mathrm{d}\,({*_\sigma}\alpha) & = & -\mathrm{d}^7_1\alpha\,{*_\sigma}1 & & & \\
\mathrm{d}\,\beta & = & & \frac{1}{4}\,{*_\sigma}(\mathrm{d}^{14}_7\beta\wedge\sigma) & & +\mathrm{d}^{14}_{27}\beta \\
\mathrm{d}\,({*_\sigma}\beta) & = & & {*_\sigma}\,\mathrm{d}^{14}_7\beta & & \\
\mathrm{d}\,\gamma & = & & \frac{1}{4}\,\mathrm{d}^{27}_7\gamma\wedge\sigma & & +{*_\sigma}\,\mathrm{d}^{27}_{27}\gamma \\
\mathrm{d}\,({*_\sigma}\gamma) & = & & -\frac{1}{3}\,\mathrm{d}^{27}_7\gamma\wedge{*_\sigma}\sigma & -{*_\sigma}\,\mathrm{d}^{27}_{14}\gamma &
\end{array}
$$

Table 1. Exterior derivative formulae

$$
\begin{array}{llll}
 & \mathrm{d}^7_7\, d^1_7 = 0 & \mathrm{d}^7_{14}\,\mathrm{d}^1_7 = 0 & \\
\mathrm{d}^7_1\,\mathrm{d}^7_7 = 0 & \begin{array}{l}\mathrm{d}^{14}_7\,\mathrm{d}^7_{14} = \frac{2}{3}(\mathrm{d}^7_7)^2\\ \mathrm{d}^{27}_7\,\mathrm{d}^7_{27} = (\mathrm{d}^7_7)^2 + \frac{12}{7}\mathrm{d}^1_7\,\mathrm{d}^7_1\end{array} & \mathrm{d}^7_{14}\,\mathrm{d}^7_7 + 2\,\mathrm{d}^{27}_{14}\,\mathrm{d}^7_{27} = 0 & \begin{array}{l}3\,\mathrm{d}^{14}_{27}\,\mathrm{d}^7_{14} + \mathrm{d}^7_{27}\,\mathrm{d}^7_7 = 0\\ 2\,\mathrm{d}^{27}_{27}\,\mathrm{d}^7_{27} - \mathrm{d}^7_{27}\,\mathrm{d}^7_7 = 0\end{array} \\
\mathrm{d}^7_1\,\mathrm{d}^{14}_7 = 0 & \mathrm{d}^7_7\,\mathrm{d}^{14}_7 + 2\,\mathrm{d}^{27}_7\,\mathrm{d}^{14}_{27} = 0 & & \mathrm{d}^7_{27}\,\mathrm{d}^{14}_7 + 4\,\mathrm{d}^{27}_{27}\,\mathrm{d}^{14}_{27} = 0 \\
 & \begin{array}{l}3\,\mathrm{d}^{14}_7\,\mathrm{d}^{27}_{14} + \mathrm{d}^7_7\,\mathrm{d}^{27}_7 = 0\\ 2\,\mathrm{d}^{27}_7\,\mathrm{d}^{27}_{27} - \mathrm{d}^7_7\,\mathrm{d}^{27}_7 = 0\end{array} & \mathrm{d}^7_{14}\,\mathrm{d}^{27}_7 + 4\,\mathrm{d}^{27}_{14}\,\mathrm{d}^{27}_{27} = 0 &
\end{array}
$$

Table 2. Second order identities

6. Deformation and Evolution of G_2-structures

The material in this section was the result of a joint project with Steve Altschuler and was carried out between 1992 and 1994. Our goal was to understand the long time behavior of the Laplacian heat flow defined below for closed G_2-structures on compact 7-manifolds, specifically, to understand conditions under which one could prove that this

$$
\begin{aligned}
\Delta f &= \mathrm{d}^7_1\,\mathrm{d}^1_7\, f\\
\Delta \alpha &= \big((\mathrm{d}^7_7)^2 + \mathrm{d}^1_7\,\mathrm{d}^7_1\big)\ \alpha\\
\Delta \beta &= \big(\tfrac{5}{4}\mathrm{d}^7_{14}\,\mathrm{d}^{14}_7 + \mathrm{d}^{27}_{14}\,\mathrm{d}^{14}_{27}\big)\ \beta\\
\Delta \gamma &= \big(\tfrac{7}{12}\mathrm{d}^7_{27}\,\mathrm{d}^{27}_7 + \mathrm{d}^{14}_{27}\,\mathrm{d}^{27}_{14} + (\mathrm{d}^{27}_{27})^2\big)\ \gamma
\end{aligned}
$$

