October 30, 2003 HUTP-03/A074
hep-th/0310272

Matrix Models, Geometric Engineering and Elliptic Genera

Timothy Hollowood[1], Amer Iqbal[2], Cumrun Vafa[2]

[1]Department of Physics,
University of Wales Swansea,
Swansea, SA2 8PP, UK.

[2]Jefferson Physical Laboratory,
Department of Physics,
Harvard University,
Cambridge, MA 02138, U.S.A.

Abstract

We compute the prepotential of $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions obtained by toroidal compactifications of gauge theories from 6 dimensions, as a function of Kähler and complex moduli of $\mathbf{T}^2$. We use three different methods to obtain this: matrix models, geometric engineering and instanton calculus. Matrix model approach involves summing up planar diagrams of an associated gauge theory on $\mathbf{T}^2$. Geometric engineering involves considering F-theory on elliptic threefolds, and using topological vertex to sum up worldsheet instantons. Instanton calculus involves computation of elliptic genera of instanton moduli spaces on $\mathbf{R}^4$. We study the compactifications of $\mathcal{N}=2^*$ theory in detail and establish equivalence of all these three approaches in this case. As a byproduct we geometrically engineer theories with massive adjoint fields. As one application, we show that the moduli space of mass deformed M5-branes wrapped on $\mathbf{T}^2$ combines the Kähler and complex moduli of $\mathbf{T}^2$ and the mass parameter into the period matrix of a genus 2 curve.

1 Introduction

String theory has been rather successful in providing insights into the dynamics of supersymmetric gauge theories in 4 dimensions. In particular essentially all questions involving vacuum geometry can be settled exactly for a large class of gauge theories; all the F-terms are exactly computable. Topological strings on Calabi-Yau geometries have played a key role in this regard. In particular consideration of type IIA (and topological A-model) strings on local Calabi-Yau threefolds leads to exact results, via geometric engineering, to questions involving a large class of $\mathcal{N} = 2$ supersymmetric gauge theories in 4 dimensions [1, 2]. Also consideration of type IIB (and topological B-model) geometries with wrapped and space-time filling branes leads to exact results for $\mathcal{N} = 1$ supersymmetric gauge theories [3, 4], which is also equivalent to the matrix model realization of a perturbative window into non-perturbative dynamics of these theories. This approach can also be used to address questions involving $\mathcal{N} = 2$ supersymmetric theories, as this is a special case of $\mathcal{N} = 1$ supersymmetric gauge theories. There has been another approach developed recently [5] for answering F-term questions involving $\mathcal{N} = 2$ supersymmetric gauge theories. This involves the development of an instanton calculus, and can be viewed as an efficient method to do the relevant integration over the instanton moduli space.

One can also ask questions about the dynamics of higher dimensional supersymmetric gauge theories, which will be the main focus of this paper. Moreover we will focus mainly on the overlap of these approaches that relate to theories with 8 supercharges. All these three approaches can be extended to higher dimensions and more specifically to dimensions 5 and 6. In the geometric engineering approach to go from $4 \to 5 \to 6$ one has to consider the chain of duality between type IIA on Calabi-Yau $\mathbf{X}$ with M-theory on $\mathbf{X} \times \mathbf{S}^1$ and F-theory on $\mathbf{X} \times \mathbf{T}^2$. The latter duality requires ellipticity of $\mathbf{X}$ [6–8] and this gets related to the fact that only special 6D gauge theories with 8 supercharges are anomaly free. In the Matrix model approach to go from $4 \to 5 \to 6$ one considers associated gauge theories in $0, 1, 2$ dimensions respectively, corresponding to geometry of point, $\mathbf{S}^1$, $\mathbf{T}^2$ [9]. In the instanton calculus approach one replaces (for the case of $\mathcal{N} = 2^*$) the measure from 1 to arithmetic genus χ and then to elliptic genus, in going from 4 to 5 and then 6 dimensions (the last point will be explained in this paper).

We will restrict to a special class of gauge theories, namely those which do exist as anomaly free theories in 6 dimensions. In particular we will focus mainly on $U(N)$ coupled with an adjoint matter, known as $\mathcal{N} = 2^*$; we also discuss as a further example how these generalize to the theory with $2N$ fundamental hypermultiplets in the terminology of $\mathcal{N} = 2$ supersymmetric theories in 4 dimensions.

In the course of implementing these ideas we solve a number of related problems: we find

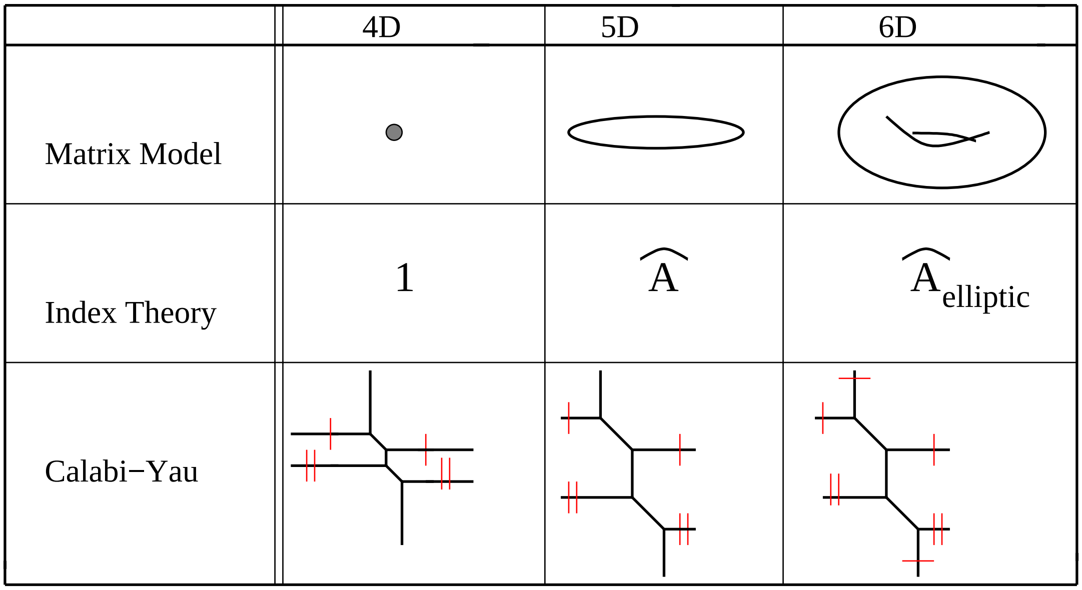

Figure 1: F-term computations for supersymmetric gauge theories from the view point of matrix models, instanton calculus and geometric engineering, in 4, 5 and 6 dimensions. The vertical and horizontal line segments on the external line of the web shown in the figure indicate gluing of those lines.

a nice way to summarize the integrality predictions [10, 11] of topological string free energy F in terms of the integrality of the partition function $Z = \exp F$. We also use the more refined information of instanton calculus [5] to shed light on the meaning of it in terms of curve counting for toric Calabi-Yau. We apply the topological vertex to double elliptically fibered Calabi-Yau (the possibility of doing this was noted in [12]) and in doing so we end up geometrically engineering theories with adjoint matter ($\mathcal{N} = 2^*$) on the one hand, and lifting theories from 5-dimensional M-theory, to 6 dimensional F-theory (with elliptic 3-folds) on the other. Moreover we show that, for the simplest gauge theory with gauge group $U(1)$, the relevant local model involves combining the Kähler class of the two elliptic fibrations as the elliptic moduli of the "two tori" of a genus 2 curve. In relating these to the instanton calculus approach we end up studying the (equivariant) elliptic genus on the moduli space of instantons on $\mathbf{R}^4$. For the case of $U(1)$ gauge theory this gets related to the elliptic genus for symmetric products of $\mathbf{R}^4$. Elliptic genera of symmetric products have been studied [13] and it turns out that there, the double ellipticity (coming from the elliptic genus on the one hand, and the parameter counting the number of copies of the symmetric product on the other) and the appearance of genus 2 curve was already apparent. The study of elliptic genera of symmetric products of instanton moduli spaces in [13] was motivated by the question of 5D black hole entropy [14] (see also [15]). As for the matrix model approach, going from $4 \to 5 \to 6$ involves changing the spectral plane from $\mathbf{C}$ to $\mathbf{C}^*$ and then to $\mathbf{T}^2$. In the case of the $\mathcal{N} = 2^*$ theory with gauge group $U(1)$, a genus 2 curve arises naturally as well.

The organization of this paper is as follows. In Section 2 we apply the matrix model techniques to study aspects of gauge theories in 5 and 6 dimensions. In Section 3 we review basic aspects of geometric engineering in 4, 5 and 6 dimensions, including theories with adjoint matter. In Section 4 we review topological A-model strings, and the integrality structure of its partition function. We also discuss how to use topological vertex to compute these amplitudes. In Section 5 we apply topological vertex techniques to calculate prepotentials for gauge theories in 4, 5 and 6 dimensions. As examples we take $\mathcal{N} = 2^*$ gauge theories as well as $U(N)$ theories with $2N$ fundamentals in these dimensions (for explicit example we take the cases of $N = 1, 2$). In Section 6 we review aspects of instanton calculus and apply it to the theories under consideration. We explain how elliptic genus of moduli space of instantons arises in studying gauge theory questions in 6 dimensions. In Section 7 we relate the $\mathcal{N} = 2^*$ theory lifted to 6 dimensions to the deformed theory of the M5-brane, or NS5-brane wrapped on $\mathbf{T}^2$, but deformed with a mass parameter. We discuss the implication of the appearance of the genus 2 curve from this perspective.

2 The matrix model approach

In this section we discuss how we can obtain results for prepotential of $\mathcal{N} = 2$ supersymmetric theories in 4 dimensions, obtained from compactification of gauge theories in 6 dimensions on $\mathbf{T}^2$ using matrix model techniques [16] adapted to higher dimensional gauge theories [9]. The idea is to consider deformations of $\mathcal{N} = 2$ theory by an $\mathcal{N} = 1$ preserving superpotential. This superpotential is just a convenience which allows one to probe a particular point on the Coulomb branch and at the end its strength may be taken to zero [17]. Thus we start with a gauge theory on $\mathbf{T}^2$ which encodes the superpotential of the corresponding $\mathcal{N} = 1$ theory, as in [9] and compute the glueball superpotential by studying the planar diagrams of that theory. We then extremize it to find the superpotential and the $U(1)$ gauge theory coupling constants which are encoded by the geometry (period matrix) of the resolvent curve. Since in this paper we would be mostly interested in the $\mathcal{N} = 2$ aspects of the theory, we will mainly keep track of the geometry of the curve because it is a feature that survives the limit when the superpotential is turned off and so pertains to the $\mathcal{N} = 2$ theory [17]. We will consider one main example with gauge group $U(N)$, to illustrate these ideas: $\mathcal{N} = 2^*$ (*i.e.* the $\mathcal{N} = 2$ theory with a massive adjoint hypermultiplet). Note that the choice of the gauge theory should be such that it is anomaly free for the 6 dimensional chiral theory and these two classes are consistent with that. These techniques can be easily generalized to many other examples, which we leave to the interested reader.

2.1 Engineering the Curve from the Matrix Model

In this section, we show how the curve for the six-dimensional theory can be engineered from a matrix model applying the techniques developed in [9,17]. More precisely, we will consider the six-dimensional $U(N)$ gauge theory with $\mathcal{N}=(1,1)$ supersymmetry compactified on a torus $\mathbf{T}^2$ defined by

$$\mathbf{T}^2=\big\{y\ \big|\ y\sim y+\frac{\beta}{2\mathrm{Im}\rho}(p+q\rho)\ ,\quad p,q\in\mathbf{Z}\big\}\ , \tag{2.1}$$

where β is a length scale and ρ is the complex structure of the torus. The effective theory in four dimensions will be the $\mathcal{N}=4$ gauge theory. However, we can also incorporate a mass for an adjoint hypermultiplet in the compactification. If $\mathbf{T}^2$ has finite volume then the effective theory in four dimensions with be generalization of the $\mathcal{N}=2^*$ theory involving all the Kaluza-Klein modes of the fields on the torus.

However, if we break this effective four-dimensional to $\mathcal{N}=1$ by adding an arbitrary superpotential for the one massless adjoint chiral multiplet then we can use the higher-dimensional generalization of the holomorphic matrix integral approach, described in [9], to find the effective superpotential. In other words, we need to generalize [18], which considered the five-dimensional lift of the $\mathcal{N}=1^*$ theory, to the six-dimensional lift. Correspondingly, we have to lift the matrix quantum mechanics to a two-dimensional matrix field theory, *i.e.* a two-dimensional gauge theory.

From the point-of-view of the effective four-dimensional theory there are 3 adjoint chiral fields Φ_i, $i=1,2,3$. One of the fields, say Φ_3, is now interpreted as the holomorphic component of the six-dimensional gauge field along the compactification torus. According to the general procedure of [9], after breaking to $\mathcal{N}=1$, the superpotential of the effective four-dimensional theory is determined by a two-dimensional gauge theory involving the fields $\Phi_i(y,\bar{y})$ and defined by the partition function

$$Z=\int\prod_{i=1}^{3}[d\Phi_i]\ \exp\big(-g_s^{-1}\int d^2y\,W(\Phi_i)\big)\ , \tag{2.2}$$

where g_s is a coupling constant. The action of the matrix model is a generalization of the one that describes the $\mathcal{N}=1^*$ deformation of the four dimensional theory [19,20]:

$$W(\Phi_i)=\mathrm{Tr}\big(\Phi_1D_{\bar{y}}\Phi_2+m\Phi_1\Phi_2+V(\Phi_3)\big) \tag{2.3}$$

where the covariant derivative is $D_{\bar{y}}\Phi_2=\partial_{\bar{y}}\Phi_2+[\Phi_3,\Phi_2]$. If we want to engineer the Seiberg-Witten curve of the six-dimensional theory on a torus, then the potential $V(\Phi_3)$ has to be chosen to be suitably generic in order that its critical points allow one to track across the

Coulomb branch of the $\mathcal{N}=2^*$. At the end, the strength of $V(\Phi_3)$ can then be taken to zero and results regarding the $\mathcal{N}=2^*$ theory are obtained. We will make a suitable choice for $V(\Phi_3)$ later.

In order to complete the description of the theory we need to specify the measure for the integrals in (2.2). Part of the matrix model approach involves interpreting the integrals in a holomorphic way. To be concrete, we can subject the matrices to particular reality conditions. In the present case, we take $\Phi_1^\dagger=\Phi_2$, or equivalently $\Phi_1+\Phi_2$ and $i(\Phi_1-\Phi_2)$ are Hermitian. In particular, the measure for the latter combination of fields is the appropriate measure for Hermitian fields. The gauge field component $\Phi_3(y,\bar{y})$ is treated in a somewhat different manner since it is the anti-holomorphic component of a gauge field on $\mathbf{T}^2$. First of all, local gauge transformations on the torus can be used to transform Φ_3 into a constant diagonal matrix:

$$UD_{\bar{y}}U^{-1}=\mathrm{diag}\big(\phi_1,\dots,\phi_N\big)\ . \tag{2.4}$$

This fixes all of the gauge group apart from permutations of the diagonal elements and large gauge transformations on the torus $\mathbf{T}^2$ in the abelian $U(1)^N$ subgroup. These latter group elements are

$$U_i=\exp\Big(\frac{2i\pi}{\beta}\big((p_i+\rho q_i)\bar{y}-(p_i+q_i\bar{\rho})y\big)\Big)\ ,\qquad p_i,q_i\in\mathbf{Z} \tag{2.5}$$

for $i=1\dots,N$. These transformations have the effect of shifting

$$\phi_i\to\phi_i+\frac{2\pi i}{\beta}(p_i+q_i\rho)\ . \tag{2.6}$$

In other words, the ϕ_i are naturally defined on the dual to the compactification torus which we denote $\tilde{\mathbf{T}}^2$:

$$\tilde{\mathbf{T}}^2=\big\{x\ \big|\ x\sim x+\frac{2\pi i}{\beta}(p+q\rho)\ ,\quad p,q\in\mathbf{Z}\big\}\ . \tag{2.7}$$

This torus also has a complex structure ρ.

Following the logic of [21,66,67], we integrate out the fields $\Phi_1(y,\bar{y})$ and $\Phi_2(y,\bar{y})$ since they appear Gaussian in (2.2) and gauge fix Φ_3 in the way described above. We end up with a (zero-dimensional) matrix integral involving the quantities ϕ_i:

$$Z=\int\prod_{i=1}^N d\phi_i\,\frac{\mathrm{Det}'(D_{\bar{y}})}{\mathrm{Det}(D_{\bar{y}}+m)}\exp\big(-g_s^{-1}v\sum_{i=1}^N V(\phi_i)\big)\ , \tag{2.8}$$

where v is the volume of $\mathbf{T}^2$. The determinant in the numerator is the gauge-fixing Jacobian while the one in the denominator arises from integrating out $\Phi_{1,2}$. For consistency, we now see that the probe potential $V(\phi)$ must respect the double-periodicity of the torus $\tilde{\mathbf{T}}^2$ (2.6):

$$V(x) = V(x + \frac{2\pi i}{\beta}(p + q\rho)) \ , \qquad p, q \in \mathbf{Z} \ . \tag{2.9}$$

It is straightforward to evaluate the ratio of determinants in (2.8). To start with, consider the simplified quantity

$$\mathrm{Det}(\partial_{\bar{y}} + C) \ , \tag{2.10}$$

where C is a constant. Take the eigenvalue equation

$$\big(\partial_{\bar{y}} + C\big)\psi(y, \bar{y}) = \lambda\psi(y, \bar{y}) \ . \tag{2.11}$$

The eigenvectors and eigenvalues can be found explicitly:

$$\psi(y, \bar{y}) = \exp\Big(\frac{2i\pi}{\beta}\big((p + q\rho)\bar{y} - (p + q\bar{\rho})y\big)\Big) \qquad p, q \in \mathbf{Z} \ , \tag{2.12}$$

and

$$\lambda = C + \frac{2\pi i}{\beta}(p + q\rho) \ . \tag{2.13}$$

Therefore the determinant, up to an infinite factor which will cancel between the denominator and numerator in (2.8), is

$$\mathrm{Det}(\partial_{\bar{y}} + C) \sim \prod_{p,q}\Big(C + \frac{2\pi i}{\beta}(p + q\rho)\Big) \ . \tag{2.14}$$

Using the identities

$$\sin x = x\prod_{n=1}^{\infty}\big(1 - \tfrac{x^2}{\pi^2 n^2}\big) \ , \qquad \theta_1(z|\tau) = q^{1/4}e^{iz}\prod_{n=1}^{\infty}(1 - q^{2n})(1 - q^{2n-2}e^{-2iz})(1 - q^{2n}z^{2iz}) \tag{2.15}$$

we can write the ratio of the determinants in (2.8) in terms of elliptic theta functions:

$$\frac{\mathrm{Det}'(D_{\bar{y}})}{\mathrm{Det}(D_{\bar{y}} + m)} \sim \frac{\prod_{i\neq j}\theta_1\big(\frac{\beta}{2i}(\phi_i - \phi_j)\big|\rho\big)}{\prod_{ij}\theta_1\big(\frac{\beta}{2i}(\phi_i - \phi_j + m)\big|\rho\big)} \ , \tag{2.16}$$

up to a ϕ_i independent multiplicative factor.

