Optimal system, symmetry reductions and new closed form solutions for the geometric average Asian options

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ABSTRACT

In this paper, the Lie group analysis method is applied to the geometric average Asian option pricing equation in financial problems. Firstly, the complete Lie symmetry group and infinitesimal generators of this equation are derived. Then the optimal system with one parameter for the Lie symmetry algebra are obtained, which gives the possibility to describe a complete set of invariant solutions to the pricing equation. Finally, based on the optimal system the symmetry reductions and corresponding closed form solutions for the pricing equation are proposed.

1. Introduction

An Asian option is a special type of option contract, which is an averaging option whose terminal payoff is determined by the average underlying price over some pre-set period of time [1–13]. Because of this fact, Asian options have a lower volatility and hence render them cheaper relative to the usual European option and American option, where the payoff of the option contract depends on the price of the underlying instrument at exercise. They are commonly traded on currencies and commodity products which have low trading volumes. Asian options are so called because they were introduced in Tokyo, Japan, in 1987 when Banker’s Trust Tokyo office used them for pricing average options on crude oil contracts. Thus Asian options are one of the basic forms of path-depending exotic options.

Asian options are broadly segregated into three categories: arithmetic average Asian options, geometric average Asian options and both these forms can be averaged on a weighted average basis. In general, there are not general explicit pricing formulae for the arithmetic averaged Asian options because the distribution of the arithmetic average of a set of lognormal distributions is not explicit. People think that the distribution of the geometric average of a set of lognormal distribution is also lognormal. So along this line Kemna and Vorst [6] and Conze and Viswanathan [7] obtain an explicit pricing formula for the Asian options on geometric average. Moreover, Bouaziz et al. [1] use a simple linearization procedure and propose an approximate closed-form solution to the pricing of “floating-strike” Asian options. The approximation suggested by Turnbull and Wakeman (TW) [8] makes use of the fact that the distribution under arithmetic averaging is approximately lognormal, and they put forward the first and second moments of the average in order to price the option. Caverhill and Clewlow [9] use the fast Fourier transform to obtain numerical approximations of the price of Asian options. Geman and Yor [10] propose an analytical study of Asian options. In particular, they characterize the case where an Asian call option price is higher than a standard European call option price. Alziary et al. [11] the Asian options analytically and numerical by a P.D.E. approach. Ju [12] produces an analytical approximation to price Asian options by assuming that even though the weighted average of
lognormal variables is not lognormal, one can still be able to approximate the weighted average by a lognormal variable if the first two moments of moments are true. Most recently, Devreese et al. [13] derive a closed-form solution for the price of an average strike as well as an average price geometric Asian option, by making use of the path integral formulation.

There are two main classes of Asian options, the average price options and the average strike options. The corresponding terminal call payoff for the two classes are \( \max[J_T - K, 0] \) and \( \max[S_T - J_T, 0] \) for a call option, respectively. Here, \( S_T \) is the asset price at expiry, \( K \) is the strike price and \( J_T \) denotes some form of average of the price of the underlying asset over the averaging period \( [0, T] \), either the arithmetical or geometrical average of the asset price. The value of \( J_T \) depends on the realization of the asset price path. The average price Asian options cost less than plain vanilla options, which are useful in protecting the owner from sudden short-lasting price changes in the market for example due to order imbalances.

The modern analysis of option pricing begins with the work of Black and Scholes [14] and Merton [15] in the early 1970s. The resulting model, called the Black–Scholes equation is a linear partial differential equation whose solution gives the fair price of a contingent claim. Under suitable assumptions, the Asian options can also be described by a linear partial differential equation [5]. There are two ways to solve the option pricing problems: numerical treatments [16–18] and analytical methods [19,18–21]. Recent years, there are an increasing number of researches about the use of symmetry analysis for the option pricing differential equations [20,21]. It is worth pointing out that the pioneering paper of Bordag and Chmakova [20] investigates the evaluation of an option hedge-cost under relaxation of the price-taking assumption by Lie group method. Specifically, they find some explicit solutions of the nonlinear Black–Scholes equation which incorporates the feedback-effect of a large trader in case of market illiquidity and show that these typical solutions would have a payoff which approximates a strangle. For a given differential equation, one first uses Lie point symmetry analysis to obtain its symmetry groups. Then under some mild conditions one can reduce the written equation for the invariant solution with respect to a subgroup. However, there is almost always an infinite number of the subgroups so we need an optimal system to classify all possible group-invariant solutions to the option pricing differential equations [20,21]. The optimal systems and their corresponding group-invariant solutions have been discussed for a number of partial differential equations [20–24].