Table 3. Laplacians

flow converged to a G_2-structure that is both closed and coclosed. Nowadays, this flow is called the *Hitchin flow* after Hitchin's fundamental paper [12].

We were never able to prove long-time existence under any reasonable hypotheses, so we wound up not publishing anything on the subject, although we did get some interesting results and formulae that I have not seen so far in the literature.[3]

6.1. The deformation forms

It turns out to be quite easy to describe deformations of G_2-structures. The following result is well-known and can be found most explicitly in Joyce's treatment [14, §10.3], though the notation is somewhat different. It is included here to establish notation for the discussion to follow.

Proposition 4 (Deformation forms). *Let $\sigma_t \in \Omega^3_+(M)$ be a smooth 1-parameter family of G_2-structures on M. Let g_t and $*_t$ denote the underlying metric and Hodge star operator associated to σ_t. Then there exist three differential forms $f^0_t \in \Omega^0(M)$, $f^1_t \in \Omega^1(M)$, and $f^3_t \in \Omega^3_{27}(M,\sigma_t) \subset \Omega^3(M)$ that depend differentiably on t and that are uniquely characterized by the equation (in which the t-dependence has been suppressed for notational clarity)*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\sigma\big) = 3f^0\,\sigma + *_\sigma\big(f^1\wedge\sigma\big) + f^3\,. \tag{6.1}$$

Moreover, the associated metric and dual 4-forms satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(g\big) = 2f^0\,g + \tfrac{1}{2}\,\mathrm{j}\big(f^3\big) \tag{6.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(*_\sigma\sigma\big) = 4f^0\,*_\sigma\sigma + f^1\wedge\sigma - *_\sigma f^3\,. \tag{6.3}$$

Definition 5 (The deformation forms). The forms f^0_t, f^1_t, and f^3_t associated to the family σ_t will be referred to as the *deformation forms* of the family.

[3] I would be happy to learn of any places where these results have appeared so that I can properly acknowledge them in future versions of this article.

One immediate consequence of Proposition 4 is a formula for the variation of the volume form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(*_\sigma 1\big) = 7f^0 *_\sigma 1\,. \tag{6.4}$$

By the same techniques, one can derive a second order expansion:

Proposition 5 (Taylor expansion formula)**.** *Let $\phi \in \Omega^3_+(M)$ be a G_2-structure. Then for all $b_0 \in \Omega^0(M)$, $b_1 \in \Omega^1(M)$, and $b_3 \in \Omega^3_{27}(M,\phi)$ of sufficiently small C^0-norm, the 3-form*

$$\sigma = \phi + (3b_0\,\phi + *_\phi(b_1 \wedge \phi) + b_3) \tag{6.5}$$

is definite. Moreover, there is an expansion of the form

$$\begin{aligned} *_\sigma\sigma = *_\phi\phi + \big(4b_0 *_\phi\phi + b_1\wedge\phi - *_\phi b_3\big) + \big(\,2(b_0)^2 + \tfrac{2}{21}\,|b_1|^2_\phi - \tfrac{1}{42}\,|b_3|^2_\phi\,\big) *_\phi\phi \\ + Q_1(b_0,b_1,b_3)\wedge\phi + *_\phi Q_3(b_0,b_1,b_3) + R(b_0,b_1,b_3) \end{aligned} \tag{6.6}$$

where Q_1 (a 1-form) and Q_3 (a 3-form in $\Omega^3_{27}(M,\phi)$) are quadratic in the coefficients of the b_i and R is a 4-form that vanishes to order 3 in the coefficients of the b_i. Consequently, there is an expansion of the form

$$*_\sigma 1 = \big(1 + 7b_0 + \big(\,14(b_0)^2 + \tfrac{2}{3}|b_1|^2_\phi - \tfrac{1}{6}|b_3|^2_\phi\,\big) + r(b_0,b_1,b_3)\,\big) *_\phi 1 \tag{6.7}$$

where r vanishes to order 3 in (b_0,b_1,b_3).