Now we are ready to perform a large-N saddle-point evaluation of the remaining matrix model around a critical point. In order to engineer the Seiberg-Witten curve for this theory,

$V(x)$ must have at least N critical points. Given this, one expands around a critical point where there is one eigenvalue ϕ_i in a subset of N of the critical points. We will make a convenient choice for $V(x)$ later. As usual in the matrix model we replace $N \to \hat{N}$ and introduce a degeneracy $\hat{N}_i$ at each of the N critical points inhabited by a field theory eigenvalue. We then take the limit $N_i \to \infty$, $g_s \to 0$ with $S_i = g_s\hat{N}_i$ fixed. In the large-$\hat{N}$ limit, the eigenvalues ϕ_i form a continuum and condense onto N open contours on the dual torus $\tilde{\mathbf{T}}^2$. We define these contour by specifying the end-points:

$$\mathcal{C}_i = [a_i, b_i] \ . \tag{2.17}$$

We also define the union

$$\mathcal{C} = \bigcup_{i=1}^{N} \mathcal{C}_i \ . \tag{2.18}$$

The configuration is described by the density of eigenvalues $\varrho(x)$, a function which has support only along the N contours, and which we normalize according to

$$\int_{\mathcal{C}} \varrho(x)\, dx = 1 \ . \tag{2.19}$$

The saddle-point equation is most conveniently formulated after defining the resolvent function

$$\omega(x) = \int_{\mathcal{C}} dy\, \varrho(y) \partial_x \log \theta_1\big(\tfrac{\beta}{2i}(x-y)\big|\rho\big) \ . \tag{2.20}$$

This function is a multi-valued function on the torus $\tilde{\mathbf{T}}^2$,

$$\omega(x + 2\pi i/\beta) = \omega(x) \ , \qquad \omega_2(x + 2\pi i\rho/\beta) = \omega(x) - \beta^{-1} \ , \tag{2.21}$$

except cuts along the N contours $\mathcal{C}_i$. The matrix model spectral density $\rho(x)$ is then equal to the discontinuity across the cut

$$\omega(x+\epsilon) - \omega(x-\epsilon) = 2\pi i \varrho(x) \ , \qquad x \in \mathcal{C} \ . \tag{2.22}$$

In this, and following equations, ϵ is a suitable infinitesimal such that $x \pm \epsilon$ lies infinitesimally above and below the cut at x. The saddle-point equation expresses the condition of zero force on a test eigenvalue in the presence of the large-N distribution of eigenvalues along the cut:

$$\frac{vV'(x)}{S} = \omega(x+\epsilon) + \omega(x-\epsilon) - \omega(x+m) - \omega(x-m) \ , \qquad x \in \mathcal{C} \ . \tag{2.23}$$

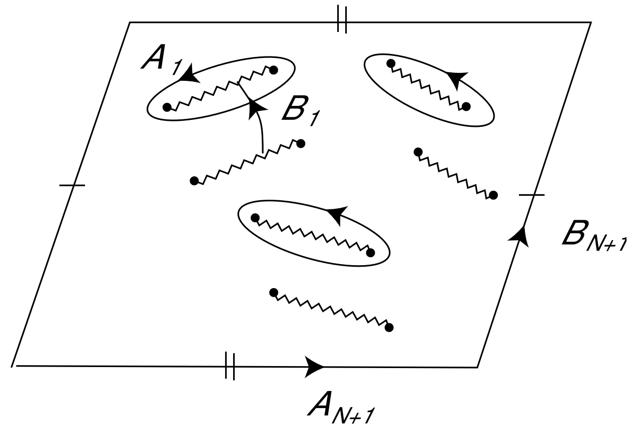

Figure 2: The cut torus on which $G(x)$ is defined for the case $N = 3$. Each pairs of cuts is identified. The cycles A_i and B_i, $i = 1, \dots, N$ are associated to each pair of cuts and A_{N+1} and B_{N+1} are the cycles of the torus $\tilde{\mathbf{T}}^2$.

This equation can be re-written in terms of the function

$$G(x) = U(x) + iS\big(\omega(x + \tfrac{m}{2}) - \omega(x - \tfrac{m}{2})\big) \ , \tag{2.24}$$

where $U(x)$ is determined by the finite-difference equation

$$U(x + \tfrac{m}{2}) - U(x - \tfrac{m}{2}) = ivV'(x) \ . \tag{2.25}$$

From its definition, one can see that $G(x)$ is now single-valued on $\tilde{\mathbf{T}}^2$ with N pairs of cuts

$$\mathcal{C}_i^{\pm} = [a_i \pm \tfrac{m}{2}, b_i \pm \tfrac{m}{2}] \ . \tag{2.26}$$

This is illustrated in Fig. 2. In terms of $G(x)$, the matrix model saddle-point equation (2.23) is

$$G(x + \tfrac{m}{2} \pm \epsilon) = G(x - \tfrac{m}{2} \mp \epsilon) \qquad x \in \mathcal{C} \ . \tag{2.27}$$

These equations can be viewed as conditions which glue the top (bottom) of $\mathcal{C}_i^+$ to the bottom (top) of its partner $\mathcal{C}_i^-$. This generates a handle as illustrated in Fig. 3. In other words $G(x)$ naturally defines a genus $N + 1$ Riemann surface $\Sigma_{\rm mm}$ defined as the torus $\tilde{\mathbf{T}}^2$ with N pairs of cuts $\mathcal{C}_i^{\pm}$ which are glued together in pairs to create N additional handles.

It appears that the resulting Riemann surface has $2N$ moduli provided by the positions $\{a_i, b_i\}$ of the ends of the cuts C_i. In fact, let us call $\mathcal{M}$ the moduli space of surfaces

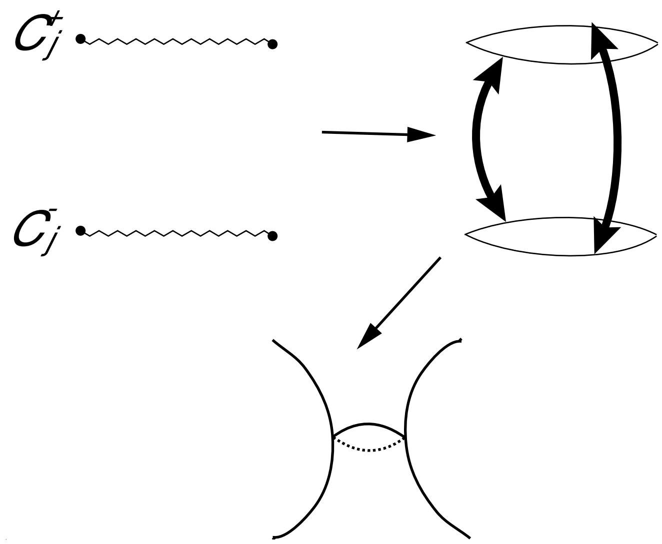

Figure 3: The top (bottom) of $\mathcal{C}_j^+$ is identified with the bottom (top) of $\mathcal{C}_j^-$. The figure shows how this generates a handle in the surface on which $G(x)$ is defined.

defined in this way with (complex) dimension $\dim \mathcal{M} = 2N$.[1] However, the requirement that a meromorphic function $G(x)$ exists on the surface with a suitable polar divisor actually means that the actual moduli space of the matrix model curve is only an N (complex) dimensional subspace $\mathcal{M}_{\rm mm} \subset \mathcal{M}$. To see this notice that $V'(x)$ has by hypothesis at least N zeros and hence a polar divisor of order at least N on the torus $\tilde{\mathbf{T}}^2$. This can be arranged, for example, by taking $U(x)$ to have a pole of order $N+2$ at a single point x_0 on the torus $\tilde{\mathbf{T}}^2$. It follows that $V'(x)$ will have a polar divisor of order $2N+4$ and hence have $2N+4$ zeros. The reason for this choice is purely based on convenience as will emerge shortly. We can write our choice for $U(x)$ explicitly as

$$U(x) = \frac{\prod_{l=1}^{N+2} \theta_1(\frac{\beta}{2i}(x-c_l)|\rho)}{\theta_1(\frac{\beta}{2i}(x-x_0)|\rho)^{N+2}} \ , \qquad \sum_{l=1}^{N+2} c_l = (N+2)x_0 \ . \tag{2.28}$$

Note that this function is single-valued on $\tilde{\mathbf{T}}^2$. With the above choice, $G(x)$ must also have a pole of order $N+2$ at x_0 on Σ. For generic x_0, the Riemann-Roch Theorem, guarantees that $G(x)$ will be unique up to an overall scaling. Hence, matching the singular part of $G(x)$ with $U(x)$ at x_0 leads to N conditions on the moduli of the surface. Consequently, the dimension of the moduli space of matrix model curves $\mathcal{M}_{\rm mm}$ is N as claimed. Of course, the same counting of moduli will also work for other choices of $U(x)$ for which $V'(x)$ has at least N zeros, but our choice was a convenient one.

[1] We count all dimensions as complex.

The N moduli of the surface are encoded in the quantities $S_j = g_s\hat{N}_j$ which can be expressed as the following contour integrals:

$$S_j = S\int_{\mathcal{C}_i} dx\,\varrho(x) = -\frac{S}{2\pi i}\oint_{A_j} dx\,\omega(x) = -\frac{1}{2\pi}\oint_{A_j} G(x)dx\ , \qquad j=1,\ldots,N\ , \tag{2.29}$$

where A_j encircles the cut $\mathcal{C}_j^+$ as in Fig. 2.

The other ingredient required to determine the glueball superpotential of the six-dimensional QFT compactified on the torus is the variation of the genus zero free energy $\mathcal{F}_0$ of the matrix model in transporting a test eigenvalue from infinity to one of the original N cuts $\mathcal{C}_j$. This is obtained by integrating the force on a test eigenvalue, which can be expressed in terms of the function $G(x)$ as

$$-i\big(G(x+\tfrac{m}{2})-G(x-\tfrac{m}{2})\big)\ , \tag{2.30}$$

from infinity to a point on the cut $\mathcal{C}_j$. This can be written as an integral of $G(x)$ itself along a contour starting at a point on the lower cut $\mathcal{C}_j^-$ going off to infinity and then back to a point on the upper cut $\mathcal{C}_j^+$. This can be deformed into the contour running from a point on $\mathcal{C}_j^-$ to the image point on C_j^+ related by a shift in x by m. Since the 1-form $G(x)dx$ is single-valued on $\Sigma_{\rm mm}$ this integral is in fact around the closed cycle B_j on $\Sigma_{\rm mm}$ conjugate to the cycle A_j defined above: see Fig. 2. Hence,

$$\partial\mathcal{F}_0\partial S_j = -i\oint_{B_j} G(x)dx\ , \qquad j=1,\ldots,N\ . \tag{2.31}$$

According to the matrix model approach [3,21,16,9], the effective glueball superpotential in this vacuum where the degeneracies are $N_i=1$ is given by

$$W_{\rm eff}(S_i) = \sum_{j=1}^{N}\Big(\partial\mathcal{F}_0\partial S_j - 2\pi i\tau S_j\Big)\ , \tag{2.32}$$

where τ is the usual complexified coupling of the supersymmetric gauge theory in four dimensions.

A critical point of $W_{\rm eff}(S_j)$ corresponds to

$$\sum_{j=1}^{N}\frac{\partial^2\mathcal{F}_0}{\partial S_k\partial S_j} = 2\pi i\tau \qquad k=1,\ldots,N\ . \tag{2.33}$$

This equation can be written in a more suggestive way by noticing that $\omega_j = -\frac{1}{2\pi}\frac{\partial}{\partial S_j}G(x)dx$ $j=1,\ldots,N$ are a subset of the holomorphic 1-forms on $\Sigma_{\rm mm}$. The reason is that the singular part of $G(x)dx$ at x_0 depends only on $U(x)$ and so is manifestly independent of the moduli $\{S_j\}$. Furthermore, the ω_j are normalized so that

$$\oint_{A_j} \omega_k = \delta_{jk} \ . \tag{2.34}$$

Hence

$$\frac{\partial^2 \mathcal{F}_0}{\partial S_k \partial S_j} = 2\pi i \oint_{B_j} \omega_k = 2\pi i \Pi_{jk} \ , \tag{2.35}$$

where Π_{jk} are elements of the period matrix of $\Sigma_{\rm mm}$ excluding the last row and column. Consequently the critical point equations are

$$\sum_{j=1}^{N} \Pi_{jk} = \tau \qquad k = 1, \dots , N \ . \tag{2.36}$$

Given that $\mathcal{M}_{\rm mm}$ is N-dimensional, these N conditions completely fix the geometry of the Riemann surface $\Sigma_{\rm mm}$ in terms of the parameters of the probe potential $V(x)$.

The remaining elements of the period matrix Π are fixed in the following way. Notice that the remaining holomorphic 1-form ω_{N+1} is identified with $\beta dx/(2\pi i)$ since

$$\oint_{A_j} dx = 0 \ , \quad j = 1, \dots , N \ , \quad \oint_{A_{N+1}} dx = \frac{2\pi i}{\beta} \ . \tag{2.37}$$

Hence

$$\Pi_{N+1,j} = \Pi_{j,N+1} = \oint_{B_j} \omega_{N+1} = \frac{\beta}{2\pi i} \int_{x-\frac{m}{2}}^{x+\frac{m}{2}} dx = \frac{\beta m}{2\pi i} \ , \quad j = 1, \dots , N \ , \tag{2.38}$$

while

$$\Pi_{N+1,N+1} = \oint_{B_{N+1}} \omega_{N+1} = \frac{\beta}{2\pi i} \int_0^{2\pi i \rho/\beta} dx = \rho \ . \tag{2.39}$$

Hence, the period matrix of $\Sigma_{\rm mm}$ at a critical point is

$$\Pi = \begin{pmatrix} \Pi_{11} & \cdots & \Pi_{1N} & \frac{\beta m}{2\pi i} \\ \vdots & \ddots & \vdots & \vdots \\ \Pi_{N1} & \cdots & \Pi_{NN} & \frac{\beta m}{2\pi i} \\ \frac{\beta m}{2\pi i} & \cdots & \frac{\beta m}{2\pi i} & \rho \end{pmatrix} \ , \quad \sum_{j=1}^{N} \Pi_{jk} = \tau \ . \tag{2.40}$$

2.2 Extracting the Seiberg-Witten curve

We now show how to extract the Seiberg-Witten curve for the compactified six-dimensional theory Σ. The idea is that this curve at some point in its moduli space is simply identified with the matrix model curve $\Sigma_{\rm mm}$. By changing the potential we can move around in the moduli space of the curve Σ. In other words, the Seiberg-Witten curves Σ are the curves in $\mathcal{M}$ subject to the N conditions (2.36).

The crucial observation is that the curve Σ admits the two multi-valued functions. Firstly, the critical point equations (2.33) imply that z defined by

$$z(P) = \int_{P_0}^{P} \sum_{j=1}^{N} \omega_N \ , \tag{2.41}$$

for an arbitrary point P_0, is a multi-valued function on $\Sigma_{\rm mm}$ with

$$\begin{array}{llll} A_j: \quad z \to z+1 \ , & B_j: \quad z \to z+\tau \ , & j=1,\ldots,N \\ A_{N+1}: \quad z \to z \ , & B_{N+1}: \quad z \to z + \frac{N\beta m}{2\pi i} \ . \end{array} \tag{2.42}$$

In addition to this we also have the multi-valued function x

$$x(P) = \frac{\beta}{2\pi i} \int_{P_0'}^{P} \omega_{N+1} \ , \tag{2.43}$$

defined with respect to some other, possibly different, base point P_0', with

$$\begin{array}{llll} A_j: \quad x \to x \ , & B_j: \quad x \to x+m \ , & j=1,\ldots,N \\ A_{N+1}: \quad x \to x + \dfrac{2\pi i}{\beta} \ , & B_{N+1}: \quad x \to x + \dfrac{2\pi i}{\beta}\rho \ . \end{array} \tag{2.44}$$

From these monodromy properties it follows that Σ is holomorphically embedded in a slanted 4-torus $\mathbf{T}^4$. Introducing complex coordinates for $\mathbf{C}^2$

$$z_1 = z \ , \qquad z_2 = \frac{\beta N}{2\pi i} x \ , \tag{2.45}$$

then we can write

$$\mathbf{T}^4 = \Big\{ z_i \in \mathbf{C}^2 \ \Big| \ z_i \sim z_i + \sum_{\alpha=1}^{4} \Omega_{i\alpha} p_\alpha \ , \quad p_\alpha \in \mathbf{Z} \Big\} \ , \tag{2.46}$$

where the 2×4-dimensional period matrix is

$$\Omega = \begin{pmatrix} 1 & 0 & \tau & \frac{N\beta m}{2\pi i} \\ 0 & N & \frac{N\beta m}{2\pi i} & N\rho \end{pmatrix} \ . \tag{2.47}$$

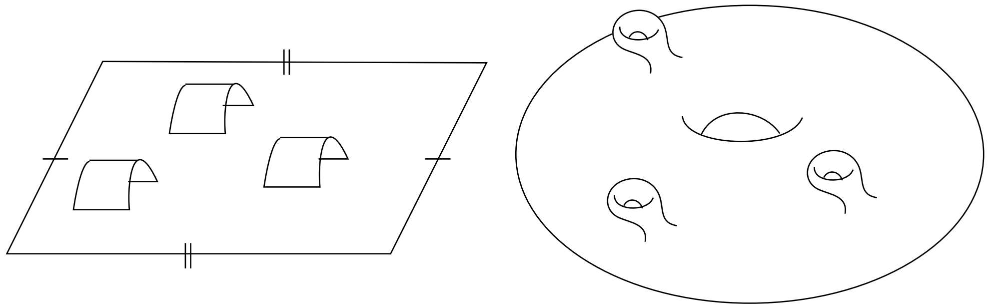

Figure 4: On the left, the surface Σ realized as the cut x-torus $\tilde{\mathbf{T}}^2$. The cuts in each of the N pairs are separated by m and are glued together as in Fig. 3. On the right, an impression of the surface realized as N handles on the x-torus.

In fact, the form of the period matrix implies that $\mathbf{T}^4$ is an abelian surface, or 2-dimensional abelian variety [22, 23].

We can picture the curve in two ways. Firstly, as already presented in the matrix model, as a torus in the x-plane with periods $(2\pi i/\beta, 2\pi i\rho/\beta)$ and with N pairs of cuts across which x jumps by $\pm m$ whose edges are identified to create a handle as in Fig. 3. This is illustrated in Fig. 4. The second representation consists of N copies of a torus in the z-plane with periods $(1,\tau)$ joined by $N-1$ branch cuts. On the face of it, such a surface would have genus N but on one of the sheets there is a pair of cuts across which z jumps by $\pm N\beta m/(2\pi i)$ whose edges are identified to create an extra handle. This is illustrated in Fig. 5.

Since $\mathbf{T}^4$ is an abelian surface, it turns out that there is an explicit realization of the curve in terms of generalized theta-functions associated to $\mathbf{T}^4$ [22]. In our conventions, these are defined as

$$\Theta\left[\delta\epsilon\right](Z|\Pi) = \sum_{m\in\mathbf{Z}^g} \exp\big(\pi i(m+\delta)\cdot\Pi\cdot(m+\delta) + 2\pi i(Z+\epsilon)\cdot(m+\delta)\big) \ . \tag{2.48}$$

In this definition, Z, δ, ϵ and m are g-vectors and Π is a $g\times g$ matrix. In the present case, $g=2$ and the curve can then be written as[2]

$$\sum_{j=0}^{N-1} A_j \Theta\begin{bmatrix} 0 & \frac{j}{N} \\ 0 & 0 \end{bmatrix}\left(z \quad \frac{N\beta x}{2\pi i}\Big|\begin{matrix}\tau & \frac{N\beta m}{2\pi i} \\ \frac{N\beta m}{2\pi i} & N\rho\end{matrix}\right) = 0 \ . \tag{2.49}$$

[2]For more details, see [24].

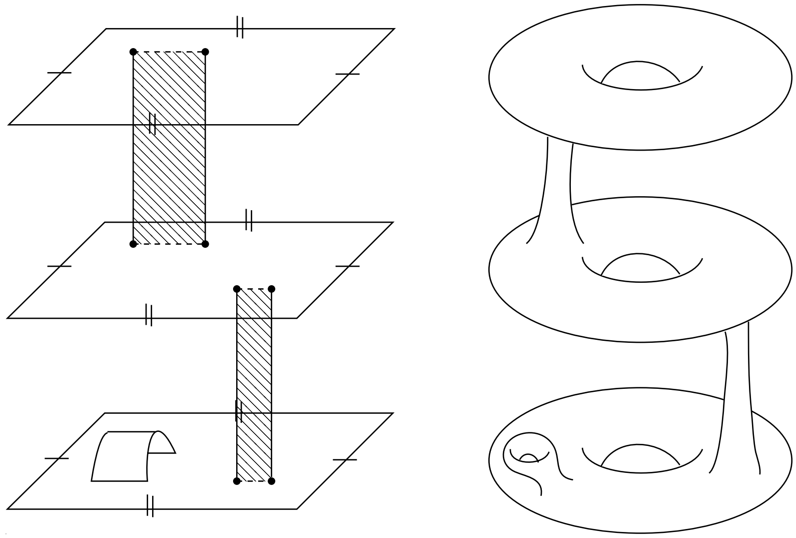

Figure 5: On the left, the surface Σ realized as N copies of the z-torus connected by $N-1$ branch cuts. On one of the sheets there is an additional pairs of cuts separated by $N\beta m/(2\pi i)$ which are glued together as in Fig. 3. On the right, is an impression of the surface illustrating the N copies of the z-torus plumbed together along with the additional handle on one of the sheets.