In the present paper, we construct the group-invariant optimal system of the Asian option pricing equation, from which the interesting exact closed form solutions are obtained. The paper is organized as follows. We present the geometric average Asian option pricing Black–Scholes equation in Section 2. In Section 3, the complete Lie symmetry group and infinitesimal generators of this pricing equation are derived, and the optimal system with one parameter for the Lie symmetry algebra is given. In Section 4, the similarity variables and closed form solutions of the pricing equation are obtained by using the optimal system. The conclusions are stated in Section 5.

2. Geometric average Asian options

In this section we derive the governing differential equation for the price of a geometric average Asian option using the Black–Scholes approach. We consider a call option contract that is written at time \( \tau = 0 \). The holder of the contract will have a right to claim the difference between the average rate \( J_T \) and a strike price \( K \) at maturity \( T \), i.e., the payoff of this Asian call option is \( \max[J_T - K, 0] \). The average rate \( J_T \) is determined by the geometric mean of the underlying asset price

\[
J_T = \left( \prod_{i=1}^{n} S_{\tau_i} \right)^{1/n} = e^{\frac{1}{n} \sum_{i=1}^{n} \ln S_{\tau_i}}
\]  

for discrete geometric averaging and

\[
J_t = e^{\int_{\tau}^{t} \ln S_{\tau} d\tau}
\]  

for continuous geometric averaging, where \( S_{\tau_i} \) is the asset price at discrete time \( \tau_i \).

Let \( C(S, J, \tau) \) denote the value of the Asian option, which is a function of time and the two state variables, asset price \( S \) and average asset value \( J \). Consider a portfolio that contains one unit of the Asian option and \( -\Delta \) unit of the underlying asset. We then choose \( \Delta \) such that the stochastic components associated with the option and the underlying asset cancel each other out. Assume the asset price dynamics to be given by

\[
\frac{dS}{S} = \mu d\tau + \sigma dW,
\]

where \( W \) is the standard Brownian process, \( \mu \) and \( \sigma \) are the expected rate of return and volatility of the asset price, respectively. Let \( \Pi \) denote the value of the above portfolio, so the portfolio value is given by

\[
\Pi = C(S, J, \tau) - \Delta S
\]

and assuming \( \Delta \) to be kept instantaneously “frozen”. After considering the dividend yield \( q \) on the asset, the differential of \( \Pi \) can be found by Ito’s formula [5] as

\[
d\Pi = dC(S, J, \tau) - \Delta dS - q \Delta S d\tau = \left( \frac{\partial C}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - q \Delta S + \frac{\partial C}{\partial f} \frac{\partial f}{\partial \tau} \right) d\tau + \left( \frac{\partial C}{\partial S} \right) dS.
\]
where the term $-q\Delta Sd\tau$ corresponds to the contribution of the dividend dollar amount from the asset to the portfolio value. The absence of arbitrage shows

$$d\Pi = r\Pi d\tau = r(C(S, J, \tau) - \Delta S)d\tau,$$

(6)

where $r$ is the riskless interest rate. Thus let $\Delta = \frac{\partial C}{\partial S}$, the price of the geometric average Asian option can be determined by the following partial differential equation of $C(S, J, \tau)$ as

$$\frac{\partial C}{\partial \tau} + \frac{\partial C}{\partial J} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S \frac{\partial C}{\partial S} - rC = 0,$$

(7)

where $J = e^{\int f^J_{inS,dt}}$. Upon substituting $\frac{\partial C}{\partial S} = \frac{\ln J}{C0}$, the Asian option pricing Eq. (7) is written as

$$\frac{\partial C}{\partial \tau} + \frac{\ln J}{\tau} S - \ln J \frac{\partial C}{\partial J} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S \frac{\partial C}{\partial S} - rC = 0.$$

(8)

In order to simplify the pricing Eq. (8), we introduce the transformation

$$x = \frac{1}{\tau} \ln \frac{J}{S}, \quad t = T - \tau, \quad C = \tilde{U}(t, x)$$

(9)

to convert Eq. (8) into the following partial differential equation

$$F(t, x, u) = \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \frac{2}{2} (\rho + x) \frac{\partial u}{\partial x} + qu = 0,$$

(10)

with $\rho = \frac{1}{2} (r - q + \frac{\sigma^2}{\tau})$, which is a variable-coefficient linear heat equation.

If we can derive the explicit solutions $u(t, x)$ of the partial differential Eq. (10), we can obtain the explicit solutions $C(S, J, \tau)$ to the geometric average Asian option pricing Black–Scholes Eq. (8) from the following transformation

$$C(S, J, \tau) = \tilde{U}(T - \tau, \frac{1}{\tau} \ln \frac{J}{S}).$$

Thus in the next section we restrict ourself on studying the explicit solutions of the partial differential Eq. (10).