6.2. The Laplacian evolution

A natural evolution equation for G_2-structures is the (nonlinear) Laplacian evolution equation for $\sigma \in \Omega^3_+(M)$ defined as follows:

$$\frac{d}{dt}(\sigma) = \Delta_\sigma\sigma\,. \tag{6.8}$$

This equation is diffeomorphism invariant and hence cannot be elliptic in the strict sense. However, it is not difficult to compute the linearization and see that it is transversely elliptic, i.e., elliptic transverse to the action of the diffeomorphism group.

Thus, the by-now standard methods of DeTurck and Hamilton can be applied to show that, if M is compact, then for any smooth $\sigma_0 \in \Omega^3_+(M)$ there exists an extended number T satisfying $0 < T \leq \infty$ and a 1-parameter family $\sigma(t) \in \Omega^3_+(M)$ defined for all t such that $0 < t < T$ so that the family satisfies (6.8) and so that $\sigma(t)$ approaches σ_0 uniformly as t approaches 0 from above. The fundamental issue then becomes to understand the behavior of the family as t approaches T.

For general σ, the formula for the Laplacian in terms of the torsion forms is not too illuminating:

$$\Delta_\sigma\sigma = \mathrm{d}\big(\tau_2 - 4\,\tau_1^\sharp \lrcorner\, \sigma\big) + *_\sigma \mathrm{d}\big(\tau_0\,\sigma + 3\,\tau_1^\sharp \lrcorner *_\sigma\sigma + \tau_3\big). \tag{6.9}$$

This can be further expanded, but the general formula becomes unwieldy rather quickly.

6.2.1. Evolution of closed forms

Suppose now that the initial form σ is closed, i.e., that τ_0, τ_1 and τ_3 are all zero initially. It is not difficult to show that the Laplacian flow preserves this condition, i.e., that the family $\sigma(t)$ consists of closed forms.

For notational simplicity, for the rest of this section, τ_2 will be denoted simply as τ. Also, in the calculations to follow, t will be treated as a parameter, i.e., I will regard $\mathrm{d}t$ as zero when computing exterior derivatives. Thus, the assumptions are that

$$\begin{aligned} \mathrm{d}\sigma &= 0 \\ \mathrm{d}{*_\sigma}\sigma &= \tau\wedge\sigma \end{aligned} \tag{6.10}$$

and that

$$\frac{d}{dt}(\sigma) = \mathrm{d}\tau. \tag{6.11}$$

As has already been shown in (4.35),

$$d\tau = \tfrac{1}{7}|\tau|^2\,\sigma + \gamma \tag{6.12}$$

for some $\gamma \in \Omega^3_{27}(M,\sigma)$. In particular, it follows from Proposition 4 that

$$\frac{d}{dt}({*_\sigma}\sigma) = \tfrac{4}{21}|\tau|^2\,{*_\sigma}\sigma - {*_\sigma}\gamma = \tfrac{1}{3}|\tau|^2\,{*_\sigma}\sigma - {*_\sigma}\mathrm{d}\tau\,. \tag{6.13}$$

Moreover, (6.4) now becomes

$$\frac{d}{dt}({*_\sigma}1) = \tfrac{1}{3}|\tau|^2\,{*_\sigma}1. \tag{6.14}$$

In particular, note that the associated volume form ${*_\sigma}1$ is *pointwise* increasing.[4]

Finally, combining (6.11) with the formulae (4.39) and (6.2), one gets the evolution of the metric g_σ in the form

$$\frac{d}{dt}(g_\sigma) = -2\,\mathrm{Ric}(g_\sigma) + \tfrac{8}{21}|\tau|^2\,g_\sigma + \tfrac{1}{4}\,\mathsf{j}\big({*_\sigma}(\tau\wedge\tau)\big). \tag{6.15}$$

Remark 16 (Hitchin's interpretation). Hitchin [12] has given the following interpretation of this flow. Suppose that ϕ is a closed definite 3-form and on a compact 7-manifold M. Let

$$[\phi]_+ = \{\ \phi + \mathrm{d}\beta \in \Omega^3_+(M)\ \mid\ \beta\in\Omega^2(M)\ \} \tag{6.16}$$

be the open set in the cohomology class $[\phi] = \{\phi + \mathrm{d}\beta \mid \beta\in\Omega^2(M)\}$ that consists of definite 3-forms.