The coefficients A_j are moduli of the curve. Since the overall scale of the A_j is unimportant, the moduli are actually valued in $\mathbf{P}^{N-1}$. There are two other moduli corresponding to moving the curve as a whole in $\mathbf{T}^4$. In all there are $N+1$ moduli which matches the number of moduli of the matrix model curve when we include P_0 the arbitrary fixed point in the definition of z in (2.41). It can be shown that Σ is in the homology class dual to

$$N dy_1 \wedge dy_3 + dy_2 \wedge dy_4 \ , \tag{2.50}$$

where y_α are real coordinates, $0 \leq y_\alpha < 1$, with $z_i = \sum_{\alpha=1}^4 \Omega_{i\alpha} y_\alpha$. This is interpreted as meaning that the curve is wrapped N times around the z torus and once around the x torus, as in clear from Figs. 4 and 5. Similar curves which wrap k times around the x torus describe the $U(N)^k$ quiver theories in six dimensions. It is interesting to note that the construction of our curve is identical to the curve that appears in [25] describing instantons in non-commutative gauge theory on $\mathbf{T}^4$. The relation between the two problems can be made by compactifying our effective four-dimensional theory down to 3 dimensions, in other words the six-dimensional theory is on a 3-torus [26]. This is precisely the philosophy of [27] which formulates the problem of finding the vacuum states of the theory when broken to $\mathcal{N} = 1$ in terms of equilibrium configurations of an integrable system [28]. This line of thinking leads to the question of what integrable system lies behind the compactified six-dimensional theory which generalizes the N-body elliptic Calogero-Moser system, for the four-dimensional theory, and the N-body elliptic Ruijsenaars-Schneider system, for the compactified five-dimensional theory? It turns out that the resulting system is not the "doubly elliptic system" of [29, 30], rather it is an N-body system where the momenta and positions (q_i, p_i) as complex 2-vectors lie in the 4-torus $\mathbf{T}^4$ [24].

The form of the curve (2.49) can be re-caste in the following way which makes the reduction to five and four dimensions more immediate [24]:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{m}{2\pi i} \Big)^n \partial_z^n \theta_1 \big(\pi z \big| \tau \big) \partial_x^n H(x) = 0 \ , \tag{2.51}$$

where

$$H(x) = \prod_{j=1}^{N} \theta_1 \big(\tfrac{\beta}{2i} (x - \zeta_i) \big| \rho \big) \ . \tag{2.52}$$

Here, ζ_i are N of the $N+1$ moduli and the remaining one corresponds to shifting z by a constant. To go from the six to the five-dimensional curve one takes $\rho \to i\infty$ in which case

$$H(x) \to \prod_{j=1}^{N} \sinh \tfrac{\beta}{2} (x - \zeta_i) \ , \tag{2.53}$$

and from the five to the four-dimensional curve one takes $\beta \to 0$ giving rise to

$$H(x) \to \prod_{j=1}^{N} (x - \zeta_i) \ . \tag{2.54}$$

The curve of the four-dimension theory is identical to the curve described by Donagi and Witten [31]. It is well-known that this is the spectral curve of the N-body elliptic Calogero-Moser integrable system [32,33]. The curve of the five-dimensional theory can be shown to be the spectral curve of the Ruijsenaars-Schneider integrable system as predicted by Nekrasov [34]. The relation between this integrable system and the matrix quantum mechanical system has already been established in [18].

The form of the curve (2.51) is very natural from Type IIA/M Theory elliptic brane construction [35]. Using the representation

$$\theta_1(z|\tau) = \sum_{n \in \mathbf{Z}} (-1)^{n-1/2} e^{i\pi\tau(n+1/2)^2} e^{i(2n+1)z} \tag{2.55}$$

(2.51) can be written

$$\sum_{n \in \mathbf{Z}} (-1)^{n-1/2} e^{i\pi\tau(n+1/2)^2} e^{i(2n+1)\pi z} H(x + m(n+1/2)) = 0 \ . \tag{2.56}$$

For the four-dimensional case where $H(x) = \prod_{i=1}^{N}(x - \zeta_i)$ we recognize z and x with the spacetime coordinates as follows

$$z = (x_{10} + ix_6)/R_{10} \ , \qquad x = x_4 + ix_5 \ , \tag{2.57}$$

where R_{10} is the size of the M-theory circle. The parameters ζ_i are nothing but the positions of the N D4-branes and the curve takes account of the periodicity in the x_6 direction by including an infinite set images each shifted by an integer multiple of m which identifies m as the hypermultiplet mass. The five and six-dimensional curves result from compactifying $x = x_4 + ix_5$ on a circle and torus $\tilde{\mathbf{T}}^2$, respectively. The replacement $H_{4D}(x) \to H_{5D}(x) \to H_{6D}(x)$ takes account of the compactification by including all the images of the D4-branes.

To summarize, the Seiberg-Witten curve of the six-dimensional $\mathcal{N} = (1,1)$ theory compactified on a torus is a Riemann surface embedded holomorphic in the abelian surface. In the M-theory formulation, the M5-brane is wrapped on this Riemann surface.

3 Geometric engineering of gauge theories

Calabi-Yau manifolds have played an important role in the study of supersymmetric gauge theories in various dimensions. The geometry of CY3-folds has been the source of important insights for gauge theories. The geometries we will consider in this paper give rise to gauge theories with $U(N)$ gauge group and fundamental or adjoint hypermultiplets via geometric engineering as we will explain later. In this section we will review the geometric engineering of four [1,2], five [36–39] and six [7,8] dimensional SYM gauge theories with eight supercharges from CY3-folds. The basic idea is to use F-theory compactification on elliptic threefolds times $\mathbf{T}^2$, and its equivalence to M-theory on the 3-fold times an $\mathbf{S}^1$ and type IIA on the 3-fold. Moreover one has to choose special threefolds which admit appropriate loci of A_{N-1} singularities, to engineer $U(N)$ gauge theories with some matter content encoded by the geometry. We will also solve a puzzle in the geometric engineering approach by showing how theories with massive adjoint matter can be engineered. In order to motivate this it is convenient to also review the (p,q) 5-brane web construction of some of these theories [40] and how to realize adjoint matter in the brane constructions [35] and reading off the equivalent CY geometry from the resulting webs [39].

3.1 $\mathcal{N} = 4$ $D = 4$

Let us begin by considering the well known case of pure $U(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry. Type IIA superstrings in the background of A_{N-1} singularity inside a $K3$

realizes $U(N)$ gauge theory with 16 supercharges in 6 dimensions. The $D2$ branes wrapped over the 2-cycles of the blown up geometry realize the charged fields of the vector multiplet. Type IIB on the A_{N-1} singularity leads to tensionless strings and is equivalent to N copies of NS5-branes of IIA [41]. Now, consider compactifying type IIA strings in the background of A_{N-1} to 4 dimensions. Depending on how the A_{N-1} geometry is fibered over the extra 2 dimensions we get various kinds of gauge theories.

If we consider a trivial fibration on $\mathbf{T}^2$ we get $\mathcal{N}=4$ supersymmetric theory in 4 dimensions. The gauge coupling constant in 4 dimensions is given by the volume of $\mathbf{T}^2$. Note that this is also equivalent to type IIB on the same geometry by doing a T-duality on $\mathbf{T}^2$ exchanging Kähler and complex structures on $\mathbf{T}^2$. The Montonen-Olive duality is realized in this context by the modular group $SL(2,\mathbf{Z})$ acting on the complex structure of $\mathbf{T}^2$. Perhaps the most well known way to realize this theory is on a set of N coincident D3-branes in flat space. By a chain of dualities this configuration of D3-branes is related to the set of type IIA NS5-branes wrapped on a $\mathbf{T}^2$.

The prepotential of this 4D theory gets only classical contributions which, in terms of geometry of A_{N-1}, is proportional to the triple intersection numbers of the 4-cycles, which include 2-cycles of A_{N-1} times $\mathbf{T}^2$. To see this note that $H_2(A_{N-1},\mathbf{Z})$ is isomorphic to the root lattice of A_{N-1} Lie algebra. The holomorphic curves in A_{N-1} 2-fold are in one to one correspondence with positive roots of A_{N-1} algebra. Let us denote by a_i the moduli of the Coulomb branch such that $\sum_{i=1}^{N} a_i = 0$ and by $\phi_i = a_i - a_{i+1}$ the area of the curve F_i corresponding to the i^{th} simple root, $1 \leq i \leq N-1$. The intersection number of F_i is given by the Cartan matrix A_{ij} *i.e.* ,

$$F_i \cdot F_i = -2\,,\;\; F_i \cdot F_{i+1} = 1\,,\;\; i = 1, \ldots, N-2\,. \tag{3.1}$$

Then the prepotential is given by

$$\mathcal{F} = \tfrac{\tau}{2}\, F \cdot F\,,\;\; F = \sum_{i,j} \phi_i (A^{-1})_{ij} F_j\,. \tag{3.2}$$

Thus the geometry of the 2-fold encodes the prepotential is a simple way. This is also holds for $\mathcal{N}=2$ 4D theories: the classical contribution to the prepotential is given by classical intersection numbers of the CY geometry. In the case of $\mathcal{N}=4$ the classical result is exact.

3.2 $\mathcal{N}=2$, $D=4$ pure $SU(N)$ theory:

After this brief review of $\mathcal{N}=4$ theory let us consider $\mathcal{N}=2$ 4D pure $SU(N)$ theory. The engineering of an $\mathcal{N}=2$ $SU(N)$ gauge theory requires a singularity of A_{N-1} type to produce the appropriate gauge symmetry and another two dimensional space over which

A_{N-1} is fibered to get four non-compact dimensions. However, the 2D space cannot be arbitrary since the total space has to be CY3-fold. In the case of $\mathcal{N}=4$ this was $\mathbf{T}^2$ and the CY3-fold was a product $A_{N-1}\times\mathbf{T}^2$ space. To break supersymmetry down to $\mathcal{N}=2$ (eight supercharges) the surface should have no holomorphic one forms and therefore has to be a $\mathbf{P}^1$. However, the total geometry cannot be a product of A_{N-1} and $\mathbf{P}^1$ anymore since it is not Calabi-Yau threefold. To obtain a CY3-fold the A_{N-1} is fibered non-trivially over the $\mathbf{P}^1$. The details of the $\mathcal{N}=2$ theory obtained by type IIA compactification on such a CY3-fold depends on the way A_{N-1} is fibered over the $\mathbf{P}^1$. In the 4D field theory limit, which we will describe later, all such 3-folds give the same theory after appropriate identification of parameters.

This theory can also be realized using NS5-branes similar to the case of $\mathcal{N}=4$. In this case the NS5-branes are wrapped on $\mathbf{P}^1$ inside $T^*\mathbf{P}^1$ ($\mathcal{O}(-2)$ bundle over $\mathbf{P}^1$, (Fig. 6)).

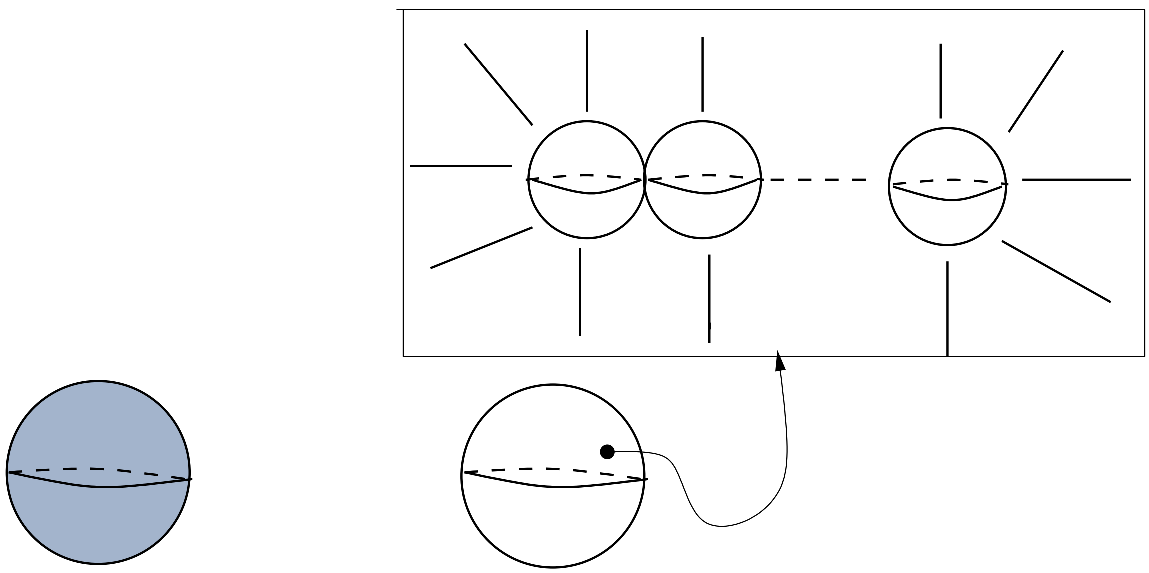

Figure 6: Realization of pure $\mathcal{N}=2$ D=4 theory on the worldvolume of N NS5-branes wrapped on a $\mathbf{P}^1$ (inside $T^*\mathbf{P}^1$) (a) and its dual description in terms of A_{N-1} fibered CY3-fold (b).

The four dimensional field theory limit is obtained by taking the string scale to infinity. By the relations of the base and fiber Kähler parameters to the gauge coupling and W-boson masses, these parameters must be scaled as [1]

$$Q_b := e^{-T_b} = \left(\frac{\beta\Lambda}{2}\right)^{2N}, \;\; Q_{F_i} := e^{-T_{F_i}} = e^{-\beta(a_i-a_{i+1})} \; i=1,\dots,N-1\,, \qquad (3.3)$$

where T_b denotes the volume of the base $\mathbf{P}^1$ and T_{F_i} denote the volumes of the fiber $\mathbf{P}^1$'s. Λ in the above denotes the quantum scale in four dimensions, a_i are the moduli of the Coulomb branch and the parameter β is introduced such that the four dimensional field theory limit corresponds to $\beta \to 0$.

The $\mathcal{N} = 2$ prepotential has both 1-loop perturbative and non-perturbative (instanton) contributions,

$$\mathcal{F} = \mathcal{F}_{\text{classical}+\mathcal{F}_{\text{1-loop}+\sum_{k=1}^{\infty} c_k(a_i)\Lambda^{2Nk}}} . \tag{3.4}$$

In the geometric engineering picture this is obtained from the genus zero topological string amplitude of the CY3-fold on which type IIA string theory is compactified. By considering (3.3), it then becomes clear that the k-th gauge instanton contributions stem from worldsheet instantons that wrap the base $\mathbf{P}^1$ of our geometries k-times. As we will discuss later the instanton contribution are encoded in the classical geometry of type IIB on the mirror Calabi-Yau, which in turn is equivalent to NS5-branes compactified on a Riemann surface Σ [42]. This can also be viewed, from the M-theory perspective as M5-brane with worldvolume of Σ. In the next section we will review the brane construction that directly leads to the M5-brane.

Brane description: The D-brane construction of this theory is well known [35] and involves D4-branes and NS5-branes. Consider type IIA string theory with two NS5-branes and N D4-branes. The NS5-branes are extended in the $x^{0,1,2,3,4,5}$ directions, being located at equal values in $x^{7,8,9}$ and separated in the x^6 direction by a distance L. The D4-branes span the $x^{0,1,2,3}$ and x^6 directions, being finite in the x^6 direction in which they are suspended between the NS5-branes as shown in Fig. 7. When the D4-branes are coincident the effective

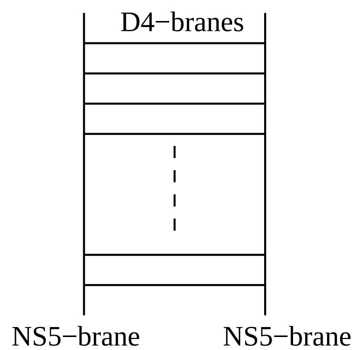

Figure 7: The brane configuration giving rise to four dimensional pure $\mathcal{N} = 2$ $U(N)$ theory.

worldvolume theory is $D = 4$ $\mathcal{N} = 2$ $SU(N)$ gauge theory with coupling constant given by $1/L$. The picture of the geometry shown above is only approximate and is a good description for small g_s since in this limit we can ignore the effect of D4-brane ending on the NS5-brane. In general the NS5-brane will get curved due to the D4-brane. In this limit it is more useful to lift the above configuration to M-theory and so that the above configuration of D4-brane and

NS5-branes becomes a single M5-brane wrapped on a Riemann surface (the Seiberg-Witten curve) of genus $N-1$ given by

$$y^2 = \prod_{i=1}^{N}(x-\phi_i)^2 - 4\left(\tfrac{\Lambda}{2}\right)^{2N}. \tag{3.5}$$

Five dimensional: As mentioned before in the field theory limit ($\beta \to 0$) all CY3-folds which are given by A_{N-1} fibration over $\mathbf{P}^1$ give the same four dimensional field theory. However, it is possible to distinguish between these different CY3-folds if we instead consider the five dimensional $SU(N)$ gauge theory obtained via M-theory compactification on the same Calabi-Yau times an $\mathbf{S}^1$. The parameter β gets identified with the perimeter of $\mathbf{S}^1$. Recall that in this case these different CY3-folds are distinguished by the Chern-Simons coefficient, k, of the five dimensional theory *i.e.* , the coefficient of the term

$$\int_{\mathbf{R}^5} \mathrm{Tr} A \wedge F \wedge F\,, \tag{3.6}$$

where A is the gauge field and F is the corresponding field strength. The cubic part of the prepotential of the five dimensional $SU(N)$ gauge theory with Chern-Simons coefficient k, in the limit $\beta \to \infty$, is given by the triple intersection numbers of the corresponding CY3-fold [38],

$$\mathcal{F}^{\mathrm{cubic}_{5D}=\frac{1}{6}\left[\sum_{i,j}\phi_i(A^{-1})_{ij}S_j\right]^3=\frac{k}{6}\sum_i a_i^3+\frac{1}{6}\sum_{i>j}|a_i-a_j|^3\,.} \tag{3.7}$$

S_i are the various non-trivial 4-cycles in the CY3-fold. The Chern-Simons coefficient takes values from $-N$ to $+N$. The geometry of the corresponding CY3-fold can be seen easily from the toric diagram or the corresponding dual web of (p,q) 5-branes of type IIB [40, 39] (Fig. 8). Recall that for non-compact toric threefolds the five dimensional theory obtained via M-theory compactification is dual to the five dimensional theory living on a (p,q) 5-brane web in type IIB. This is a consequence of the duality between M-theory on $\mathbf{T}^2$ and type IIB on a circle. Since the non-compact toric CY3-folds have a $\mathbf{T}^2$ fibration which degenerates on a planar tri-valent graph therefore using the M-theory/IIB duality adiabatically one can replace the degeneration locus with 5-branes. The (p,q) charge of the 5-brane is determined by the degenerate cycle of the $\mathbf{T}^2$. Holomorphicity of the CY3-fold implies that the orientation of the 5-brane is correlated with its charge *i.e.* , (p,q) 5-brane is oriented in the direction (p,q) (for type IIB coupling constant $\tau = i$). This web diagram can be obtained directly from the toric diagram as its dual.

As an example consider the case of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ over $\mathbf{P}^1$. This is a non-compact toric 3-fold with one Kähler parameter, the size of the $\mathbf{P}^1$, which we will denote as r. The linear

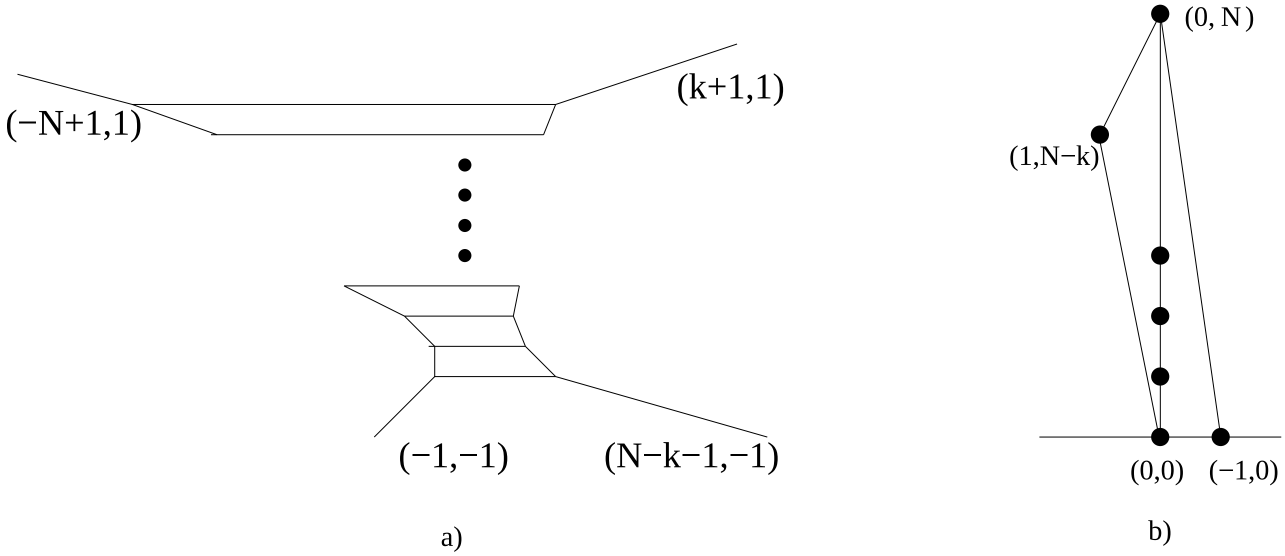

Figure 8: (p,q) 5-brane web (a), and the corresponding toric diagram (b), which realizes five dimensional pure $\mathcal{N}=1$ theory with Chern-Simons coefficient k.

sigma model description [43] of this geometry is given by

$$|\Phi_1|^2+|\Phi_2|^2-|\Phi_3|^2-|\Phi_4|^2=r\,,$$
$$(\Phi_1,\Phi_2,\Phi_3,\Phi_4)\sim(\Phi_1e^{i\alpha},\Phi_2e^{i\alpha},\Phi_3e^{-i\alpha}\Phi_4e^{-i\alpha})\,. \tag{3.8}$$

The base of this geometry parameterized by $(|\Phi_1|^2,|\Phi_3|^2,|\Phi_4|^2)$ is shown in Fig. 9(a). The base is the three dimensional convex region bounded by the planes $P_{1,2,3,4}$.These 2 dimensional planes P_1,P_2,P_3,P_4 are given by $\Phi_4=0,\Phi_2=0,\Phi_3=0$ and $\Phi_1=0$ respectively. We can project this geometry onto a two dimensional plane and since the locus where the various planes intersect each other has a degenerate $\mathbf{T}^2$ we see that in the two dimensional plane the (p,q) cycle of the $\mathbf{T}^2$ fibration degenerate over line which is projection of the intersection of two planes and is oriented in the (p,q) direction. This is the corresponding web diagram and is shown in Fig. 9(b).