**Remark 1.** It is remarked that the variable-coefficient linear heat Eq. (10) can be transformed into the constant-coefficient heat equation

$$\frac{\partial U}{\partial t} + \sigma^2 \frac{\partial^2 U}{\partial x^2} + U = 0,$$

by the similarity transformation

$$u = \tilde{u}(\tilde{t}, \tilde{x}),$$

with $\tilde{u} = e^{\tilde{x}^2 \tilde{t} q - \tilde{t}}, \tilde{t} = C_0 \tilde{t}$ and $\tilde{x} = \sqrt{\frac{C_0}{\tilde{C}_0}} \tilde{t} (4x - 2q + 2r + \sigma^2)$. Here $C_0$ is a nonzero constant. Moreover, we also remark that Bordag et al. [25,26] have proposed a more advanced theory of group-invariant reductions to investigate the variable-coefficient nonlinear Black–Scholes equations, and obtained a complete set of possible reductions and families of exact solutions to the nonlinear models.

### 3. Lie symmetry analysis and optimal system

In this section, we propose the Lie symmetry group and one-dimensional optimal system of the partial differential Eq. (10). By Lie symmetry theory [22], the construction of the symmetry group is equivalent to determination of its infinitesimal transformation

$$\nabla = \xi \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial \tilde{t}} + \phi \frac{\partial}{\partial \tilde{u}},$$

(11)

with $\xi$, $\tau$ and $\phi$ being functions of the variables $(x, t, u)$. Thus the second prolongation $pr^{(2)}\nabla$ is

$$pr^{(2)}\nabla = \xi \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial \tilde{t}} + \phi \frac{\partial}{\partial \tilde{u}} + \phi^x \frac{\partial}{\partial \xi_x} + \phi^\theta \frac{\partial}{\partial \theta_{\tilde{u}}} + \phi^{xx} \frac{\partial}{\partial \xi_{xx}}.$$

where

$$\phi^j = D_j(\phi - \xi u_x - \theta u_t) + \xi u_{xx} + \theta u_{\tilde{u}x}, \quad j = \{t, x, \tilde{x}\},$$

with “D” a total derivative, i.e.
\[ D_4(P) = \frac{\partial}{\partial x} P + u_s \frac{\partial}{\partial u} P + u_{xx} \frac{\partial}{\partial u_{xx}} P + \cdots \text{ with } P = P(x, t, u, u_t, u_{xx}, \ldots). \]

Solving the invariance condition \( pr^2 \nabla \overline{F}(t, x, u)|_{\chi(t, x, u) = 0} = 0 \) [22] with \( F(t, x, u) \) in Eq. (10) we obtain the Lie algebra admitted by Eq. (10) and formulate the results in the following theorem.

**Theorem 1.** The partial differential Eq. (10) admits a six-dimensional Lie algebra with the following generators

\[
\begin{align*}
\nu_1 &= t^2 \frac{\partial}{\partial x}, \\
\nu_2 &= u \frac{\partial}{\partial u}, \\
\nu_3 &= t^3 \frac{\partial}{\partial x} + \frac{5(x + \rho)}{t^2 \sigma^2} u \frac{\partial}{\partial u}, \\
\nu_4 &= \frac{\partial}{\partial t} - \frac{1}{2}(x + \rho) \frac{\partial}{\partial u} - q t \frac{\partial}{\partial u}, \\
\nu_5 &= t^6 \frac{\partial}{\partial t} + 2 t^5 (x + \rho) \frac{\partial}{\partial x} - q t^6 \frac{\partial}{\partial u}, \\
\nu_6 &= t^4 \frac{\partial}{\partial x} - \frac{3(x + \rho)}{t^2} \frac{\partial}{\partial u} \left( \frac{25 \sigma^2 + 50 \sigma t \rho + 25 t \rho^2 + 2 t \sigma^2 q - 5 \sigma^2}{2 t^2 \sigma^4} \right) u \frac{\partial}{\partial u},
\end{align*}
\]

and infinite-dimensional subalgebra with generator \( \nu_7 = \psi(x, t) \frac{\partial}{\partial \psi} \) where \( \psi(x, t) \) being an arbitrary solution of Eq. (10) satisfying \( F(t, x, \psi(t, x)) = 0 \).