Define the volume function $V : [\phi]_+ \to \mathbb{R}^+$ by $V(\sigma) = \int_M {*_\sigma}1 > 0$ for $\sigma\in[\phi]_+$. Hitchin shows that $\sigma\in[\phi]_+$ is a critical point of V if and only if σ is coclosed (as well as closed) and he shows that the flow (6.8) is the gradient flow of the functional V (in the L^2 metric on $[\phi]_+$).

[4]In view of Hitchin's interpretation of this flow as the gradient flow of the volume functional on the space $[\phi]_+$, it is to be expected that the integral of ${*_\sigma}1$ over M is increasing.

Suppose that ϕ is a critical point of V, i.e., that $*_\phi\phi$ is closed. Then by Hodge theory there is a direct sum decomposition

$$\mathrm{d}\big(\Omega^2(M)\big) = \{\,\mathcal{L}_Z\,\phi \mid Z \in \mathrm{Vect}(M)\,\} \oplus \{\,\mathrm{d}\beta \mid \beta \in \Omega^2_{14}(M,\phi),\ \mathrm{d}^{14}_7\beta = 0\,\}. \tag{6.17}$$

The first summand is the tangent space to the orbit of ϕ under $\mathrm{Diff}^\circ(M)$ (i.e., the diffeomorphisms of M that act trivially on $H^*(M)$), while the second summand represents the tangent space at $\mathrm{Diff}^\circ(M)\cdot\phi$ to the 'moduli space' $\mathrm{Diff}^\circ(M)\backslash[\phi]_+$. If $\beta \in \Omega^2_{14}$ satisfies $d^{14}_7\beta = 0$, then, setting $\sigma = \phi + t\,\mathrm{d}\beta = \phi + t\,d^{14}_{27}\beta$, one finds, by (6.7), that

$$*_\sigma 1 = \big(1 - \tfrac{1}{6}\left|d^{14}_{27}\beta\right|^2 + t^3 R(t,\mathrm{d}\beta)\big)*_\phi 1. \tag{6.18}$$

for some smooth remainder term $R(t,\mathrm{d}\beta)$. By the formulae in Table 1, the equations $\mathrm{d}^{14}_7\beta = \mathrm{d}^{14}_{27}\beta = 0$ for $\beta \in \Omega^2_{14}(M,\phi)$ imply that $\mathrm{d}\beta = \delta\beta = 0$. It follows that the Hessian of V at ϕ is negative definite on $\{\,\mathrm{d}\beta \mid \beta \in \Omega^2_{14}(M,\phi),\ \mathrm{d}^{14}_7\beta = 0\,\}$. Thus, $\mathrm{Diff}^\circ(M)\cdot\phi$ is a local maximum of V on the moduli space $\mathrm{Diff}^\circ(M)\backslash[\phi]_+$.[5]

In particular, it seems reasonable to expect that, for $\sigma \in [\phi]_+$ 'sufficiently near' ϕ in a appropriate norm, the V-gradient flow (6.8) with σ as initial value would converge to a point on $\mathrm{Diff}^\circ(M)\cdot\phi$.

Remark 17 (Nonconvergence). A more likely difficulty, it seems, is posed by the possibility that there may be torsion-free G_2-structures ϕ for which the volume functional is not bounded above on $[\phi]_+$, so one would not expect the Laplacian flow to converge for most closed G_2-structures in $[\phi]_+$.

Example 2 (Fernández' closed G_2-solvmanifold). Fernández [4, 5] has constructed compact 7-dimensional manifolds M^7 that support a closed G_2-structure ϕ but that cannot, for topological reasons, support a torsion-free G_2-structure. Thus, in these cases, the above flow cannot converge, since there will be no critical points of V on $[\phi]_+$.