The various geometries which give rise to $SU(3)$ five dimensional gauge theory are shown in Fig. 10(a).

For a more concrete connection with the four dimensional gauge theory consider compactifying the five dimensional theory on a circle of radius β. Then for small β it is more useful to lift the web of 5-branes to M-theory on $\mathbf{T}^2$ (with the area of $\mathbf{T}^2$ equal to $1/\beta$) such that the web becomes a single M5-brane wrapped on a Riemann surface Σ. The Riemann surface Σ is embedded in $\mathbf{R}^2\times\mathbf{T}^2$ where $\mathbf{R}^2$ is the plane in which the original 5-brane web lived and $\mathbf{T}^2$ is dual to the circle of type IIB. The Riemann surface Σ is given by just thickening

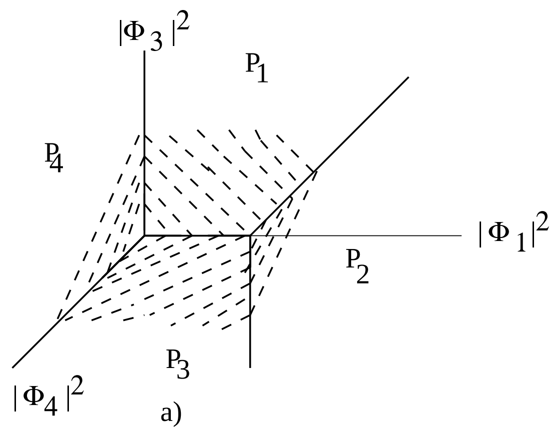

Figure 9: The linear sigma model description of the base of the deformed conifold in $\mathbf{R}_+^3$ (a) and the corresponding web in in $\mathbf{R}^2$ (b).

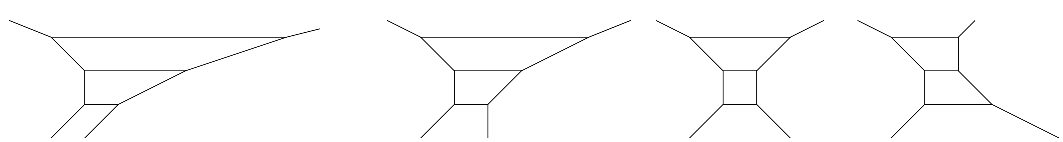

Figure 10: The web diagram of various CY3-fold geometries which realize the five dimensional pure $U(3)$ theory.

the original graph of the web and its equation can be read easily from the toric diagram,

$$Y + \alpha \, \tfrac{X^{N-k}}{Y} + P_N(X) = 0 \,, \ \ X, Y \in \mathbf{C}^* \,, \tag{3.9}$$

where $P_N(X)$ is a polynomial of degree N. This Riemann surface actually is the non-trivial part of the CY3-fold which is mirror to the CY3-fold which geometrically engineers the five dimensional theory via M-theory compactification. To see this note that the mirror of the CY3-fold [42, 44, 45] with toric diagram given by Fig. 8(b) is

$$F(X,Y) := Y + e^{-t_B} \tfrac{X^{N-k}}{Y} + P_N(X) = uv \,, \ \ X, Y \in \mathbf{C}^* \,, \ u, v \in \mathbf{C} \,. \tag{3.10}$$

The Calabi-Yau can be viewed as $\mathbf{C}^*$ fibration over the X, Y plane where the circle fibration degenerates over the Riemann Surface $F(X,Y) = 0$. The periods of this CY3-fold reduce to integrals of a 1-form over 1-cycles of Σ. The complex structure parameters of the mirror CY, the complex coefficients in the above equation, are related to the Kähler parameters t_B (size of the base $\mathbf{P}^1$) and t_{F_i} (size of the i-th fiber $\mathbf{P}^1$). The geometry of the degeneration of the circle fibration maps this to the geometry of type IIA NS5-brane wrapped on the same Riemann surface [42], which can then be lifted up to an M-theory M5-brane. Thus we see that the M5-brane wrapped on the mirror Riemann surface gives the compactified five dimensional theory. This Riemann surface becomes the Seiberg-Witten curve if we redefine the variables X, Y suitably in the field theory limit. For this note that we can write the equation of the mirror Riemann surface as

$$y^2 = \prod_{i=1}^{N} (X - e^{\phi_i})^2 - 4e^{-t_B} \, X^{N-k} \,. \tag{3.11}$$

Now define

$$X = e^{\beta x} \,, \ \ e^{\phi_i} = e^{\beta(a_i - a_{i+1})} \,, \ e^{-t_B} = (\tfrac{\beta\Lambda}{2})^{2N} \,. \tag{3.12}$$

then in the limit $\beta \to 0$ (4D limit) the equation of the mirror Riemann surface becomes the equation for the SW curve,

$$y^2 = \prod_{i=1}^{N} (x - a_{i,i+1})^2 - 4(\tfrac{\Lambda}{2})^{2N} \,. \tag{3.13}$$

Note that since X was a $\mathbf{C}^*$ variable, x takes value on a cylinder of radius $1/\beta$. Thus in the limit $\beta \to 0$, x becomes a $\mathbf{C}$ variable. Also the integer k has disappeared from the equation reflecting the fact that in this limit the geometries becomes equivalent. From Eq. (3.12) it is clear that in the limit of four dimensional theory the base $\mathbf{P}^1$ grows and the fiber $\mathbf{P}^1$ shrink with a scaling given by (3.3).

3.3 $\mathcal{N}=2$, $SU(N)$ with $N_f=2N$

Once pure $SU(N)$ gauge theory has been engineered it is relatively simple to modify the CY3-fold geometry to include hypermultiplets in the fundamental representation. A fundamental hypermultiplet of mass m appears in the gauge theory if we blowup the CY3-fold of the pure $SU(N)$ gauge theory such that the mass of the hypermultiplet m is proportional to the area of the blown up curve.

Let us begin by considering the case of 5-dimensional $U(1)$ theory with $N_f = 2$. The geometry giving rise to pure $U(1)$ theory is that of resolved conifold *i.e.* , the total space of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ bundle over $\mathbf{P}^1$. The size of the $\mathbf{P}^1$, t, is proportional to the length of the internal line as shown in Fig. 11(a) and is inversely proportional to the gauge theory coupling constant. To introduce fundamental hypermultiplets we need to blow up this geometry at two points as shown in Fig. 11(b). In this case the 4 dimensional field theory limit is given

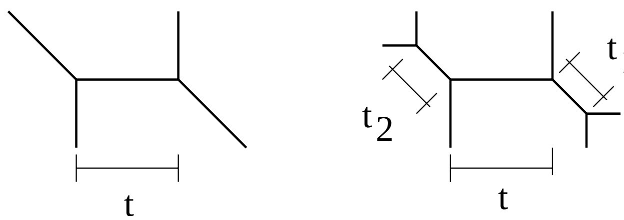

Figure 11: a) $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ bundle over $\mathbf{P}^1$, b) blow up of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ bundle over $\mathbf{P}^1$ at two points.

by

$$e^{-t_a} = e^{-\beta m_a}\,,\;\; q = e^{-\beta}\,,\;\; \beta \to 0\,, \tag{3.14}$$

where $m_{1,2}$ is the mass of the two hypermultiplets in the 4 dimensional field theory and $t_{1,2}$ is the area of the blown up rational curves. If we do not take this limit we obtain five dimensional theory compactified on a circle of size β.

Generalization to 5-dimensional $U(N)$ with matter is straightforward. All we have to do is blow up the geometry giving rise to pure $U(N)$ theory at N_f points as shown in Fig. 12(b) for the case $N_f = 2N$. The corresponding (p,q) 5-brane web in type IIB is shown in Fig. 12(a) and consists of intersecting D5-branes and NS5-branes. This is the case when the hypermultiplets have zero mass. To introduce non-zero mass one has to resolve the intersection locus of D5-brane and the NS5-brane. In terms of CY3-fold geometry each exceptional curve generated by the blow up gives rise to a fundamental hypermultiplet such that the mass of the hypermultiplet is proportional to the area of the corresponding curve. The limit of 4 dimensional field theory is given by

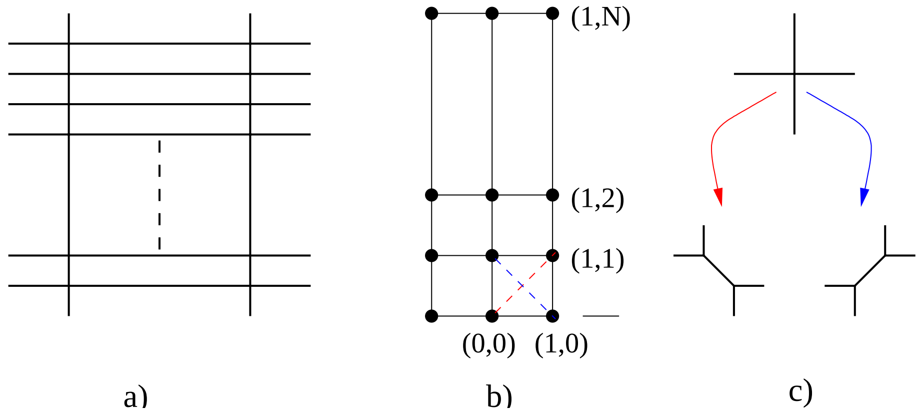

Figure 12: a) The web diagram of the geometry relevant for five dimensional $U(N)$ theory with $N_f = 2N$, b) the corresponding toric diagram and c) the flop transition reflecting the choice of the triangulation of the toric diagram.

$$e^{-T_b} = (\tfrac{\beta\Lambda}{2})^{2N-N_f}\,, \;\; e^{-T_{F_i}} = e^{-\beta(a_i - a_{i+1})}\,, \;\; e^{-t_a} = e^{-\beta m_a}\,, \;\; a = 1, \ldots, N_f\,, \;\; \beta \to 0\,, \quad (3.15)$$

where t_a is the area of the a^{th} exceptional curve generated by the blow up.

From the toric diagram, Fig. 12(b), it is clear that there are $4N$ distinct triangulations of the diagram. Different triangulations of the toric diagram give geometries related to each other by flop transitions. In the web description this corresponds to two possible resolutions into tri-valent graph at each of the $2N$ points where the $(1,0)$ line meets the $(0,1)$ line. In the gauge theory language this is given by the choice of the sign of the mass term. Thus for zero size of the blown up $\mathbf{P}^1$ the geometry is unique. The case of $SU(2)$ is illustrated in Fig. 13. The various geometries which give rise to $SU(2)$ theory (Fig. 13(a)) are related to the same

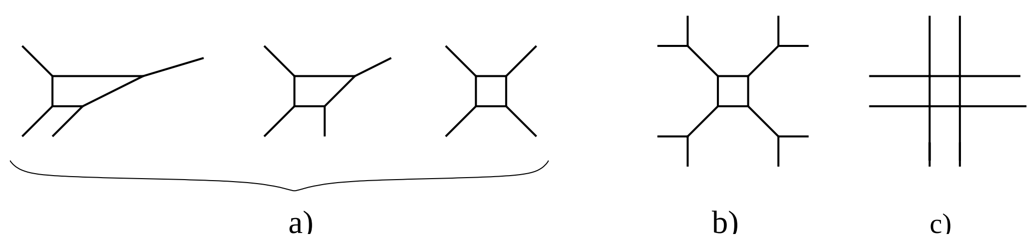

Figure 13: a) The web diagrams of the CY3-fold which realize pure 5D $U(2)$ theory, b) its blow up at four points for a particular choice of the triangulation and c) the unique web diagram obtained by blowing up four points with zero area.

geometry (Fig. 13(b)) by flop transitions once four points have been blown up. In all these case the classical part of the prepotential is given by the triple intersection numbers of the corresponding CY3-fold.

6D Theory: In the previous section we saw that 4D theory can be obtained as a limit of the 5D theory. A natural question that arises here is whether the 5D theory can be obtained as some limit of a compactified 6D theory. One way of answering this is to consider F-theory on $\mathbf{X} \times \mathbf{T}^2$. But for this to work the CY3-fold $\mathbf{X}$ must be an elliptic fibration with a section. This is related to the fact that the 6D theory must be anomaly free and therefore must have a very specific matter content. For the case of $U(N)$ theory with N_f fundamental hypermultiplets it requires that $N_f = 2N$. The elliptic 3-folds which give rise to these theories where constructed in [46].

To construct the non-compact 3-folds with elliptic fibration relevant for $U(N)$ theory with $N_f = 2N$ let us first consider the simple geometry of the deformed conifold given by

$$x_1 x_2 - x_3 x_4 = \varepsilon \,. \tag{3.16}$$

We can write the above as

$$x_1 x_2 = z \,, \;\; x_3 x_4 = z - \varepsilon \,. \tag{3.17}$$

Then we see that the deformed conifold is given by two $\mathbf{C}^*$ fibrations over the complex z-plane as shown in Fig. 14(a). The $\mathbf{S}^3$ in the geometry, formed by the two circles in the $\mathbf{C}^*$ fibration and the line segment joining the points $z = 0$ and $z = \varepsilon$, has size given by ε. In terms of the (p, q) 5-brane web the $\varepsilon \neq 0$ deformation is given by separating D5-brane and the NS5-brane from each other by a length ε as shown in Fig. 14(b).

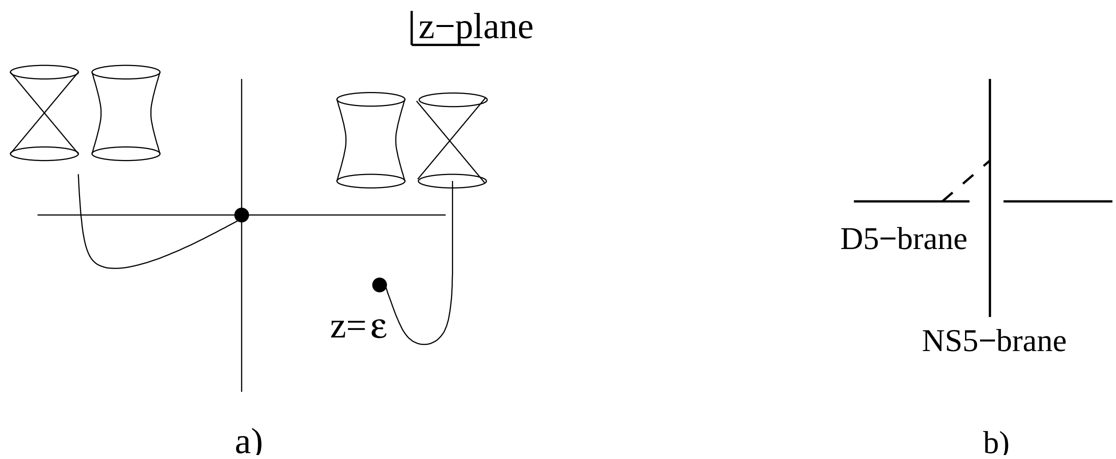

Figure 14: a) The deformed conifold as the double $\mathbf{C}^*$ fibration over the z-plane and b) the corresponding 5-brane web configuration

Now given this picture of the geometry in terms of two $\mathbf{C}^*$ fibration it is easy to see that one can get an elliptically fibered CY3-fold by compactifying one of the $\mathbf{C}^*$ fibers to a $\mathbf{T}^2$, as shown in Fig. 15(a), such that

$$\begin{aligned} x_1 x_2 &= z\,, \\ y^2 &= x^3 + f(z,\varepsilon)x + g(z,\varepsilon)\,. \end{aligned} \tag{3.18}$$

The second equation above defines the elliptic fibration over the z-plane which degenerates at $z = \varepsilon$. The corresponding type IIB configuration of 5-branes is now such that the NS5-brane is wrapped on a circle and for $\varepsilon = 0$ the D5-brane intersects the circle as shown in Fig. 15(b). The circle on which the NS5-brane is wrapped is exactly the circle created by the compactification of the $\mathbf{C}^*$ fiber to $\mathbf{T}^2$ and its size is related to the Kähler class of the compactified $\mathbf{T}^2$. Now given this web description we can consider two NS5-branes wrapped

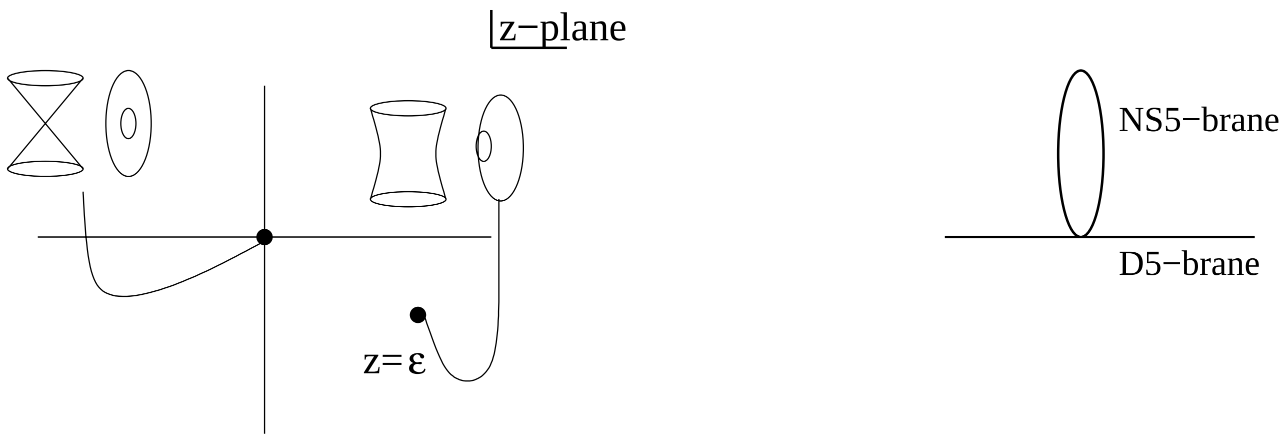

Figure 15: a) The partial compactification of the deformed conifold geometry by replacing one of the $\mathbf{C}^*$ with a $\mathbf{T}^2$ and b) the corresponding 5-brane web description.

on the circle but separated in along the D5-brane. In this case it is easy to see that when the radius of the circle goes to infinity we get 5-dimensional $U(1)$ theory with $N_f = 2$ living on the D5-brane and intersecting the two NS5-branes. Therefore the six dimensional theory compactified on a circle is given by a (p,q) 5-brane web which consists of a D5-brane and two NS5-branes wrapped on a circle such that the D5-brane intersects the circle at a point and the NS5-brane are separated along the D5-brane as shown in Fig. 16. The distance between the two NS5-branes, along the D5-brane, is inversely proportional to the coupling constant of the 6 dimensional gauge theory.

Generalization to geometries giving rise to $U(N)$ theory with $N_f = 2N$ is straightforward. Instead of single D5-brane we consider N D5-branes intersecting the circle on which two NS5-branes are wrapped as shown in Fig. 17.

In the limit that the size of the circle goes to infinity we get back the (p,q) web configuration giving rise to 5 dimensional $U(N)$ theory with $N_f = 2N$.