Let us consider a point transformation \( G : (t, x, u) \rightarrow (t', x', u') \) with \( u' = u(x', t') \) a solution of

\[
\frac{\partial u'(x', t')}{\partial t'} = \frac{1}{2} \sigma^2 \frac{\partial^2 u'(x', t')}{\partial x'^2} + \frac{2(\rho + x')}{t'} \frac{\partial u'(x', t')}{\partial x'} + q u'(x', t') = 0.
\]

Based on the six vector elements in (12) and infinite-dimensional generator \( \nu_7 = \psi(x, t) \frac{\partial}{\partial \psi} \), solving the following system of ordinary differential equations with initial condition

\[
\begin{align*}
\frac{dx'}{dc} &= \xi(x', t', u'), \\
\frac{dt'}{dc} &= \theta(x', t', u'), \\
\frac{du'}{dc} &= \phi(x', t', u'),
\end{align*}
\]

we have the corresponding seven one-parameter groups of symmetries for the Eq. (10) as

\[
\begin{align*}
G_1 : (t, x, u) &\rightarrow (t, x + \epsilon_1 t^2, u), \\
G_2 : (t, x, u) &\rightarrow (t, x, e^{\epsilon_2} u), \\
G_3 : (t, x, u) &\rightarrow (t, x + \epsilon_3 \frac{x + \rho + x^2 \rho}{t^2 \sigma^2}), \\
G_4 : (t, x, u) &\rightarrow (e^{\epsilon_4} t, e^{\epsilon_4 / 2}(\rho + x) - \rho, e^{-\epsilon_4}(e^{\epsilon_4} - 1)), \\
G_5 : (t, x, u) &\rightarrow \left( e^{\epsilon_5} \frac{(5 \sigma t + \rho)^{5/2}}{5 \sigma t^5 + 1}, \frac{\rho + x}{5 \sigma t^5 + 1} \right), \\
G_6 : (t, x, u) &\rightarrow (\sqrt[5]{5 \sigma t + \rho} + t^2 \rho + x^2 \rho, \frac{t}{5 \sigma t^5 + 1} - \rho \frac{u \sqrt{5 \sigma t + \rho} + t^2 \rho + x^2 \rho}{t^2 / 2}), \\
G_7 : (t, x, u) &\rightarrow (t, x, u + \epsilon_7 \psi(x, t)),
\end{align*}
\]

with \( \epsilon_j = 1, 2, \ldots, 7 \) being arbitrary constants.

Working on the generators \( \nu_i \) of the Lie-point transformations in (12), we can derive the exact solutions of the partial differential Eq. (10) via the symmetry reduction approach. We can also reduce the number of independent variables of the Eq. (10) by using the invariants associated with the subgroup of the symmetry group. In general, each subgroup corresponds a family of group-invariant solutions of the system. Since there are almost always an infinite number of subgroup, it is impossible to list all the group-invariant solutions. So it is necessary to find an “optimal system” of group-invariant solutions from which every other such solution can be obtained. Following the procedure in [22], we derive the optimal system of group-invariant solutions to the partial differential Eq. (10).
As is explained in [22], the infinite-dimensional subalgebra \(<\nu_i>\) does not lead to any group invariant solutions. Consequently it will not be considered in the classification problem. From now on we consider the Lie algebra \(g\) spanned by \{ \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6 \}. We will classify one-dimensional subalgebras of \(g\) up to the adjoint representation. Applying the commutator operators \([\nu_m, \nu_n] = \nu_m \nu_n - \nu_n \nu_m\), we get the commutator relations listed in Table 1 with the \((i,j)\)-th entry indicating \([\nu_m, \nu_n]\). Thus the following proposition holds.

**Theorem 2.** The generators \(\nu_i \ (i = 1, 2, \ldots, 6)\) in Theorem 1 form a Lie algebra \(g\), which is a six-dimensional symmetry algebra.

Using the commutator relations in Table 1 and the formula

\[
\text{Ad}(\exp(\epsilon V))W_0 = W_0 - \epsilon [V, W_0] + \frac{\epsilon^2}{2} [V, [V, W_0]] - \cdots,
\]

we obtain the adjoint representation in Table 2 with the \((i,j)\)-th entry indicating \(\text{Ad}(\exp(\epsilon \nu_i)) \nu_j\). Note that Eq. (13) is an infinite series, but after some calculations it is found that in all the adjoint representations of Lie algebra \(g\) the terms larger than \(\epsilon^3\) are zero after using the commutating relations in Table 1.

Here \(\alpha = 1 + \frac{25\epsilon^2}{4}, \beta = 1 + \frac{25\epsilon^2}{2}, \Gamma_1 = \nu_5 - 10\epsilon \nu_4 - \frac{25\nu_5 \nu_6}{2} + 25\epsilon^2 \nu_6, \Gamma_2 = \nu_6 + 10\epsilon \nu_4 + \frac{25 \epsilon}{2} \nu_2 + 25\epsilon^2 \nu_5\).