It is instructive to look at one of her examples: Let $G \subset \mathrm{GL}(5,\mathbb{R})$ be the subgroup that consists of matrices of the form

$$g = \begin{pmatrix} 1 & 0 & x^2 & x^4 & x^6 \\ 0 & 1 & x^3 & x^5 & x^7 \\ 0 & 0 & 1 & 0 & x^1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.19}$$

where $x^i \in \mathbb{R}$ for $1 \le i \le 7$. Write the left-invariant form on G in the form

$$g^{-1}\,\mathrm{d}g = \begin{pmatrix} 0 & 0 & \omega^2 & \omega^4 & \omega^6 \\ 0 & 0 & \omega^3 & \omega^5 & \omega^7 \\ 0 & 0 & 0 & 0 & \omega^1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.20}$$

[5] Hitchin says that V is a 'Morse-Bott' functional on $[\phi]_+$, i.e., that V has nondegenerate critical points on the moduli space.

where $\mathrm{d}\omega^i = 0$ for $1 \le i \le 5$ while $\mathrm{d}\omega^6 = \omega^1{\wedge}\omega^2$ and $\mathrm{d}\omega^7 = \omega^1{\wedge}\omega^3$.

Let $\Gamma = G{\cap}\mathrm{GL}(5,\mathbb{Z})$ and note that Γ is a co-compact discrete subgroup of G. Let $M^7 = \Gamma\backslash G$ be the space of right cosets of Γ in G. Then the ω^i are well-defined on M and it is easy to verify that the 3-form

$$\sigma = \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{347} - \omega^{356} \tag{6.21}$$

is a closed G_2-structure on M. It is not coclosed, but satisfies

$$\mathrm{d}{*_\sigma}\sigma = \big(\omega^{27} - \omega^{36}\big) \wedge \sigma. \tag{6.22}$$

Since

$$\mathrm{d}\big(\omega^{27} - \omega^{36}\big) = 2\,\omega^{123}, \tag{6.23}$$

it follows that the flow satisfies

$$\sigma(t) = \mathrm{e}^{2t}\omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{347} - \omega^{356}. \tag{6.24}$$

The associated metric is

$$g(t) = \mathrm{e}^{4t/3}\big((\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2\big) + \mathrm{e}^{-2t/3}\big((\omega^4)^2 + (\omega^5)^2 + (\omega^6)^2 + (\omega^7)^2\big). \tag{6.25}$$

In particular, note that, under this flow (which exists for all time, both past and future), the volume of the metric increases without bound.

By the way, M cannot carry a metric with holonomy a subgroup of G_2 for the following reason: As Fernández shows, the first Betti number of M is 5. If there were a metric g on M with holonomy in G_2, then it would be Ricci-flat and hence the harmonic representatives of the first cohomology group would give five linearly independent g-parallel 1-forms on M. However, this would imply that the holonomy of M is trivial, which would imply that there exist seven linearly independent parallel 1-forms on M, which would in turn imply that the first Betti number was at least 7.

6.2.2. Further calculations

Return now to the flow of a general closed G_2-structure. Taking the exterior derivative of (6.13) yields

$$\frac{d}{dt}(\tau\wedge\sigma) = \tfrac{1}{3}\mathrm{d}\big(|\tau|^2\big)\wedge{*_\sigma}\sigma + \tfrac{1}{3}|\tau|^2\,\tau\wedge\sigma - \mathrm{d}{*_\sigma}\mathrm{d}\tau\,. \tag{6.26}$$

Expanding the left hand side of this equation and using (6.11) yields

$$\frac{d}{dt}(\tau)\wedge\sigma = \tfrac{1}{3}\mathrm{d}\big(|\tau|^2\big)\wedge{*_\sigma}\sigma + \tfrac{1}{3}|\tau|^2\,\tau\wedge\sigma - \mathrm{d}{*_\sigma}\mathrm{d}\tau - \tau\wedge\mathrm{d}\tau\,. \tag{6.27}$$