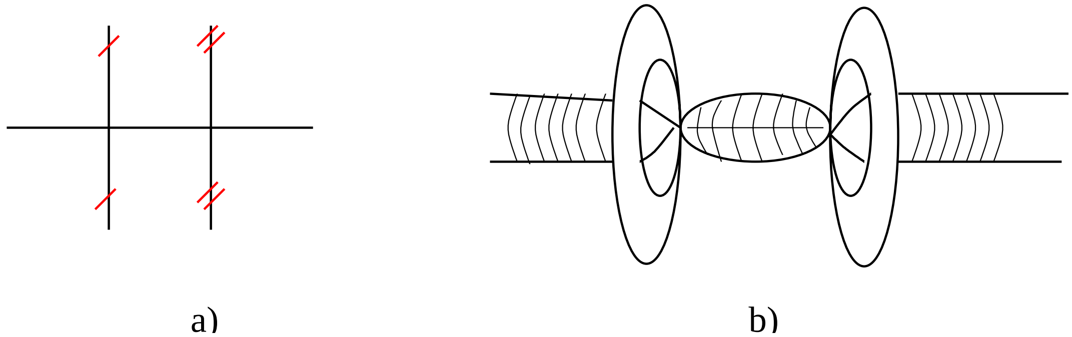

Figure 16: a) The 5-brane web description of the partial compactification of the resolved conifold blown up at two points and b) the geometry as an elliptic fibration which degenerates on the base $\mathbf{P}^1$.

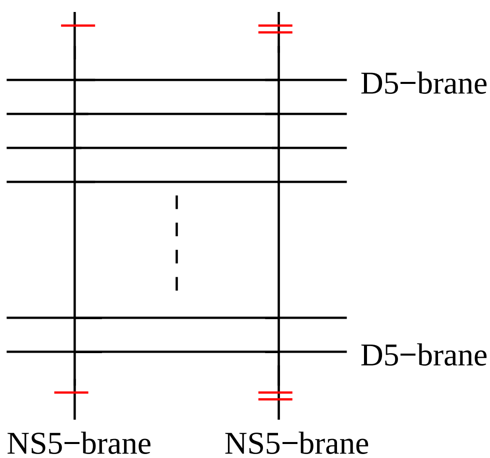

Figure 17: The 5-brane web description of the compactified 6D $U(N)$ theory with $N_f = 2N$.

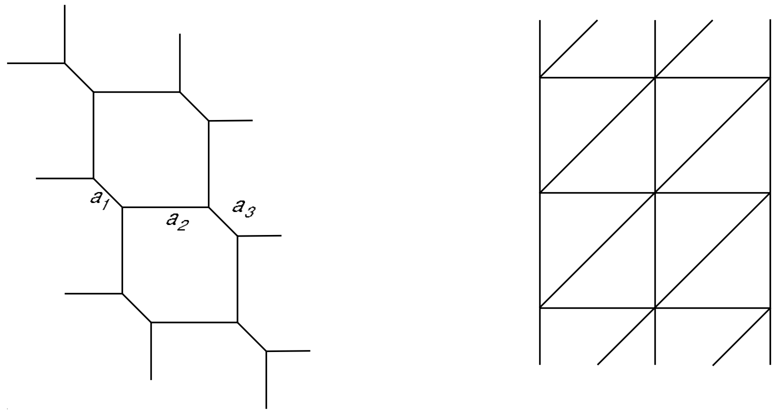

Figure 18: The web diagram for the 6D $N = 1$ and $N_f = 2$ theory where periodicity is implemented by including an infinite set of images. On the right is the associated grid diagram.

The exact form of the curve can be extracted from the web diagrams using the rules of [40] or that of local mirror symmetry applied to toric threefolds [2, 44, 45]. To apply this method to the 6D theories we have to take account of the periodicity of the web by including all its images under the periodic shift. For the $N = 1$ the web diagram with all its images is shown in Fig. 18. On the right is the associated grid diagram. The associated curve can be written $F(X, Y) = 0$, where $F(X, Y)$ is the sum of monomials, one for each vertex of the grid diagram. If a vertex is at (k, l) then the monomial is simply $A_{kl}X^kY^l$. The A_{kl} is a potential modulus of the curve; however, there are conditions that must be satisfied that restrict them. These conditions arise as follows. Consider a link on the grid that joins (k, l) with (u, v). Each such link is uniquely associated to a link on the web to which it is orthogonal. Suppose the link of the web is described by the equation $py = qx + \alpha$, then the orthogonality condition is

$$(k, l) - (u, v) = (-q, p) . \tag{3.19}$$

For each such link, we then have the constraint

$$py = qx + \alpha : \qquad A_{kl} = e^{\beta\alpha} A_{uv} . \tag{3.20}$$

In the present case, we have monomials $A_{kl}X^kY^l$, $l \in \mathbf{Z}$ and $k = 0, 1, 2$. Applying the rules, we find that all the coefficients are fixed in terms of the parameters $a_{1,2,3}$ in Fig. 18, up to an unimportant overall scaling and a choice of origin:

$$A_{0,l} = e^{\beta l(l-1)L/2} , \quad A_{1,l} = e^{\beta l(l-1)L/2+\beta a_1(1-l)} , \quad A_{1,l} = e^{\beta l(l-1)L/2+\beta a_1(2-l)+\beta a_2+\beta a_3(1-l)} , \tag{3.21}$$

where L is the period of the web in the vertical direction. This parameter is related to ρ via

$$L = \frac{2\pi i\rho}{\beta} . \tag{3.22}$$

We will identify the parameters a_1 and a_3 in terms of the masses,

$$a_1 = -\beta m_1 \ , \qquad a_3 = \beta m_2 \ . \tag{3.23}$$

This leaves one remaining degree-of-freedom in a_2 which corresponds to the bare coupling. So the curve becomes

$$F(X,Y) = \sum_{l=-\infty}^{\infty} e^{2\pi i l(l-1)\rho} Y^l \big(e^{-\beta m_1(l-1/2)} + X + e^{\beta a_2 - \beta(m_1+m_2)/2} e^{-\beta m_2(l-1/2)} X^2\big) = 0 \ . \tag{3.24}$$

Identifying $Y = e^{\beta x}$ and $X = e^{2\pi i z}$, and with some re-scalings of X and Y, the curve becomes

$$\frac{c}{4}\theta_1(\tfrac{\beta}{2i}(x-m_1)|\rho) + X\theta_1(\tfrac{\beta}{2i}x|\rho) + X^2\theta_1(\tfrac{\beta}{2i}(x-m_2)|\rho) = 0 \ , \quad \frac{c}{4} = e^{\beta a_2 - \beta(m_1+m_2)/2} \ . \tag{3.25}$$

Defining $y = 2X + \theta_1(\frac{\beta}{2i}x)/\theta_1(\frac{\beta}{2i}(x-m_2)$, the curve becomes

$$y^2 = \theta_1(\tfrac{\beta}{2i}x|\rho)^2 - c\,\theta_1(\tfrac{\beta}{2i}(x-m_1)|\rho)\theta_1(\tfrac{\beta}{2i}(x-m_2)|\rho) \ . \tag{3.26}$$

The constant $c = c(a_2)$ represents the freedom to change the bare coupling of the theory.

For $N > 1$, using an identical approach we find a curve that can be written

$$y^2 = \prod_{i=1}^{N} \theta_1\big(\tfrac{\beta}{2i}(x-\zeta_i)\big|\rho\big)^2 - c\prod_{f=1}^{2N}\theta_1\big(\tfrac{\beta}{2i}(x-m_f)\big|\rho\big) \ . \tag{3.27}$$

where the m_f are the masses and the ζ_i are the moduli of the Coulomb branch. In the four-dimensional limit, this reduces to the well-known hyper-elliptic geometry. We note that our curve is very simply related to the spectral curve of an XYZ spin chain suggested in [61].

3.4 $\mathcal{N} = 2$, $SU(N)$ with adjoint hypermultiplet

In this section we will review the brane configurations and the CY3-fold geometry that realizes the $\mathcal{N} = 2$ $U(N)$ gauge theory with an adjoint hypermultiplet in 4, 5 and 6 dimensions.

Brane construction: The basic type IIB setup which realizes the 5-dimensional theory is similar to the elliptic models which realize 4 dimensional theory with adjoint hypermultiplet [35]. The only difference is that instead of D4-branes we have D5-branes in this case. There is a single NS5-brane and the D5-branes are wrapped on a circle: the NS5-brane is extended in the $x^{0,1,2,3,4,5}$ direction. The D5-branes span the $x^{0,1,2,3,4}$ and x^6 direction. The direction

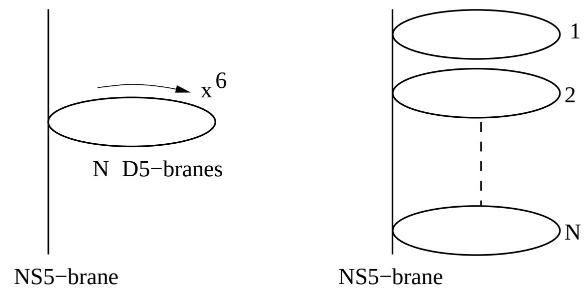

Figure 19: Elliptic models involving 5-branes which realize 5D $U(N)$ theory with a massless adjoint.

x^6 is taken to be compact so that the D5-branes wrap the circle and intersect the NS5-brane which is a point on the circle (Fig. 21(a)). On the 5-dimensional non-compact part of the D5-brane worldvolume there is an $\mathcal{N}=1$ (8 supercharge) $U(N)$ gauge theory with a massless adjoint *i.e.* , it has $\mathcal{N}=2$ (16 supercharge) supersymmetry reflected by the fact that since there is a single NS5-5brane it can be moved away from the circle leaving behind N D5-branes wrapped on a circle. The theory on such a set of N D5-branes is $\mathcal{N}=2$ $U(N)$ gauge theory. Turning on the mass of the adjoint breaks the supersymmetry down to $\mathcal{N}=1$ and therefore the corresponding branes configuration must be such that NS5-brane cannot be moved away from the circle. This is achieved as in [35] by changing the geometry so that as one goes around the x^6 circle x^5 shifts by m.[3] Obviously this corresponds to resolving the intersection of D5-branes and the NS5-branes into a tri-valent web of (p,q) 5-branes in the $x^{5,6}$ plane as shown in Fig. 20(b,c). In this case the separation distance is equal to the mass of the adjoint. Once the adjoint acquires non-zero mass we can Higgs the gauge group

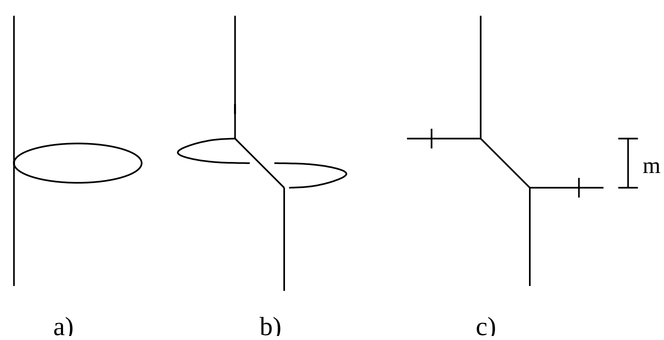

Figure 20: a) Elliptic model realizing $U(1)$ theory with massless adjoint, b) adjoint can be made massive by deforming the brane configuration and c) the same web diagram drawn with identifications.

[3]In the five-dimensional theory m is real.

by separating the D5-branes from each other (Fig. 21). The (p, q) 5-brane web given above

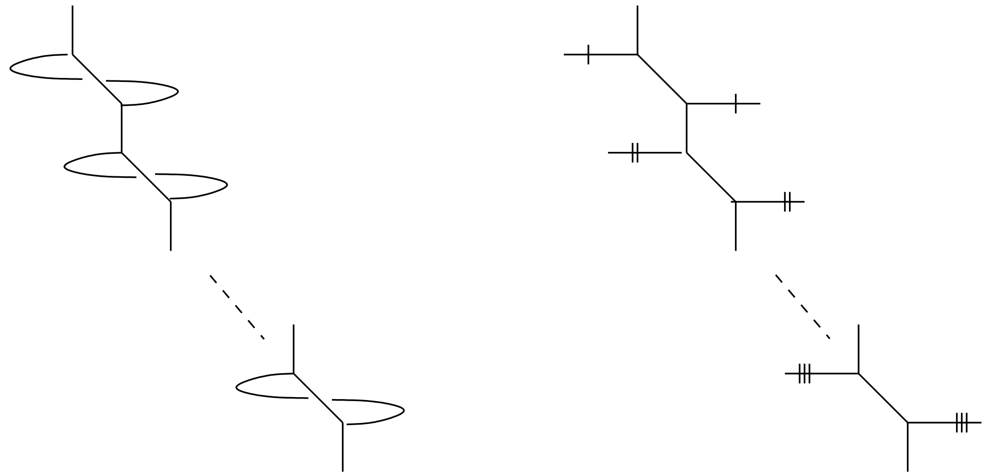

Figure 21: Brane configuration which realizes $U(N)$ theory with adjoint mass.

also defines the CY3-fold geometry which in this case has a elliptic fibration since the web is compactified on a circle. By compactifying one of the direction perpendicular to the web, say x^4, on a circle of size β we can take a limit in which we get the 4 dimensional field theory:

$$e^{-A(\mathbf{T}^2)} = e^{2\pi i\tau}\,,\;\; q := e^{-g_s} = e^{-\beta}\,,\;\; e^{-t} = e^{-\beta m}\,,\; \beta \to 0\,, \tag{3.28}$$

where $A(\mathbf{T}^2)$ is the area of the compactified $\mathbf{T}^2$ in the CY3-fold geometry, t is the area of the exceptional curves in the geometry and τ is the coupling of the four-dimensional theory.

To describe the 6 dimensional geometry we start with the $U(1)$ case. The corresponding 5-dimensional geometry was discussed in the last section and is given by compactifying one of the two $\mathbf{C}^*$ fibers so that we have a $\mathbf{C}^* \times \mathbf{T}^2$ fibration over the z-plane. The $\mathbf{C}^*$ fiber degenerates at $z = 0$ and the elliptic fibration degenerates at $z = \varepsilon$. The two fibrations together define an $\mathbf{S}^3$ in the geometry. Shrinking this $\mathbf{S}^3$ produces a singularity which when resolved gives the picture dual to the picture of D5-brane and NS5-brane intersecting and then deformed into a tri-valent web of (p, q) 5-branes. The $(1, 1)$ 5-brane introduced by this resolution is dual to the exceptional curve produced by the resolution of singularity. The 6 dimensional compactified theory can be obtained by compactifying both the NS5-brane and the D5-brane on two different circles so that the plane of the web is a 2-torus. This corresponds to compactifying both $\mathbf{C}^*$ fibrations as shown in Fig. 22(a). Recall that the compactified 5-dimensional theories on the (p, q) 5-brane webs can be lifted to M-theory where the theory lives on an M5-brane with worldvolume $\mathbf{R}^4 \times \Sigma$. The Riemann surface Σ is obtained from the (p, q) 5-brane web by "thickening" the lines and is embedded in the 4-dimensional space $x^{4,5,6,10}$, where x^{10} is the M-theory dimension. This is locally of the form

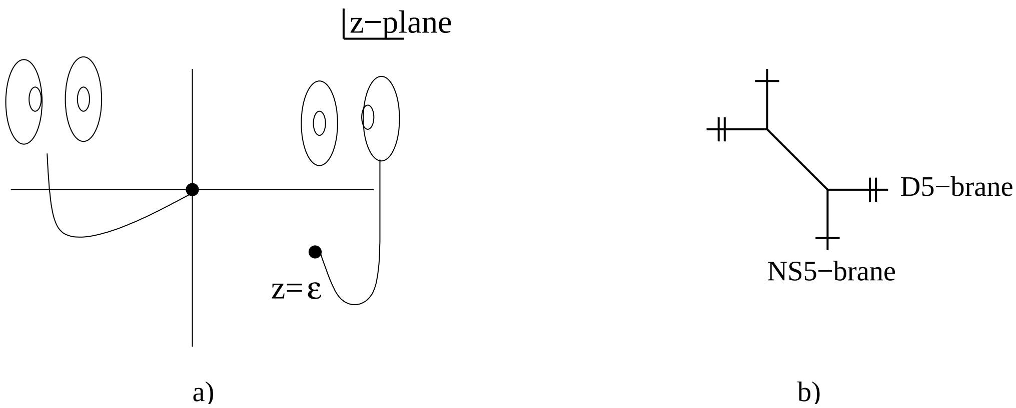

Figure 22: a) The elliptically fibered CY which gives $U(1)$ theory with massive adjoint and b) the corresponding 5-brane web diagram.

$\mathbf{S}^1 \times \mathbf{R} \times \mathbf{T}^2_\tau$, where $\mathbf{T}^2_\tau$ is the torus with complex structure τ in the $x^{6,10}$ directions and $\mathbf{S}^1$ is the T-dual of the original x^4 circle, *i.e.* now having size β^{-1}. Globally one must take account of the mass so that as one goes around the x^6-cycle of $\mathbf{T}^2_\tau$, x^5 shifts by m. In the six dimensional case, where both directions along the web are now periodic, we can lift the (p,q) 5-brane web to obtain an M5-brane wrapped on a Riemann surface Σ. In this case Σ can also be obtained from the web by thickening it but it now is embedded in a slanted 4-torus in the $x^{4,5,6,10}$ space. This 4-torus is identified with the abelian surface $\mathbf{T}^4$ in Eq. (2.46).[4] It is easy to see that Σ is a genus two curve which degenerates into two elliptic curves when the mass of the adjoint goes to zero (Fig. 23). This is precisely as one would predict from the analysis in Section 2.

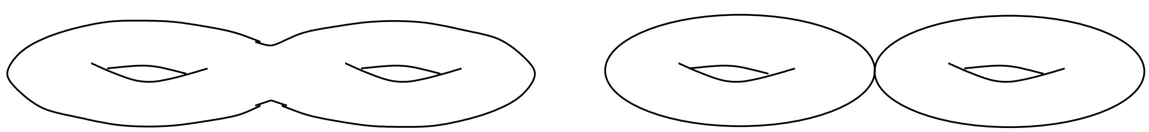

Figure 23: a) The genus 2 curve which describes the geometry of the 6D theory with an adjoint for $N = 1$, b) when the mass $m \to 0$, the genus 2 curve degenerates into 2 elliptic curves with complex structures τ and ρ.

The exact form for the curve can be obtained from the web diagram using the rules established in [40] or local mirror symmetry applied to toric Calabi-Yau [2, 44, 45] that we summarized at the end of Section 3.3. As in Section 3.3, in order to apply them to webs with periodicities we employ the method of images, although now we have double periodicity.

[4] In this context m can be taken to be complex.

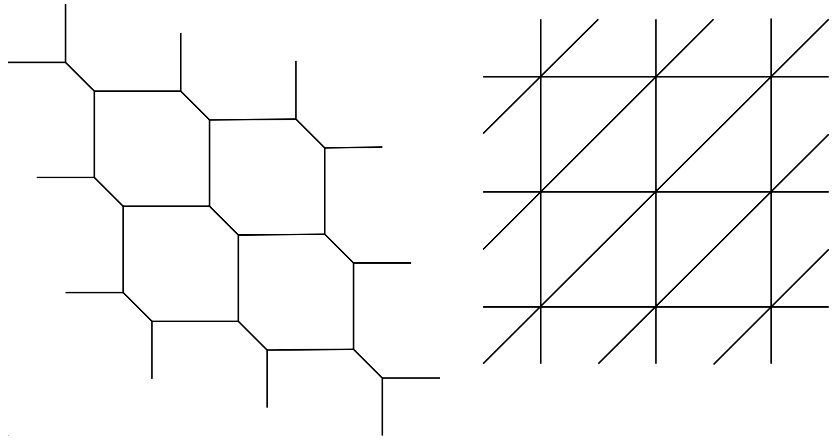

Figure 24: The web diagram for the 6D $N = 1$ theory with adjoint including all the images under the two periodic directions. On the right is the associated grid diagram.

We will only consider the $N = 1$ case in any detail but it should be clear to reader how to extend the method to $N > 1$. When we include all the images, the web tessellates the plane as illustrated in Fig. 24. This figure also shows the associated grid diagram. The associated curve can be written $F(X, Y) = 0$, where $F(X, Y)$ is the sum of monomials $A_{kl}X^kY^l$, $k, l \in \mathbf{Z}$, one for each vertex of the grid diagram. The A_{kl} are potential moduli which can be fixed using the connection with the web diagram as in Section 3.3, that fixes them, up to an overall scaling. Using the rule (3.20), we find the following recursion relation for the elements:

$$A_{k+1,l} = e^{\beta(L_h k + ml)} A_{kl} \ , \quad A_{k,l+1} = e^{\beta(L_v l + mk)} A_{kl} \ , \tag{3.29}$$

where L_h and L_v are the periods of the web in the vertical and horizontal direction. These parameters are related to ρ and τ via

$$L_v = \frac{2\pi i\rho}{\beta} \ , \qquad L_h = \frac{2\pi i\tau}{\beta} \ , \tag{3.30}$$

at least for ρ and τ purely imaginary. However, the results extend holomorphically to arbitrary values of ρ and τ. Solving (3.29), we find

$$A_{kl} = e^{2\pi ik(k-1)\tau + 2\pi il(l-1)\rho + kl\beta m} \tag{3.31}$$

up to an unimportant overall factor. Notice that once the rule (3.20) has been imposed there are no remaining moduli apart from the overall position of the web: just as one expects for the $N = 1$ curve. So the curve becomes

$$F(X, Y) = \sum_{kl=-\infty}^{\infty} e^{2\pi ik(k-1)\tau + 2\pi il(l-1)\rho + kl\beta m} X^k Y^l = 0 \ , \tag{3.32}$$

which, after identifying $X = e^{2\pi i z}$ and $Y = e^{\beta x}$, and after suitable re-scalings, is simply

$$\Theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left(z \quad \frac{\beta x}{2\pi i} \Big| \begin{matrix} \tau & \frac{\beta m}{2\pi i} \\ \frac{\beta m}{2\pi i} & \rho \end{matrix} \right) = 0 \,. \tag{3.33}$$

This is precisely the $N = 1$ version of (2.49). Notice that the form of the curve ensures that under the identification

$$X \to X e^{2i\pi\tau} \quad Y \to Y e^{m}$$
$$Y \to Y e^{2i\pi\rho} \quad X \to X e^{m}$$

$F(X, Y) = 0$ is invariant, since $F(X, Y) \to X^a Y^b F(X, Y)$ for some a, b (note that $X, Y \neq 0$).