Let \(G\) be the Lie symmetry group of the Eq. (10) associated with the Lie algebra \(g\). For a given nonzero vector \(V = a_1 \nu_1 + a_2 \nu_2 + a_3 \nu_3 + a_4 \nu_4 + a_5 \nu_5 + a_6 \nu_6\), we try to simplify as many of the coefficients \(a_i\) as possible by applying the adjoint maps to \(V\). Note that the function \(f(V) = a_4^2 - 4a_5a_6\) is an invariant of the full adjoint action \(f(\text{Ad}(g)) = f(V), V \in g\). If we act on \(V\) by \(\text{Ad}(\exp(\epsilon \nu_4))\) and \(\text{Ad}(\exp(\epsilon \nu_5))\) then

\[
\tilde{V} = \sum_{i=1}^{6} \tilde{a}_i \nu_i = \text{Ad}(\exp(\epsilon \nu_4)) \cdot \text{Ad}(\exp(\epsilon \nu_5)) V,
\]

with coefficients

\[
\tilde{a}_4 = 10a_6 \epsilon \epsilon_1 - 50a_4 \epsilon_1 \epsilon_2 - 250a_6 \epsilon_1^2 \epsilon_2 + a_4 - 10a_5 \epsilon_2,
\]

\[
\tilde{a}_5 = 25a_6 \epsilon_1^2 + 5a_4 \epsilon_1 + a_5,
\]

\[
\tilde{a}_6 = 125a_4 \epsilon_1 \epsilon_2 - 50a_6 \epsilon_1 \epsilon_2 - 5a_4 \epsilon_2 + 625a_6 \epsilon_1^2 \epsilon_2 + 25a_5 \epsilon_1^2 + a_6.
\]

If \(f(V) > 0\), then we can choose

\[
\epsilon_1 = \frac{\sqrt{a_4^2 - 4a_5a_6} - a_4}{10a_6}, \quad \epsilon_2 = \frac{a_6}{2\sqrt{a_4^2 - 4a_5a_6}}.
\]

so that \(\tilde{a}_5 = \tilde{a}_6 = 0\) while \(\tilde{a}_4 = \sqrt{f(V)} \neq 0\), so \(V\) is equivalent to the vector \(\tilde{V} = \tilde{a}_1 \nu_1 + \tilde{a}_2 \nu_2 + \tilde{a}_3 \nu_3 + \nu_4 + \nu_5\). Acting further on \(\tilde{V}\) by the adjoint maps generated by \(\nu_1\) and \(\nu_3\) we can let the coefficients of \(\nu_1\) and \(\nu_3\) in vector \(V\) vanish. Thus the original vector \(V\) equivalent to \(\nu_4 + \nu_5 + \beta \nu_2\) for some \(\beta \in R\) under the adjoint representation.

If \(f(V) < 0\), then we can choose \(\epsilon_1 = 0, \epsilon_2 = \frac{a_4}{a_6}\) to make \(\tilde{a}_4 = 0\). It is easy to see that the coefficients of \(\nu_5\) and \(\nu_6\) are not zero in this case. Moreover, acting on \(\tilde{V}\) by the group generated by \(\nu_5\) and \(\nu_6\) we can make the coefficients of \(\nu_5\) and \(\nu_6\) to be 0, so vector \(V\) is equivalent to the vector \(\tilde{V} = \tilde{a}_1 \nu_1 + \tilde{a}_2 \nu_2 + \tilde{a}_3 \nu_3 + \nu_5 + \nu_6\). Acting further by the adjoint maps generated by \(\nu_1\) and \(\nu_3\) we find that the original vector \(V\) is equivalent to \(\nu_5 + \nu_6 + \beta \nu_2\) for some \(\beta \in R\).

Finally, if \(f(V) = 0\) we can simplify the vector \(V\) from two aspects. On one hand, if not all of the coefficients \(a_4, a_5, a_6\) are zero, we can choose \(\epsilon_1 = (a_4 - 10a_5 \epsilon_2)/(25a_4 \epsilon_2)\) in Eq. (16) so that \(\tilde{a}_4 = \tilde{a}_6 = 0\) and \(\tilde{a}_5 \neq 0\). Thus vector \(V\) is equivalent to the vector \(\tilde{V} = \tilde{a}_1 \nu_1 + \tilde{a}_2 \nu_2 + \tilde{a}_3 \nu_3 + \nu_5 + \nu_6\) generated by \(\nu_1, \nu_3, \nu_5, \nu_6\). Assuming \(\tilde{a}_3 \neq 0\) we can act on \(\tilde{V}\) by the adjoint maps generated by \(\nu_1, \nu_3, \nu_5, \nu_6\) to make \(\tilde{a}_3 = \tilde{a}_2 = 0\). Thus the vector \(V\) is equivalent to a multiple of \(\nu_5 \pm \nu_6\). If \(\tilde{a}_3 = 0\) we can act on \(\tilde{V}\) by \(\text{Ad}(\exp(\epsilon \nu_3))\) with \(\epsilon_3 = (a_1a_4 - 2a_5a_6)/(10a_4a_2)\) to reduce \(V\) to a vector of form \(\nu_5 + \gamma \nu_2\) for some \(\gamma \in R\). On the other hand, if \(a_4 = a_5 = a_6 = 0\), Table 1