(This equation can be solved for the time-derivative of τ since wedging with σ is an isomorphism between Ω^2 and Ω^5.) Recalling that $\tau{\wedge}{*_\sigma}\sigma = 0$ and $\tau{\wedge}\tau{\wedge}\sigma = -|\tau^2|{*_\sigma}1$, this yields

$$\frac{d}{dt}(\tau)\wedge\tau\wedge\sigma = -\tfrac{1}{3}|\tau|^4\,{*_\sigma}1 - \tau\wedge\mathrm{d}{*_\sigma}\mathrm{d}\tau - \tau\wedge\tau\wedge\mathrm{d}\tau\,. \tag{6.28}$$

Finally, this can be used in the following computation

$$\begin{aligned}\frac{d}{dt}(|\tau^2| *_\sigma 1) &= \frac{d}{dt}(-\tau\wedge\tau\wedge\sigma) = -2\frac{d}{dt}(\tau)\wedge\tau\wedge\sigma - \tau\wedge\tau\wedge\mathrm{d}\tau \\ &= \tfrac{2}{3}|\tau|^4 *_\sigma 1 + 2\tau\wedge\mathrm{d}{*_\sigma}\mathrm{d}\tau + \tau\wedge\tau\wedge\mathrm{d}\tau \\ &= \left(\tfrac{2}{3}|\tau|^4 - 2|\mathrm{d}\tau|^2\right) *_\sigma 1 + \mathrm{d}\left(2\tau\wedge *_\sigma\mathrm{d}\tau + \tfrac{1}{3}\tau^3\right).\end{aligned} \tag{6.29}$$

Integrating this equation over M yields

$$\frac{d}{dt}\int_M |\tau^2| *_\sigma 1 = \int_M \left(\tfrac{2}{3}|\tau|^4 - 2|\mathrm{d}\tau|^2\right) *_\sigma 1. \tag{6.30}$$

This equation can be rewritten by using (4.39), which yields

$$\frac{d}{dt}\int_M (|\tau^2| *_\sigma 1) = \int_M \left(\tfrac{2}{3}|\tau|^4 - 2\left|\tfrac{3}{14}|\tau|^2\sigma + \tfrac{1}{2}*_\sigma(\tau\wedge\tau) - \tfrac{1}{2}\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big)\right|^2\right) *_\sigma 1. \tag{6.31}$$

Now, going back to (4.45) and integrating this over M yields

$$0 = \int_M \tfrac{2}{7}|\tau_2|^4 *_\sigma 1 - \tfrac{1}{2}\left({\tau_2}^2 + \tfrac{1}{7}|\tau_2|^2 *_\sigma\sigma\right)\wedge\mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big), \tag{6.32}$$

i.e.,

$$\int_M \left\langle *_\sigma\left({\tau_2}^2 + \tfrac{1}{7}|\tau_2|^2 *_\sigma\sigma\right), \mathsf{i}\big(\mathrm{Ric}^0(g_\sigma)\big)\right\rangle *_\sigma 1 = \frac{4}{7}\int_M |\tau_2|^4 *_\sigma 1. \tag{6.33}$$

Using this relation and the algebraic identities (2.30) and (2.28), one sees that (6.31) can be rewritten in the form

$$\frac{d}{dt}\int_M (|\tau^2| *_\sigma 1) = 4\int_M \left(\tfrac{11}{21}\,\mathrm{Scal}(g_\sigma)^2 - \left|\mathrm{Ric}^0(g_\sigma)\right|^2\right) *_\sigma 1. \tag{6.34}$$

This equation is suggestive. One of the reasons for wanting to study the Laplacian flow on closed G_2-structures is that it might provide a means of constructing metrics with holonomy G_2 by starting with a closed G_2-structure $\sigma \in \Omega^3_+(M)$ with 'sufficiently small' torsion and then running the Laplacian flow to move it closer to a G_2-structure that is both closed and coclosed.

However, if such a procedure is to work, then the volume function along the flow line must approach a constant and one would certainly expect the second derivative to become negative if the volume were to approach the 'local maximum' target volume. However, (6.34) shows that, in this case, the relative separation of the eigenvalues of the Ricci tensor cannot decrease too much during the flow. This 'forced separation' is somewhat stronger than the separation implied by Corollary 3.

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