Note that from the above construction of the curve from the toric diagram we have a cyclic symmetry between the parameters. To see this consider the transformation

$$X \mapsto X\,,\ Y \mapsto XY\,, \tag{3.34}$$

which has the effect of changing the basic grid in the toric diagram. The curve $F(X, Y)$ after this transformation becomes

$$F(X, XY) = \Theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left(z + \tau \quad \frac{\beta x}{2\pi i} \Big| \begin{matrix} \tau & \frac{\beta m}{2\pi i} - \tau \\ \frac{\beta m}{2\pi i} - \tau & \tau + \rho - \frac{\beta m}{2\pi i} \end{matrix} \right) = 0 \,. \tag{3.35}$$

The period matrix is given by

$$\Pi = \begin{pmatrix} \frac{\beta m}{2\pi i} + \widehat{\tau} & -\widehat{\tau} \\ -\widehat{\tau} & \widehat{\rho} + \widehat{\tau} \end{pmatrix} \,, \tag{3.36}$$

where $\tau = \widehat{\tau} + \frac{\beta m}{2\pi i}$ and $\rho = \widehat{\rho} + \frac{\beta m}{2\pi i}$. After an $Sp(4, \mathbf{Z})$ transformation we can write this as

$$S^{-1}\Pi S = \begin{pmatrix} \widehat{\rho} + \widehat{\tau} & \widehat{\tau} \\ \widehat{\tau} & \frac{\beta m}{2\pi i} + \widehat{\tau} \end{pmatrix} = \begin{pmatrix} \tau + \rho + 2\frac{\beta m}{2\pi i} & \tau - \frac{\beta m}{2\pi i} \\ \tau - \frac{\beta m}{2\pi i} & \tau \end{pmatrix} \,. \tag{3.37}$$

Now consider the transformation

$$X \mapsto XY\,,\ Y \mapsto Y\,, \tag{3.38}$$

which again changes the basic grid. Also note that the choices of the basic grid in the toric diagram that we used are the only possibilities. In this case the curve is given by

$$F(XY, Y) = \Theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left(z - \tau + \frac{\beta m}{2\pi i} \quad \frac{\beta x}{2\pi i} \Big| \begin{matrix} \tau + \rho - 2\frac{\beta m}{2\pi i} & \frac{\beta m}{2\pi i} - \rho \\ \frac{\beta m}{2\pi i} - \rho & \rho \end{matrix} \right) = 0 \,. \tag{3.39}$$

The period matrix is given by

$$\Pi = \begin{pmatrix} \widehat{\tau} + \widehat{\rho} & -\widehat{\rho} \\ -\widehat{\rho} & \widehat{\rho} + \frac{\beta m}{2\pi i} \end{pmatrix} \,. \tag{3.40}$$

After an $Sp(4,\mathbf{Z})$ transformation we get

$$S^{-1}\Pi S = \begin{pmatrix} \frac{\beta m}{2\pi i}+\widehat{\rho} & \widehat{\rho} \\ \widehat{\rho} & \widehat{\tau}+\widehat{\rho} \end{pmatrix} \,. \tag{3.41}$$

Thus the period matrix has a cyclic symmetry ($\widehat{\tau}\mapsto\widehat{\rho}\mapsto\frac{\beta m}{2\pi i}$) with three period matrices given by

$$\Pi := \begin{pmatrix} \widehat{\tau}+\frac{\beta m}{2\pi i} & \frac{\beta m}{2\pi i} \\ \frac{\beta m}{2\pi i} & \widehat{\rho}+\frac{\beta m}{2\pi i} \end{pmatrix}\,,\ \begin{pmatrix} \widehat{\rho}+\widehat{\tau} & \widehat{\tau} \\ \widehat{\tau} & \frac{\beta m}{2\pi i}+\widehat{\tau} \end{pmatrix}\,,\ \begin{pmatrix} \frac{\beta m}{2\pi i}+\widehat{\rho} & \widehat{\rho} \\ \widehat{\rho} & \widehat{\tau}+\widehat{\rho} \end{pmatrix}\,. \tag{3.42}$$

This symmetry is quite clear from the web diagram and is also present in the corresponding partition function as we will see in a later section.

This analysis can be extended to $N>1$ by applying the method of images as above. In this case the curve should be invariant (up to $F(X,Y)\mapsto X^aY^bF(X,Y)$) under the transformations

$$X\mapsto Xe^{2\sigma}\,,\ Y\mapsto Ye^{N\lambda}\quad \text{and}\,,X\mapsto Xe^{N\lambda}\,,\ Y\mapsto Ye^{2N\sigma}\,. \tag{3.43}$$

The curve is given by summing over all the monomials associated with the vertices of the toric diagram $(k,l)\mapsto X^{k/N}Y^l$,

$$F(X,Y) := \sum_{k,l=-\infty}^{+\infty} A_{kl}X^{\frac{k}{N}}Y^l = \sum_{j=0}^{N-1}\sum_{m,l} A^j_{ml}X^{m+\frac{j}{N}}Y^l\,, \tag{3.44}$$

where in the final expression above we have defined $j=k\,(\mathrm{mod}\ N)$. Using Eq. (3.43) we see that

$$A^j_{ml} = A_j\, e^{m(m-1)\mu+l(l-1)\sigma+mlN\lambda+\frac{2\mu jm}{N}+j\lambda n}\,. \tag{3.45}$$

So the curve becomes

$$\begin{aligned} F(X,Y) &= \sum_{j=0}^{N} A_j\sum_{m,l} e^{m(m-1)\mu+l(l-1)\sigma+mlN\lambda+\frac{2\mu jm}{N}+j\lambda n}X^{m+j/N}Y^l\,, \\ &= \sum_{j=0}^{N-1} A_j\Theta\begin{bmatrix} 0 & \frac{j}{N} \\ 0 & 0\end{bmatrix}\left(z \quad \frac{N\beta x}{2\pi i}\middle| \begin{matrix} \tau & \frac{N\beta m}{2\pi i} \\ \frac{N\beta m}{2\pi i} & N\rho \end{matrix}\right)\,, \end{aligned} \tag{3.46}$$

where $X=e^{N\beta x}$ and $Y=e^{2\pi i z}$. This agrees with the matrix model calculation of the last section.

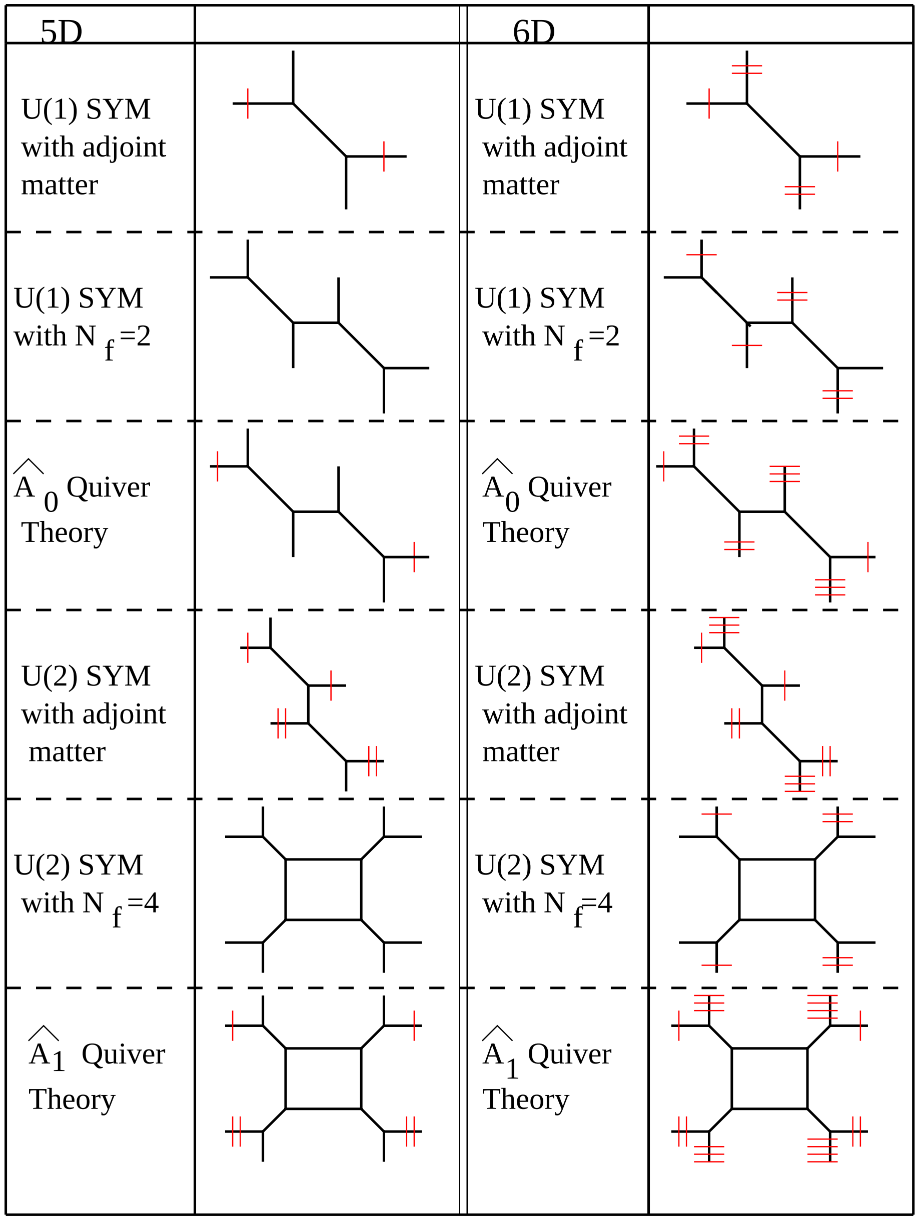

Figure 25: Field theories in 5 and 6 dimensions with various matter content and the corresponding web diagrams representing the CY3-fold geometry.

3.5 $\widehat{A}_{N-1}$ theories

In the last section we saw that 6D theories with $N_f = 2N$ can be engineered using elliptically fibered CY which can be described by semi-compact web diagrams. In this case one can ask the question about the relevance, if any, of completely compact web diagram obtained by identifying both the NS5-brane direction as well as the D5-brane direction from the web diagrams of $U(N)$ theory with $N_f = 2N$. The case of $U(1)$ with $N_f = 2N$ is shown in Fig. 26. It is easy to see that if the NS5-brane direction is not compact but the D5-brane direction is compact we get 5D $U(1) \times U(1)$ theory with bifundamental matter. Thus compactifying the NS5-brane direction gives us 6D $U(1) \times U(1)$ theory compactified on $\mathbf{T}^2$ with bifundamental matter. In case of N D5-branes we get 6D $U(N) \times U(N)$ theory with matter in $(N, \bar{N}) + (\bar{N}, N)$ compactified on $\mathbf{T}^2$ [47].

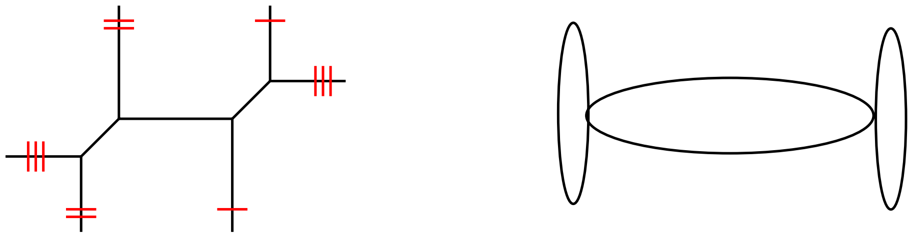

Figure 26: The web diagram of the 6-dimensional $\hat{A}_0$ theory.

4 Topological string amplitudes and BPS degeneracies and

In the previous section we reviewed the geometric engineering of compactified 4, 5 and 6 dimensional $\mathcal{N} = 2$ theories from CY3-folds via type IIA, M-theory or F-theory compactifications. An important ingredient of the geometric engineering recipe is the calculation of the gauge theory prepotential from the genus zero topological string amplitude of the corresponding CY3-fold [1]. In this section we will review the interpretation of topological string amplitudes as the generating functions of the BPS degeneracies of wrapped M2-branes (or D2-brane and D0-branes) which give rise to particles in the five dimensional theory.

Consider type IIA strings compactified on a CY3-fold $\mathbf{X}$. The theory on the transverse four dimensions has $\mathcal{N} = 2$ supersymmetry. This theory in 4 dimensions has certain F-terms

$(g \geq 1)$

$$\int d^4x F_g(t_i)\, R_+^2\, F_+^{2g-2}\,, \tag{4.1}$$

which can be calculated exactly. In the above expression R_+^2 is the contraction of self-dual part of the Riemann tensor and F_+ is the self-dual part of the graviphoton field strength. The function F_g is the A-model topological string amplitude of $\mathbf{X}$ [48, 49] and depends on t_i, the Kähler parameters of $\mathbf{X}$. As mentioned before the prepotential of the theory is given by genus zero amplitude for which the F-term is given by

$$\int d^4x F_+^i \wedge F_+^j\, \partial_{t_i}\partial_{t_j} F_0\ . \tag{4.2}$$

Thus the gauge coupling of the 4D theory in terms of the genus zero amplitude is given by

$$\tau_{ij} = \partial_{t_i}\partial_{t_j} F_0(t_i)\,. \tag{4.3}$$

The topological string amplitudes F_g arise in the A-twisted topological theory as integrals over the genus g moduli space of Riemann surfaces and are related to the generating functions of the genus g Gromov-Witten invariants. Let us denote by $\omega \in H^2(\mathbf{X}, \mathbf{C})$ the complexified Kähler class of $\mathbf{X}$. Then the topological string amplitudes be compactly organized into the generating function

$$F(t_i, \lambda_s) = \sum_{g=0}^{\infty} \lambda_s^{2g-2} F_g(t_i)\,, \tag{4.4}$$

where λ_s is the constant self-dual graviphoton field strength.

From the worldsheet point of view the genus g amplitude, F_g, is the generating function of the "number" of maps from a genus g Riemann surface to CY3-fold $\mathbf{X}$. However, the target space viewpoint provides a more physical interpretation of the generating function $F(t_i, \lambda_s)$, which we now review [10, 11]. Recall that in M-theory compactification on CY3-fold $\mathbf{X}$ we get a 5-dimensional field theory with eight supercharges. The particles in this theory come from quantization of the wrapped M2-branes on various 2-cycles of $\mathbf{X}$. If we consider compactifying one direction then we can interpret the particles as wrapped D2-branes and the KK modes as bound D0-branes. In this case integrating out these various charged particles gives rise to the F-terms in the effective action. The contribution of a particle of mass m and in representation $\mathcal{R}$ of the $SU(2)_L \times SU(2)_R$ (the little group of massive particles in 5D) to F is given by

$$S = \mathrm{logdet}(\Delta + m^2 + 2e\,\sigma_L \mathcal{F}) \;= \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{R}}(-1)^{\sigma_L+\sigma_R} e^{-sm^2} e^{-2se\sigma_L\mathcal{F}}}{(2\sinh(se\mathcal{F}/2))^2}\,, \tag{4.5}$$

where σ^L is the Cartan of $SU(2)_L$ and arises because the graviphoton field strength is self-dual. e is the charge of the particle and is equal to its mass and we identify the graviphoton field strength $\mathcal{F} = \lambda_s$. The mass of the particle is given by the area of the curve on which the D2-brane is wrapped. An extra subtlety arises due to D0-branes. In the lift to M-theory we see that a wrapped M2-brane comes with momentum in the circle direction and therefore if we denote the mass of the M2-brane wrapping a curve class $\Sigma \in H_2(\mathbf{X}, \mathbf{Z})$ by T_Σ then the mass of the M2-brane with momentum n is given by taking T_Σ to $T_\Sigma + 2\pi i n/\lambda$. Let us denote by $N_\Sigma^{(j_L,j_R)}$ the number of BPS states coming from M2-brane wrapped on the holomorphic curve Σ and left-right spin content under $SU(2)_L \times SU(2)_R$ given by (j_L, j_R). Then the total contribution coming from all particles is obtained by summing over the momentum, the holomorphic curves and the left-right spin content,

$$\begin{aligned} F &= \sum_{\Sigma \in H_2(\mathbf{X},\mathbf{Z})} \sum_{n \in \mathbf{Z}} \sum_{j_L, j_R} N_\Sigma^{(j_L,j_R)} \int_\epsilon^\infty \frac{ds}{s} \frac{Tr_{(j_L,j_R)}(-1)^{\sigma_L+\sigma_R} e^{-sT_\Sigma - 2\pi i n} e^{-2s\sigma_L \lambda_s}}{(2\sinh(s\lambda_s/2))^2}, \\ &= \sum_{\Sigma \in H_2(\mathbf{X},\mathbf{Z})} \sum_{k=1}^\infty \sum_{j_L,j_R} N_\Sigma^{(j_L,j_R)} e^{-kT_\Sigma} \frac{\mathrm{Tr}_{(j_L,j_R)}(-1)^{\sigma_L+\sigma_R} e^{-2k\lambda_s \sigma_L}}{(2\sinh(k\lambda_s/2))^2}, \\ &= \sum_{\Sigma \in H_2(\mathbf{X},\mathbf{Z})} \sum_{k=1}^\infty \sum_{j_L} N_\Sigma^{j_L} e^{-kT_\Sigma} \frac{\mathrm{Tr}_{j_L}(-1)^{\sigma_L} e^{-2k\lambda_s \sigma_L}}{(2\sinh(k\lambda_s/2))^2}, \end{aligned} \tag{4.6}$$

where

$$N_\Sigma^{j_L} = \sum_{j_R} N_\Sigma^{(j_L,j_R)} (-1)^{2j_R} (2j_R + 1)\,. \tag{4.7}$$

It is useful to define a different basis of $SU(2)_L$ representations given by $I_g = (2(0) + (\frac{1}{2}))^g$ such that in terms of this basis

$$\sum_{j_L} N_\Sigma^{j_L} [j_L] = \sum_{g=0}^\infty n_\Sigma^g I_g\,. \tag{4.8}$$

The coefficients n_Σ^g are integers and given by

$$\sum_{g=0}^\infty n_\Sigma^g (-1)^g (q^{1/2} - q^{-1/2})^{2g} = \sum_{j_L} N_\Sigma^{j_L} (q^{-j_L} + \cdots + q^{+j_L})\,. \tag{4.9}$$

In terms of these integers one can write F as

$$F = \sum_{\Sigma \in H_2(\mathbf{X},\mathbf{Z})} \sum_{k=1}^\infty \sum_{g=0}^\infty n_\Sigma^g e^{-kT_\Sigma} \frac{\mathrm{Tr}_{I_g}(-1)^{\sigma_L} e^{-2k\lambda_s \sigma_L}}{(2\sinh(k\lambda_s/2))^2}, \tag{4.10}$$

It is easy to show that

$$\mathrm{Tr}_{I_g}(-1)^{\sigma_L} e^{-2k\lambda_s\sigma_L} = \Big(\mathrm{Tr}_{I_1}(-1)^{\sigma_L} e^{-2k\lambda_s\sigma_L}\Big)^g = \Big(2\sinh(k\lambda_s/2)\Big)^{2g}. \tag{4.11}$$

Thus we get

$$F = \sum_{\Sigma\in H_2(\mathbf{X},\mathbf{Z})} \sum_{k=1}^{\infty}\sum_{g=0}^{\infty} n_\Sigma^g (2\sinh(k\lambda_s/2))^{2g-2} e^{-kT_\Sigma}. \tag{4.12}$$

The target space point of view allows the topological string amplitudes to be written in terms of integers n_Σ^g which give the BPS degeneracies of the states coming from wrapped D2-branes. The fact that F has this particular form with integer n_Σ^g has been confirmed for many non-compact toric threefolds.

4.1 Second quantized strings and A-model partition function

In the previous section we saw that the target space view point when discussing the topological string amplitudes is perhaps more interesting than the worldsheet view point since it allows the amplitudes to be written in terms of invariants which are integers. However another interesting property, the integrality of e^F, becomes clear from this view point as well.