<table>
<thead>
<tr>
<th>Lie</th>
<th>(\nu_1)</th>
<th>(\nu_2)</th>
<th>(\nu_3)</th>
<th>(\nu_4)</th>
<th>(\nu_5)</th>
<th>(\nu_6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu_1)</td>
<td>0</td>
<td>0</td>
<td>(\frac{5}{12})</td>
<td>(-\frac{5}{12})</td>
<td>0</td>
<td>5(\nu_3)</td>
</tr>
<tr>
<td>(\nu_2)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\nu_3)</td>
<td>(-\frac{5}{12})</td>
<td>0</td>
<td>0</td>
<td>(\frac{5}{12})</td>
<td>5(\nu_1)</td>
<td>0</td>
</tr>
<tr>
<td>(\nu_4)</td>
<td>(\frac{5}{12})</td>
<td>0</td>
<td>(-\frac{5}{12})</td>
<td>0</td>
<td>5(\nu_5)</td>
<td>(-5\nu_6)</td>
</tr>
<tr>
<td>(\nu_5)</td>
<td>0</td>
<td>0</td>
<td>(-5\nu_1)</td>
<td>(-5\nu_5)</td>
<td>0</td>
<td>(-10\nu_4 - \frac{25\nu_5}{2})</td>
</tr>
<tr>
<td>(\nu_6)</td>
<td>(-5\nu_1)</td>
<td>0</td>
<td>0</td>
<td>5(\nu_6)</td>
<td>10(\nu_4 + \frac{25\nu_5}{2})</td>
<td>0</td>
</tr>
</tbody>
</table>
the vector \( V = v_1 + v_2 + v_3 + v_4 \). If \( a_1 \neq 0 \), we can use the group generated by \( v_1 \) and \( v_6 \) to reduce \( V \) to a multiple of \( v_1 \). If \( a_1 = 0 \) and \( a_3 \neq 0 \), we can act on \( V = a_2 v_2 + a_3 v_3 \) by \( \exp(\epsilon v_2) \) with \( \epsilon \neq 0 \) to reduce \( V \) to \( 5a_3 v_1 + a_2 v_2 + a_3 v_3 \), which can also be reduced to a multiple of \( v_1 \) by the group generated by \( v_1 \) and \( v_6 \). If \( a_1 = 0 \) and \( a_3 = 0 \), then the vector \( V \) is reduced to a multiple of \( v_2 \). Thus any general vector is equivalent to one of the following vectors:

\[
\begin{align*}
\omega_1 &= v_4 + \alpha v_2, \\
\omega_2 &= v_5 + v_6 + \beta v_2, \\
\omega_3 &= v_5 + v_3, \\
\omega_4 &= v_5 - v_3, \\
\omega_5 &= v_5 + \gamma v_2, \\
\omega_6 &= v_1.
\end{align*}
\]

So we have the following theorem:

**Theorem 3.** The operators \( \{\omega_i\}_{i=1}^7 \) generate a one-parameter optimal system for the six-dimensional symmetry algebra \( \mathfrak{g} = \{v_1, v_2, v_3, v_4, v_5, v_6\} \).

### 4. Symmetry reductions and closed form solutions

Having determined the optimal system of the infinitesimals, we can find the group-invariant variables and closed form solutions of the partial differential Eq. (10) by solving the invariant surface condition [22]:

\[
0 = \frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = \phi.
\]

Making use of optimal system in Theorem 3, we derive several types of symmetry reductions and closed form solutions of Eq. (10) by solving the corresponding invariant surface condition and subsidiary equation.