To see this note that we can write e^F, which we will call the partition function from now on, as

$$Z(\omega, g_s) = e^{F(\omega,g_s)} = \prod_{\Sigma\in H_2(\mathbf{X})} \prod_{j_L} \prod_{k=-j_L}^{+j_L} \prod_{m=0}^{\infty} (1 - q^{2k+m+1} Q^\Sigma)^{(-1)^{2j+1}(m+1)N_\Sigma^{j_L}}, \tag{4.13}$$

where $Q^\Sigma = e^{-T_\Sigma}$ and $q = e^{-i\lambda_s}$. This expression looks very much like 'counting' the states in a Hilbert space (this was also noted in [10] for the case of $j_L = 0$ BPS states). In a sense we have already explained how partition function counts $M2$-branes. So the integrality of Z must be directly related to this integrality in F. We can in fact see Z as the partition function of a second quantized theory built purely out of fields creating $M2$-branes. Let $\Phi_{\Sigma,m_1,m_2}(z_1, z_2)$ denote a field creating an $M2$-brane BPS state, where z_i denote the two complex coordinates of the 4 dimensional space, Σ denotes the BPS charge and m_i denote the internal spins of the BPS particle with respect to $U(1)\times U(1) = SO(2)\times SO(2) \subset SO(4)$. Consider only *holomorphic* configurations of the BPS fields. This is what we usually do in the context of 2d chiral block of a conformal theory. If we do this we have the natural decomposition

$$\Phi_{\Sigma,m_1,m_2}(z_1, z_2) = z_1^{m_1} z_2^{m_2} \sum_{n_1,n_2\geq 0} \alpha_{n_1+m_1,n_2+m_2}(\Sigma) z_1^{n_1} z_2^{n_2}$$

where $\alpha_{n_1+m_1,n_2+m_2}(\Sigma)$ are bosonic or fermionic modes depending on whether the field Φ (which is the lowest component of a superfield) is bosonic or fermionic respectively. Note also the prefactor monomial is the usual mapping of modes from cylinder to the plane for each z_i. Note that $j_L^3 = n_1 + n_2 + m_1 + m_2$ for each mode $\alpha_{n_1+m_1,n_2+m_2}(\Sigma)$.

Since $N_\Sigma^{j_L}$ is the BPS degeneracy of the states with charge Σ and $SU(2)_L$ spin j_L we can write the above partition function as

$$Z := \mathrm{Tr}_{\mathcal{H}}(-1)^{2(j_L+j_R)}\, q^{2j_L^3}\, e^{-T}\,. \tag{4.14}$$

where $\mathcal{H}$ denotes the subspace of the second quantized Hilbert space generated by holomorphic modes of the lowest component of the BPS fields and T denotes the total mass of the BPS states which is the same as the Hamiltonian of the theory. It is quite exciting that the partition function of topological string seems to be counting a second quantized hilbert space of holomorphic components of BPS states. It is also natural to believe there is an interesting algebra related to this partition function. In particular we can take the product of two holomorphic BPS fields as defining an algebra:

$$\Phi_{\Sigma_1}\Phi_{\Sigma_2} = \sum_{\Sigma_i=\Sigma_1+\Sigma_2} C_i \Phi_{\Sigma_i}$$

This would be interesting to study further. It would also be interesting to see the connection of this holomorphic OPE of BPS states to the BPS algebra defined in [50].

4.2 Non-compact toric threefolds and the topological vertex

In this section we consider the case of non-compact toric CY3-folds. These CY3-folds are extremely interesting not only because they are "simple" enough so that exact calculation of A-model partition function can be done but also because they give rise to gauge theories via geometric engineering [1] as we saw in the last section.

From the discussion of the last section we see that if the graviphoton field strength is not self-dual $F := F_+ + F_-$, then we can write the contribution of coming from integrating out the particle in representation $\mathcal{R}$ of $SU(2)_L \times SU(2)_R$ as

$$S := \int_\epsilon^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{R}}(-1)^{\sigma_L+\sigma_R} e^{-sm^2} e^{-2se(\sigma_L F_+ + \sigma_R F_-)}}{(2\sinh(seF_+/2))(2\sinh(seF_-/2))}\,. \tag{4.15}$$

Summing over the contribution from all particles as before we get

$$F(q_1, q_2) = \sum_{\Sigma \in H_2(\mathbf{X},\mathbf{Z})} \sum_{n=1}^{\infty} \sum_{j_L, j_R} \frac{N_{\Sigma}^{(j_L,j_R)} \Big((q_1 q_2)^{-nj_L} + \cdots + (q_1 q_2)^{nj_L}\Big)\Big((\frac{q_1}{q_2})^{-nj_R} + \cdots + (\frac{q_1}{q_2})^{nj_R}\Big)}{n(q_1^{n/2} - q_1^{-n/2})(q_2^{n/2} - q_2^{-n/2})} e^{-nT_{\Sigma}}\,, \tag{4.16}$$

where we have defined $q_1 = e^{F_+}, q_2 = e^{F_-}$. The integers $N_{\Sigma}^{(j_L,j_R)}$ give the degeneracy of particles with spin content (j_L, j_R) and charge Σ and are the number of cohomology classes with spin (j_L, j_R) of the moduli space of D-brane wrapped on a holomorphic curve in the class Σ [10, 11]. Because the D-brane has a $U(1)$ gauge field living on its worldvolume the moduli space of supersymmetric configurations includes not only the curve moduli but also the moduli of the flat connections on the curve coming from the gauge field. Since the moduli space of flat connections on a smooth curve of genus g is T^{2g} therefore the moduli space of the D-brane is T^{2g} fibration over the moduli space of the curve. The total space is a Kähler manifold and the Lefshetz action by the Kähler class is the diagonal $SU(2)_D \subset SU(2)_L \times SU(2)_R$ action on the moduli space. The $SU(2)_L \times SU(2)_R$ action on the moduli space is such that $SU(2)_L$ acts on the fiber direction and the $SU(2)_R$ acts in the base direction.

In the previous section when discussing the generic CY3-folds we summed over the $SU(2)_R$ action by taking the graviphoton field strength to be self-dual. This was essentially due to the fact that $N_{\Sigma}^{(j_L,j_R)}$ can change as we change the complex structure; the supersymmetry algebra allows such pairings between neighboring j_R's to give a non-reduced multiplet. But $N_{\Sigma}^{j_L}$, which sums over all j_R's with alternating signs remains invariant. For the case of non-compact toric CY3-folds there are no complex structure deformations. Therefore one would expect no jumps in the $N_{\Sigma}^{(j_L,j_R)}$ degeneracies, and so one would hope to be able to compute these as well. We will come back to this after our discussion of the topological vertex.

Topological vertex: It was shown in [12] (see also the earlier work [62,63,54,55]) that topological string amplitude for non-compact 3-folds can be calculated using the corresponding web diagrams and the topological vertex: A function of q, $C_{R_1R_2R_3}$, depends on three representation, $R_{1,2,3}$, of $U(\infty)$ associated with each tri-valent vertex of the web diagram. The topological vertex $C_{R_1R_2R_3}$ is actually an open string amplitude for a certain geometry with D-branes as we will see later. Let us briefly review the idea behind the topological vertex and the derivation of topological string amplitudes for a non-compact CY3-fold.

Recall that in the last section we saw that non-compact toric 3-folds can be represented by web diagrams which captures the non-trivial aspects of the geometry as a tri-valent graph in two dimensions. The graph is the locus of degeneration of a $\mathbf{T}^2$ fibration over the plane.

Along each edge of the web a 1-cycle of the fiber $\mathbf{T}^2$ shrinks and therefore at each point of the edge we have an $\mathbf{S}^1$, the cycle dual to the one shrinking. Given this cycle we can consider a D-brane with 3 dimensional worldvolume $\mathbf{S}^1 \times \mathbf{C}$ which wraps this cycle and fills two other directions only one of which could be in the plane of the web diagram. Such a 3-cycle is Lagrangian and can be used to define the boundary conditions for the open topological strings [51,52]. As we mentioned before the web diagrams corresponding to non-singular geometries are tri-valent graphs. All the vertices of the web diagram are $SL(2,\mathbf{Z})$ transform of each other and hence the web diagram can be "built" using the basic vertex, in which we have $(1,0),(0,1)$ and $(-1,-1)$ lines coming together, and its $SL(2,\mathbf{Z})$ transforms joined by edges which are straight lines. Such a web with only $(1,0),(0,1)$ and $(-1,-1)$ lines corresponds to threefold which is $\mathbf{C}^3$. In this case the geometry is trivial and the only contribution to the topological string amplitude comes from constant maps. However, as mentioned before we can have D-branes in this geometry which will provide boundaries for maps from worldsheet with boundaries as shown in 27 where R_i are the representations in which we take the holonomy around the circle which the D-branes wrap. The open

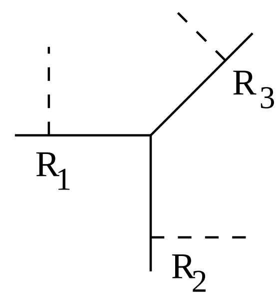

Figure 27: The topological vertex is defined as the open string amplitudes in the presence of three stacks of Lagrangian branes.

topological string amplitude of this geometry is given by the topological vertex

$$C_{R_1R_2R_3} = \sum_{Q_1,Q_2} N_{Q_1Q_2}^{R_1R_3^t}\, q^{\kappa_{R_2}/2+\kappa_{R_3}/2}\, \frac{\mathcal{W}_{R_2^tQ_1}\,\mathcal{W}_{R_2Q_2}}{\mathcal{W}_{R_2}}\,, \tag{4.17}$$

where

$$N_{Q_1Q_2}^{R_1R_3^t} = \sum_{Q} N_{QQ_1}^{R_1} N_{QQ_2}^{R_3^t}\,. \tag{4.18}$$

Here N_{ab}^c is the degeneracy of representation c in the tensor product $a\otimes b$, the $\mathcal{W}_{R_1R_2}$ is the link invariant for Hopf link for $U(\infty)$, and κ_R denotes a quadratic casimir for representation R. Another useful representation of the vertex is given using the skew-Schur functions. Let us denote the Young diagram corresponding to representation R as μ_R then [56]

$$C_{R_1R_2R_3} = (-1)^{l_{R_1}+l_{R_2}+l_{R_3}} q^{\kappa_{R_2}/2} s_{R_3^t}(q^{-\rho}) \sum_{R} s_{R_1^t/R}(q^{-\mu_{R_3}-\rho}) s_{R_2/R}(q^{-\mu_{R_3^t}-\rho})\,, \tag{4.19}$$

where $s_{R_1/R}(x)$ is the skew-Schur function defined as

$$s_{R_1/R}(x) = \sum_{R_2} N^{R_1}_{RR_2} s_{R_2}(x)\,, \tag{4.20}$$

and $q^{-\mu-\rho} = \{q^{-\mu_1+1/2}, q^{-\mu_2+3/2}, q^{-\mu_3+5/2}, \dots\}$. To calculate the partition function for any non-compact threefold we consider its web diagram and associate with each leg a representation R_i and with each vertex $C_{R_iR_jR_k}$ where $R_{i,j,k}$ are the representations on the legs joining the vertex. Then the partition function is given by multiplying all the vertices together and summing over all the representations with weights $\prod_i e^{-T_i\ell_{R_i}}$ where T_i is area of the curve associated to the i-th edge and ℓ_R is the number of boxes in the Young diagram corresponding to R. There are extra subtleties associated with orientation of the legs which leads to extra framing factors. For details of this we refer the reader to [12].

As an example consider the case of the resolved conifold as shown in 28. We will see that

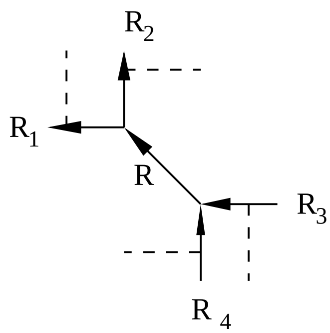

Figure 28: The web diagram of the resolved conifold with four stacks of D-branes. The representation R, which is summed over, is associated with the internal line and $R_{1,2,3,4}$ are fixed representations associated with the external lines of the web.

using the identities involving Schur and skew-Schur function it is possible to write a simple expression for the partition function of this geometry. We will use similar identities involving skew-Schur function to determine the partition functions of gauge theories in the next section.

Denote by T the area of the $\mathbf{P}^1$ then partition function is given by

$$Z_{R_1,R_2,R_3,R_4} \;\; = \sum_R e^{-T\ell_R} C_{R_1R_2R^t}(q)\, C_{R\,R_3^t\,R_4^t}(q)\,. \tag{4.21}$$

Using Eq(4.19) we can write $Z_{R_{1,2,3,4}}$ as $(Q := e^{-T})$

$$\begin{aligned} Z_{R_{1,2,3,4}} \;\; &= \;\; (-1)^{\ell_{R_1}+\ell_{R_2}+\ell_{R_3}+\ell_{R_4}} s_{R_1^t}(q^{-\rho}) s_{R_3}(q^{-\rho}) \sum_{\eta_1,\eta_2} s_{R_2^t/\eta_1}(q^{-\mu_{R_1}-\rho}) s_{R_4/\eta_2}(q^{-\mu_3^t-\rho}) \\ &\quad \sum_R Q^{-\ell_R}(-1)^{\ell_R}\, s_{R/\eta_2}(q^{-\mu_3-\rho})\, s_{R^t/\eta_1}(q^{-\mu_{R_1^t}-\rho})\,. \end{aligned}$$

The sum over R can be carried out exactly using the identity [71]

$$\sum_{R} s_{R/\eta_1}(x) s_{R^t/\eta_2}(y) = \prod_{i,j}(1+x_i y_j) \sum_{\eta_3} s_{\eta_2^t/\eta_3}(x) s_{\eta_1^t/\eta_3^t}(y)\,. \tag{4.22}$$

After summing over $R, \eta_{1,2}$ and denoting the Schur function corresponding to R in the variable $(x_1, x_2, \cdots, y_1, y_2, \cdots)$ by $s_R(x,y)$ we get

$$\begin{aligned} Z_{R_{1,2,3,4}} &= (-1)^{\ell_{R_1}+\ell_{R_2}+\ell_{R_3}+\ell_{R_4}} s_{R_1^t}(q^{-\rho}) s_{R_3}(q^{-\rho}) \prod_{i,j}(1 - Q\, q^{-\mu_{R_3,i}-\rho_i-\mu_{R_1^t,j}-\rho_j}) \\ & \sum_{\eta_3} Q^{\ell_{\eta_3}}(-1)^{\ell_{\eta_3}} s_{R_2/\eta_3}(Qq^{\mu_{R_1}+\rho}, q^{-\mu_{R_3}-\rho}) s_{R_4/\eta_3^t}(q^{-\mu_{R_3^t}-\rho}, Qq^{\mu_{R_1^t}+\rho})\,. \end{aligned} \tag{4.23}$$

The sum in the expression above is finite and can be determined easily for any given $R_{2,4}$. Consider the case $(R_1, R_2, R_3, R_4) = (\bullet, \bullet, \bullet, R)$ and $(R_1, \square, R_3, \square)$ then

$$Z_{\bullet\bullet\bullet R} = (-1)\Big(\prod_{k=1}^{\infty}(1-Qq^k)^k\Big) s_R(q^{-\rho}, Qq^{\rho})\,, \tag{4.24}$$

$$\begin{aligned} Z_{R_1,\square,R_3,\square} &= \Big(\prod_{k=1}^{\infty}(1-Qq^k)^k\Big)\Big(\prod_{k}(1-Qq^k)^{C_k(R_1^t,R_3)}\Big) s_{R_1}(q^{-\rho}) s_{R_3}(q^{-\rho}) \\ & \Big(\frac{q}{(1-q)^2}(1-Q)^2 + f_{R_3,R_3^t} - Q(f_{R_1^t R_3} + f_{R_1 R_3^t}) - Q^2 f_{R_1 R_1^t}\Big)\,. \end{aligned}$$

Where

$$\begin{aligned} f_{R_1 R_2} &= \sum_{k} C_k(R_1,R_2) q^k = s_{\square}(q^{-\mu_{R_1}-\rho}) s_{\square}(q^{-\mu_{R_2}-\rho}) - s_{\square}^2(q^{-\rho}) \\ &= q^{-1}(q-1)^2 f_{R_1} f_{R_2} + f_{R_1} + f_{R_2}\,, \quad \text{and} \\ f_R &= \sum_{(i,j)\in R} q^{j-i}\,. \end{aligned} \tag{4.25}$$

Generalized partition function: In a previous section we saw that the BPS degeneracies given by the target space viewpoint of the topological strings are obtained by summing over the j_R spin content. This is necessary in order to obtain an index invariant under the complex structure deformations of the CY3-fold. Hence for a CY3-fold with no complex structure deformation we do not have to sum over the $SU(2)_R$. As mentioned before non-compact CY3-folds are such spaces for which there are no complex structure deformations and hence a more general partition function encoding the full $SU(2)_L \times SU(2)_R$ spin content can be defined.

The generalized Partition function in such cases would be given as a trace over the second quantized Hilbert space generated by holomorphic components of the BPS fields:

$$Z = \mathrm{Tr}_{\mathcal{H}}(-1)^{2(j_L+j_R)}\, q_1^{j_L^3+j_R^3}\, q_2^{j_L^3-j_R^3}\, e^{-T} = e^{F(q_1,q_2)} =$$
$$\prod_{\Sigma\in H_2(\mathbf{X})}\prod_{j_L,j_R}\prod_{k_L=-j_L}^{+j_L}\prod_{k_R=-j_R}^{+j_R}\prod_{m_1,m_2=0}^{\infty}(1-q_1^{k_L+k_R+m_1+\frac{1}{2}}\,q_2^{k_L-k_R+m_2+\frac{1}{2}}Q^{\Sigma})^{(-1)^{2(j_L+j_R)+1}N_{\Sigma}^{j_L,j_R}}\,. \tag{4.26}$$

To determine the invariants $n_{\Sigma}^{(g_1,g_2)}$ which correspond to the basis (I_{g_1}, I_{g_2}) we can use the relation

$$\sum_{g_1,g_2\geq 0} n_{\Sigma}^{g_1,g_2}(-1)^{g_1+g_2}((q_1q_2)^{1/4}-(q_1q_2)^{-1/4})^{2g_1}\left((\tfrac{q_1}{q_2})^{1/4}-(\tfrac{q_1}{q_2})^{-1/4}\right)^{2g_2} = \tag{4.27}$$
$$\sum_{j_L,j_R} N_{\Sigma}^{j_L,j_R}(-1)^{2(j_L+j_R)}\left((q_1q_2)^{-j_L}+\cdots+(q_1q_2)^{+j_L}\right)\left((\tfrac{q_1}{q_2})^{-j_R}+\cdots+(\tfrac{q_1}{q_2})^{+j_R}\right).$$

Direct computation of such a partition function, say using some generalized topological vertex, is not known for any non-compact 3-fold. However, using the geometric engineering relation between the compactified 5D gauge theories and non-compact 3-folds one can obtain the generalized partition function for some cases using the instanton calculus developed by Nekrasov [5, 65]. Note that in the non-compact case the notion of $SU(2)_R$ is ambiguous, as it can mix with R-symmetry (there is no normalizable 4d gravity mode). In fact a particular combination of $SU(2)_R$ and the R-symmetry $SU(2)$ is what is computed in [5] which we find to correspond to the $SU(2)_R$ defined in the $D2$-brane moduli problem acting on the base of the moduli space [11].

As an example consider the pure five dimensional $U(2)$ gauge theory which can be obtained via M-theory compactification on local $\mathbf{P}^1\times\mathbf{P}^1$. In this case the gauge theory partition function was calculated in [5]. We can use their result to verify that the generalized partition function gives the degeneracies which represent the $SU(2)\times SU(2)$ action on the moduli space of the D-branes.