Case 1. We first consider the vector \( \omega_1 = v_4 + \alpha v_2 \). In this case, the invariant surface condition is

\[
t \frac{\partial u}{\partial t} - \frac{1}{2} (x + \rho) \frac{\partial u}{\partial x} = -qt u + xu.
\]

Solving the invariant surface condition, we have

\[
u = F(\eta)(x + \rho)^{-2} e^{-\eta},
\]

where \( \eta = t(x + \rho)^2 \) and the similarity function \( F(\eta) \) satisfies the reduction equation

\[
(\sigma^2 \eta - 5 \eta^2 - 4 \sigma^2 \eta x) \frac{dF(\eta)}{d\eta} + x(2 \sigma^2 x + \sigma^2 + 4 \eta) F(\eta) + 2 \sigma^2 \eta^2 \frac{d^2F(\eta)}{d\eta^2} = 0.
\]

Solving this reduction equation, we derive the closed form solution of the partial differential Eq. (10) as

\[
u = e^{\frac{\eta x \sigma^2}{4 \sigma^2 t(x + \rho)^2} - \frac{1}{2} (x + \rho)^2} \left[c_1 M \left(\frac{1}{4} - \frac{x - \frac{1}{2} + 5(t(x + \rho)^2)}{2 \sigma^2}\right) + c_2 W \left(\frac{1}{4} - \frac{x - \frac{1}{2} + 5(t(x + \rho)^2)}{2 \sigma^2}\right)\right],
\]

where \( c_1, c_2 \) and \( x \) are real constants which can be determined by the boundary conditions, and \( M \) and \( W \) are WhittakerM(\( \mu, v, z \)) and WhittakerW(\( \mu, v, z \)) functions [27], respectively, which solve the second-order differential equation

\[
y''(z) + \left[-\frac{1}{4} + \frac{\mu}{2} + \frac{1/4 - \nu^2}{z^2}\right] y(z) = 0.
\]

Here the functions \( M \left(\frac{1}{4} - \frac{z}{\frac{1}{2} + \frac{5(t(x + \rho)^2)}{2 \sigma^2}}\right) \) and \( W \left(\frac{1}{4} - \frac{z}{\frac{1}{2} + \frac{5(t(x + \rho)^2)}{2 \sigma^2}}\right) \) are always positive for \( t > 0 \), and when \( \frac{5(t(x + \rho)^2)}{2 \sigma^2} \rightarrow +\infty \) function \( M \left(\frac{1}{4} - \frac{z}{\frac{1}{2} + \frac{5(t(x + \rho)^2)}{2 \sigma^2}}\right) \rightarrow +\infty \) while \( W \left(\frac{1}{4} - \frac{z}{\frac{1}{2} + \frac{5(t(x + \rho)^2)}{2 \sigma^2}}\right) \rightarrow 0 \). We can determine the values of \( c_1 \) and \( c_2 \) by using these properties along with the boundary conditions of the Asian option pricing model.

Case 2. We now consider the vector \( \omega_2 = v_5 + v_6 + \beta v_2 \). The invariant surface condition is

\[
(t^6 + t^4) \frac{\partial u}{\partial t} + (x + \rho) \left(2 t^2 - \frac{3}{3 t^2}\right) \frac{\partial u}{\partial x} = -qt u - \frac{u [25 t (x + \rho)^2 + \sigma^2 (2 t q - 5)]}{2 t^6 \sigma^2} + \beta u.
\]
which has the following solution for parameter $\beta = 0$

$$u = e^{\frac{5(\beta^2 - 4\beta^4)}{2\sigma^2(\sigma^2 - 4\beta^2)}} \frac{t^2}{\sqrt{t^4 + 1}} F(\eta),$$

where $\eta = \frac{e^{\frac{5(\beta^2 - 4\beta^4)}{2\sigma^2(\sigma^2 - 4\beta^2)}}}{\sqrt{t^4 + 1}}$, and the similarity function $F(\eta)$ satisfies the reduction equation

$$\frac{d^2 F(\eta)}{d\eta^2} + 25\eta^2 F(\eta) = 0.$$

Solving this reduction equation, we derive the closed form solution of the partial differential Eq. (10) as

$$u = e^{\frac{5(\beta^2 - 4\beta^4)}{2\sigma^2(\sigma^2 - 4\beta^2)}} \frac{t^2}{\sqrt{t^4 + 1}} \left[ c_1 \sqrt{\eta} \left( \frac{1}{4} \frac{5\eta^2}{2\sigma^2} \right) + c_2 \sqrt{\eta} \left( \frac{1}{4} \frac{5\eta^2}{2\sigma^2} \right) \right],$$

where $\eta = \frac{e^{\frac{5(\beta^2 - 4\beta^4)}{2\sigma^2(\sigma^2 - 4\beta^2)}}}{\sqrt{t^4 + 1}}$, $c_1$ and $c_2$ are real constants, and $J$ and $Y$ are BesselJ($\nu, z$) and BesselY($\nu, z$) functions [27], respectively, which solve the second-order differential equation

$$z^2y''(z) + zy'(z) - \left( z^2 - \nu^2 \right)y(z) = 0.$$

Here the functions $J\left( \frac{1}{4} \frac{5\nu^2}{2\sigma^2} \right)$ and $Y\left( \frac{1}{4} \frac{5\nu^2}{2\sigma^2} \right)$ have zero points for large $\frac{5\nu^2}{2\sigma^2}$ but they will not reach to zeros for $\frac{5\nu^2}{2\sigma^2} < 1$, i.e. for large $t$.

Case 3. If we consider the vector $\omega_3 = v_3 + \lambda v_3$ with $\lambda \neq 0$, the invariant surface condition is

$$t^6 \frac{\partial u}{\partial t} + 2t^5(x + \rho) \frac{\partial u}{\partial x} = -qt^6u + \frac{5\lambda(x + \rho)}{t^4\sigma^2} u,$$

which has solution as follows

$$u = e^{\frac{10t^5(x + \rho) - 2t^5(\beta(x + \beta))}{10t^5\sigma^2}} F(\eta),$$

where $\eta = \frac{10t^5(x + \beta)}{10t^5\sigma^2}$ and the similarity function $F(\eta)$ satisfies the reduction equation

$$\frac{d^2 F(\eta)}{d\eta^2} - 10\dot{\lambda} F(\eta) = 0.$$

Solving this reduction equation, we derive the closed form solution of the partial differential Eq. (10) as

$$u = e^{\frac{10t^5(x + \beta) - 2t^5(\beta(x + \beta))}{10t^5\sigma^2}} \left[ c_1 Ai\left( -\sqrt{-\frac{10\lambda t^2}{\sigma^4}} \right) + c_2 Bi\left( -\sqrt{-\frac{10\lambda t^2}{\sigma^4}} \right) \right],$$

where $\eta = \frac{10t^5(x + \beta)}{10t^5\sigma^2}$, $\lambda$ is a nonzero constant, $c_1$, $c_2$ are real constants, and $Ai(z)$ and $Bi(z)$ are Airy function and Airy function of the second kind [27], respectively, which solve the second-order differential equation

$$y''(z) - \gamma y(z) = 0.$$

Here $Ai(z)$ and $Bi(z)$ are always positive for $z > 0$, and when $z \rightarrow -\infty$ function $Ai(z) \rightarrow 0$ while $Bi(z) \rightarrow +\infty$. We can determine the values of $c_1$ and $c_2$ by using these properties along with the boundary conditions of the Asian option pricing model.

Case 4. Finally we consider the vector $\omega_4 = v_4 + \gamma v_2$, where the invariant surface condition is

$$t^6 \frac{\partial u}{\partial t} + 2t^5(x + \rho) \frac{\partial u}{\partial x} = -qt^6u + \gamma u,$$

which has the following solution

$$u = e^{\frac{5\gamma t^2}{2\sigma^2}} F(\eta),$$

where $\eta = \frac{5t^2}{2\sigma^2}$ and the similarity function $F(\eta)$ satisfies the reduction equation

$$\frac{d^2 F(\eta)}{d\eta^2} + 3\sigma^2 \eta^2 \frac{d^2 F(\eta)}{d\eta^2} - 8\gamma F(\eta) = 0.$$

Solving this reduction equation, we have two closed form solutions of the partial differential Eq. (10): one is

$$u = e^{\frac{5\gamma t^2}{2\sigma^2}} \left[ c_1 \sinh \left( \frac{\sqrt{2\gamma(x + \rho)}}{\sigma t^2} \right) + c_2 \cosh \left( \frac{\sqrt{2\gamma(x + \rho)}}{\sigma t^2} \right) \right],$$

for $\gamma > 0$, and the other is
\[ u = e^{-\gamma \sigma^2 t} \left[ c_1 \sin \left( \frac{\sqrt{-1} \gamma (x + \rho)}{\sigma t^2} \right) + c_2 \cos \left( \frac{\sqrt{-1} \gamma (x + \rho)}{\sigma t^2} \right) \right] \]  

(23)

for \( \gamma < 0 \) and \( c_1, c_2 \) are real constants.

5. Conclusion

In this paper, we have applied the Lie group analysis approach to the geometric average Asian option pricing Black–Scholes equation. As a result, a six-dimensional Lie symmetry algebra of this equation is obtained, from which the optimal system with one parameter is derived. With the help of the optimal systems, the symmetry reductions and new closed form solutions for the geometric average Asian option pricing equation are obtained. The resulting solutions can be used to model option pricing in finance market.

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