The partition function for the case of local $\mathbf{P}^1\times\mathbf{P}^1$ is given by

$$Z := \sum_{R_{1,2}} Q_b^{l_1+l_2} Z_{R_1,R_2}(Q_f)\ , \tag{4.28}$$

where $T_{b,f} = -\log(Q_{b,f})$ are the Kähler parameters associated with the base and the fiber $\mathbf{P}^1$ and we sum over all pairs of Young diagrams $R_{1,2}$ such that

$$Z_{R_1,R_2}(Q_f) = \frac{C_{R_1}(q_1,q_2)C_{R_2}(q_1,q_2)C_{R_1^T}(q_2,q_1)C_{R_2^T}(q_2,q_1)\,q_1^{\kappa_{R_1}/2+\kappa_{R_2}/2}\,(q_1/q_2)^{\sum_i(\mu_{1,i}^2-\mu_{2,i}^2)/2}}{\prod_{k_1,k_2}(1-q_1^{k_1}q_2^{k_2}Q)^{C_{k_1,k_2}(R_1,R_2)}}\,.$$

In the above expression

$$C_R(q_1,q_2)=(-1)^{l_R}(q_1q_2)^{\kappa_R/8}\,q_1^{h^1_R/2}q_2^{h^2_R/2}\prod_{(i,j)\in R}(1-q_1^{h^1_R(i,j)}q_2^{h^2_R(i,j)})^{-1}\,. \tag{4.29}$$

and $\kappa_R=\sum_{(i,j)\in R}(j-i)$, $h^1_R(i,j)=\mu^t_j-i, h^2_R(i,j)=\mu_i-j+1$. The integers $C_{k_1,k_2}(q_1,q_2)$ are given by

$$\begin{aligned}\sum_{k_{1,2}}C_{k_1,k_2}(R_1,R_2)q_1^{k_1}q_2^{k_2} &= \sum_{(i,j)\in R_1}q_1^{-h^1_{R_2}(i,j)}q_2^{-h^2_{R_1}(i,j)}+\sum_{(i,j)\in R_2^T}q_1^{h^2_{R_1^T}(i,j)}q_2^{h^1_{R_2^T}(i,j)}\\ &+\sum_{(i,j)\in R_2}q_1^{h^1_{R_1}(i,j)}q_2^{h^2_{R_2}(i,j)}+\sum_{(i,j)\in R_1^T}q_1^{-h^2_{R_2^T}(i,j)}q_2^{-h^1_{R_1^T}(i,j)}\,.\end{aligned}$$

We can use the above partition function to calculate the BPS degeneracies of various states corresponding to charge $\Sigma\in H_2(\mathbf{X},\mathbf{Z})$. For example the term linear in Q_b, given by $(R_1,R_2)=(\square,\bullet),(\bullet,\square)$, determines the integers $N^{(j_L,j_R)}_\beta$ for all curves $\beta=B+kF\; k\geq 0$ and gives

$$N^{(j_L,j_R)}_{B+kF}=\delta_{j_L,0}\;\delta_{j_R,k+1/2}\,, \tag{4.30}$$

which is consistent with the fact that these are genus zero curves and therefore action in the fiber direction, which is just a point, is trivial. The moduli space of these curves is given by $\mathbf{P}^{2k+1}$ and the $SU(2)_R$ action on this is just the Lefshetz action via Kähler class therefore the cohomology classes decomposes into a spin $[k+1/2]$ representation. If ω is the Kähler class then the j^3_R on the cohomology class ω^n is given by $n-k-1/2$. And since there is one such class for each n we get a single copy of the representation $[k+1/2]$.

A more interesting example in which both the left and the right spin content is non-trivial is given by the curve $2B+2F$, the canonical class of the F_0. This is a genus one curve and therefore the corresponding moduli space will admit non-trivial $SU(2)_L$ action. To determine the spin content from the partition function we will have to expand it to order $Q_b^2Q_f^2$, take the log of the corresponding expression and subtract multicover contribution. In this case we get

$$\sum_{j_L,j_R}N^{(j_L,j_R)}_{2B+2F}(j_L,j_R)=(\tfrac{1}{2},4)+(0,\tfrac{7}{2})+(0,\tfrac{5}{2})\,. \tag{4.31}$$

To see that this is the correct result note that the moduli space of $2B+2F$ together with its jacobian is give by a $\mathbf{P}^7$ bundle over $\mathbf{P}^1\times\mathbf{P}^1$: pick a point in $\mathbf{P}^1\times\mathbf{P}^1$, the moduli space of curves passing through that point in the class $2B+2F$ is given by $\mathbf{P}^7$. Thus the diagonal $SU(2)_L\times SU(2)_R$ action which just the Lefshetz action is given by

$$(\tfrac{1}{2})\otimes(\tfrac{1}{2})\otimes(\tfrac{7}{2})=(\tfrac{5}{2})+2(\tfrac{7}{2})+(\tfrac{9}{2})\,. \tag{4.32}$$

Note that since $2B+2F$ is a genus one curve the corresponding jacobian is also genus one and therefore j_L can only be $0, \frac{1}{2}$. From this restriction on j_L and the above diagonal action we see that the unique left-right spin content is given by

$$(\tfrac{1}{2}, 4) + (0, \tfrac{7}{2}) + (0, \tfrac{5}{2})\,, \tag{4.33}$$

exactly as predicted by the partition function calculation.

It would be interesting to generalize the notion of topological vertex to depend on two parameters q_1, q_2 instead of just q. In a sense from [5] we already have a prediction for what this should be when two representations of the vertex are trivial.

5 Partition functions from the topological vertex

In this section we determine the A-model partition functions for the various CY3-fold geometries we discussed in the last section. The genus zero contribution to the partition function determines the prepotential of the corresponding gauge theory realized via geometric engineering.

The partition functions are determined mostly using the topological vertex [12]. However, in some cases it is easier to use the Chern-Simons theory [53–55].

We will discuss in detail the partition function of the CY3-folds which realize the $U(1)$ and $U(2)$ theory with an adjoint hypermultiplet as well as the CY3-folds which realize $U(1)$ and $U(2)$ theory with 2 and 4 fundamental hypermultiplets respectively. We will also give the expressions for the case of corresponding $U(N)$ theories using the Weyl symmetry present in the geometry.

5.1 $U(N)$ with massive adjoint

5.1.1 $N=1$

We start by discussing the case of 5-dimensional $U(1)$ theory. The geometry of the corresponding CY3-fold is encoded in the (p,q) 5-brane web diagram shown in Fig. 29(a). Given the web configuration we can proceed with the partition function calculation. Using the topological vertex techniques [12] the partition function in this case is given by

$$Z(T, T_m, q) := \sum_R e^{-T\,\ell_R}\,(-1)^{\ell_R}\,Z_R(T_m, q)\,, \tag{5.1}$$

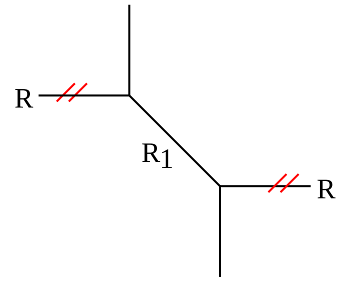

Figure 29: The web diagram for $U(1)$ theory with an adjoint field.

where

$$Z_R(T_m,q) \;=\; \sum_{R_1} e^{-T_m\,\ell_{R_1}}\,(-1)^{\ell_{R_1}} C_{\bullet\,R_1^t\,R}(q)\,C_{\bullet\,R_1 R^t}(q)\,. \tag{5.2}$$

The two Kähler parameters T and T_m are related to the coupling constant of the gauge theory and the mass of adjoint hypermultiplet respectively,

$$\begin{aligned} Q_\tau &:= e^{-T-T_m} = e^{2\pi i\tau}\,, \\ Q_m &:= e^{-T_m} = e^{\beta m}\,. \end{aligned} \tag{5.3}$$

The partition function $Z_R(T_m,q)$ can be determined using the expression of the topological vertex in terms of the Schur and skew-Schur polynomials derived in [56] and given in the last section. Let us denote the Young diagram corresponding to R and R_1 by μ_R and μ_{R_1} respectively. Then Z_R is given by,

$$\begin{aligned} Z_R &= s_R(q^{-\rho})s_{R^t}(q^{-\rho})\sum_{R_1} Q_m^{\ell_{R_1}}(-1)^{\ell_{R_1}}\,s_{R_1^t}(q^{-\mu-\rho})\,s_{R_1}(q^{-\mu^t-\rho})\,, \\ &= s_R(q^{-\rho})s_{R^t}(q^{-\rho})\sum_{R_1} s_{R_1^t}(-Q_m\,q^{-\mu_{R^t}-\rho})\,s_{R_1}(q^{-\mu_R-\rho})\,, \end{aligned} \tag{5.4}$$

where in the second equation we used the fact that $s_{R_1}(x)$ is a homogeneous function of degree l_{R_1}. Now using the identity

$$\sum_R s_{R^t}(x)s_R(y) = \prod_{i,j}(1+x_iy_j)\,, \tag{5.5}$$

we immediately get

$$\begin{aligned} Z_R &= s_R(q^{-\rho})s_{R^t}(q^{-\rho})\prod_{i,j\geq 1}(1-Q_mq^{-\mu_i-\rho_i-\mu_j^t-\rho_j})\,, \\ &= s_R(q^{-\rho})s_{R^t}(q^{-\rho})\prod_{i,j\geq 1}(1-Q_mq^{-\mu_i+i-\mu_j^t+j-1})\,, \\ &= s_R(q^{-\rho})s_{R^t}(q^{-\rho})\prod_{k=0}^{\infty}(1-Q_mq^{k+1})^{k+1}\prod_{(i,j)\in R}(1-Q_mq^{h(i,j)})(1-Q_mq^{-h(i,j)})\,, \end{aligned} \tag{5.6}$$

where

$$h_R(i,j) = \mu_{R,i} - j + \mu_{R^t,j} - i + 1 \tag{5.7}$$

is the hook length and we have used the relation

$$\begin{aligned}
\sum_{i,j\geq 1} q^{-h(i,j)} &= \frac{q}{(1-q)^2} + f_{R,R^t}(q) = \frac{q}{(q-1)^2} + \sum_{(i,j)\in R} (q^{h(i,j)} + q^{-h(i,j)})\,, \qquad (5.8)\\
f_{R,R^t}(q) &= q^{-1}(q-1)^2 f_R(q) f_{R^t}(q) + f_R(q) + f_{R^t}(q)\,,\ f_R(q) = \sum_{(i,j)\in R} q^{j-i}\,.
\end{aligned}$$

Thus the full partition function is given by ($Q = e^{-T}$)

$$Z = \prod_{k\geq 0}^{\infty} (1 - Q_m q^{k+1})^{k+1} \sum_R Q^{\ell_R} (-1)^{\ell_R} s_R(q^{-\rho})\, s_{R^t}(q^{-\rho}) \prod_{(i,j)\in R} (1 - Q_m q^{h(i,j)})(1 - Q_m q^{-h(i,j)})\,.$$

Using the definition of $s_R(q^{-\rho}) = (-1)^{\ell_R} C_{R^t \bullet\bullet}$ we get

$$\begin{aligned}
Z &= \prod_{k\geq 0}^{\infty} (1 - Q_m q^{k+1})^{k+1} \sum_R Q^{\ell_R} (-1)^{\ell_R} q^{\sum_{(i,j)\in R} h(i,j)} \prod_{(i,j)\in R} \frac{(1 - Q_m q^{h(i,j)})(1 - Q_m q^{-h(i,j)})}{(1 - q^{h(i,j)})^2}\,,\\
&= \prod_{k\geq 0}^{\infty} (1 - Q_m q^{k+1})^{k+1} \sum_R (QQ_m)^{l_R} \prod_{(i,j)\in R} \frac{(1 - Q_m q^{h(i,j)})(1 - Q_m^{-1} q^{h(i,j)})}{(1 - q^{h(i,j)})^2}\,. \qquad (5.9)
\end{aligned}$$

The first term in the above expression gives the perturbative contribution to the gauge theory prepotential. The instanton part of the above partition function is given by

$$Z_{inst} = \sum_R Q^{\ell_R} \prod_{(i,j)\in R} \frac{(1 - Q_m q^{h(i,j)})(1 - Q_m q^{-h(i,j)})}{(1 - q^{h(i,j)})(1 - q^{-h(i,j)})}\,. \tag{5.10}$$

This is the 5-dimensional partition function. To obtain the partition function of the 4 dimensional gauge theory we take the limit $\beta \to 0$ such that $q = e^{-\beta\epsilon}$,

$$Z_{inst} = \sum_R e^{2\pi i \tau \ell_R} \prod_{(i,j)\in R} \frac{(h(i,j)\epsilon + m)(h(i,j)\epsilon - m)}{(\epsilon h(i,j))^2}\,. \tag{5.11}$$

which agrees with the results of [57].

From Chern-Simons theory: We can calculate the above 5D partition function from the Chern-Simons theory also. The advantage of this approach is that we get the infinite product structure of the partition naturally.

Let us briefly review the calculation of the A-model partition function using geometric transition and the Chern-Simons theory following [12]. The geometry we are studying admits a geometric transition, *i.e.*, if we take the length of the internal line in the web diagram , given by T_m, to zero we can separate the two lines of the web diagram in the direction transverse to the plane in which the web is embedded. This is a complex structure deformation of the geometry. Since the lines in the web diagram are the loci of degeneration of the torus fibered over the plane of the web hence when the line are separated from each other we get an $\mathbf{S}^3$. Locally this is exactly the conifold transition the only difference being the globally the geometry is different from that of a conifold. The closed topological string partition function of the geometry we started from is given by the topological open string partition function of the geometry obtained after the transition. In the case of the conifold the open topological string theory partition function is given by the partition function of the $U(N)$ Chern-Simons theory on $\mathbf{S}^3$. The coupling constant, k, of the theory and rank of the gauge group, N, are related to the size of the $\mathbf{P}^1$ and the string coupling as follows

$$T_m = N\lambda_s\,,\;\; \lambda_s = \frac{2\pi}{N+k}\,. \tag{5.12}$$

But for our geometry there is an extra subtlety because of the compact circle with boundary on $\mathbf{S}^3$. As discussed in detail in [12, 55] the CS action is modified if there are finite area holomorphic maps with boundaries on the three cycles. In the case of such a holomorphic map the CS action gets modifies by the operator

$$\begin{aligned} \mathcal{O}(\tau) &= \sum_{n=1}^{\infty}\frac{Q_\tau^n}{n}\,\mathrm{Tr}U^n\mathrm{Tr}V^n\,, \\ &= \log\Big(\sum_R Q_\tau^{\ell_R}\mathrm{Tr}_R U\,\mathrm{Tr}_R V\Big)\,. \end{aligned} \tag{5.13}$$

where U, V are the holonomies around the two circles of the annulus. In the case at hand $U = V^{-1}$. Because of the geometry of the annulus in the target space we also have extra winding numbers. Thus the operator that modifies the CS theory action is given by

$$\sum_{k=1}^{\infty}\mathcal{O}(k\tau) \tag{5.14}$$

Thus the partition function is given by

$$
\begin{aligned}
Z &= \int \mathcal{D}A\, e^{S_{cs}(A)+\sum_{k=1}^{\infty}\mathcal{O}(k\tau)} = \langle e^{\sum_k \mathcal{O}(k\tau)}\rangle\,, \qquad (5.15)\\
&= \sum_{R_{1,2,\cdots}} Q_\tau^{\sum_{k=1}^{\infty} k\ell_{R_k}} \langle \prod_{k=1}^{\infty} \mathrm{Tr}_{R_k} U \mathrm{Tr}_{R_k} U^{-1}\rangle\,,\\
&= \sum_{R_{1,2,\cdots}} Q_\tau^{\sum_{k=1}^{\infty} k\ell_{R_k}} \langle \mathrm{Tr}_{\otimes_k R_k} U \mathrm{Tr}_{\otimes_k R_k} U^{-1}\rangle\,,\\
&= S^{-1}_{\bullet\bullet}(q,q^N) \sum_{R_{1,2,\cdots}} Q_\tau^{\sum_{k=1}^{\infty} k\ell_{R_k}} (-1)^{\sum_k \ell_{R_k}} W_{\otimes_k R_k \otimes R_k,\bullet}(q,q^N)\,.
\end{aligned}
$$

$W_{R,\bullet}(q,q^N)$ is the quantum dimension of R,

$$
W_{R,\bullet} = \prod_{(i,j)\in R} \frac{q^{(j-i+N)/2}-q^{-(j-i+N)/2}}{q^{h_R(i,j)/2}-q^{-h_R(i,j)/2}}\,, \quad h_R(i,j) = \mu_{R,i} - i + \mu_{R^t,j} - j + 1\,. \qquad (5.16)
$$

and $S^{-1}_{\bullet\bullet}$ is the perturbative contribution to the partition function,

$$
S^{-1}_{\bullet\bullet} = \prod_{k=1}^{\infty} (1-q^{k-N})^k\,. \qquad (5.17)
$$

The partition function can then be written as

$$
\begin{aligned}
Z: &= S^{-1}_{\bullet\bullet} \prod_{k=1}^{\infty} \sum_R Q_\tau^{k\ell_R} (-1)^{\ell_R} W_R(q,q^N) W_R(q,q^N)\,, \qquad (5.18)\\
&= \prod_{k=1}^{\infty} \sum_R Q_\tau^{\ell_R} (-1)^{\ell_R} W_R(q,q^N) W_R(q,q^N)
\end{aligned}
$$

where

$$
\begin{aligned}
K(Q,q,\lambda) &= \sum_R Q^{\ell_R}\, (-1)^{\ell_R} W_R(q,q^N)\, W_R(q,q^N) \qquad (5.19)\\
&= \exp\Big(-\sum_{n=1}^{\infty} \frac{Q^n}{n} W_{\Box}(q^n,q^{nN})\, W_{\Box}(q^n,q^{nN})\Big)\,,\\
&= \exp\Big(-\sum_{n=1}^{\infty} \frac{Q^n}{n} \Big(\frac{q^{nN/2}-q^{-nN/2}}{q^{n/2}-q^{-n/2}}\Big)^2\Big)\,,\\
&= (1-Q) \prod_{r=0}^{\infty} \Big(\frac{(1-q^{r+1+N}Q)(1-q^{r+1-N}Q)}{(1-q^r\, Q)(1-q^{r+2}\, Q)}\Big)^{r+1}\,.
\end{aligned}
$$

Since $Q_m = e^{-T_m} = q^{-N}$ the full partition function is given by

$$Z = \Big(\prod_{k=1}^{\infty}(1-Q_\tau^k)\Big)\Big(\prod_{k=1}^{\infty}(1-Q_m q^k)^k\Big)\prod_{k,r=1}^{\infty}\Big(\frac{(1-q^r Q_m Q_\tau^k)(1-q^r Q_m^{-1}Q_\tau^k)}{(1-q^{r-1}Q_\tau^k)(1-q^{r+1}Q_\tau^k)}\Big)^r. \quad (5.20)$$

The above expression agrees with toplogical vertex computation, Eq(5.9), except for the first term, $\prod_{k=1}^{\infty}(1-Q_\tau^k)$. The reason for this is that in calculating the partition function we neglected the contribution coming from the annuli which does not end on the three cycle so that U, V are trivial. The contribution of such annuli is given by

$$\sum_R Q_\tau^{\ell_R} = \prod_{k=1}^{\infty}(1-Q_\tau^k)^{-1}\,, \quad (5.21)$$

which cancels the first term in Eq(5.20). Thus the Chern-Simons computation agrees with the topological vertex calculation and moreover it naturally gives the partition function as an infinite product.

Partition function of the 6-dimensional theory: Now lets consider the case of geometry giving rise to 6 dimensional gauge theory with massive adjoint. The corresponding web diagram is shown in Fig. 30 below. The partition function is given by

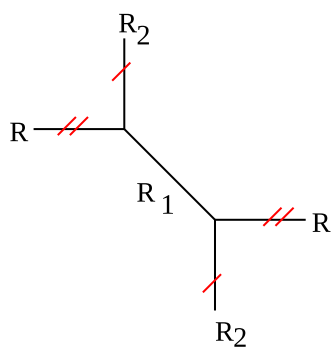

Figure 30: The web diagram of the 6-dimensional adjoint theory.

$$\begin{aligned} Z &:= \sum_{R,R_1,R_2} Q^{\ell_R} Q_1^{\ell_{R_2}} Q_m^{\ell_{R_1}} (-1)^{\ell_R+\ell_{R_1}+\ell_{R_2}} C_{R\,R_2\,R_1}\, C_{R^t\,R_2^t\,R_1^t}\,, \\ &= \sum_R Q^{\ell_R}(-1)^{\ell_R}\, Z_R(Q_1,Q_m)\,, \end{aligned} \quad (5.22)$$

where

$$Z_R(Q_1,Q_m) = \sum_{R_1,R_2} Q_1^{\ell_{R_2}} Q_m^{\ell_{R_1}} (-1)^{\ell_{R_1}+\ell_{R_2}} C_{R\,R_2\,R_1}\, C_{R^t\,R_2^t\,R_1^t} \quad (5.23)$$

Using Eq(4.19) Z_R is given by

$$\begin{aligned} Z_R &= s_R(q^{-\rho})s_{R^t}(q^{-\rho}) \sum_{R_1,R_2} Q_1^{\ell_{R_1}} Q_m^{\ell_{R_2}} \Big(\sum_{R_3} s_{R_2^t/R_3}(q^{-\mu_R-\rho}) s_{R_1/R_3}(q^{-\mu_{R^t}-\rho}) \Big) \\ &\quad \Big(\sum_{R_4} s_{R_2/R_4}(q^{-\mu_{R^t}-\rho}) s_{R_1^t/R_4}(q^{-\mu_R-\rho}) \Big), \\ &= s_R(q^{-\rho})s_{R^t}(q^{-\rho}) \sum_{R_3,R_4} s_{R_2^t/R_3}(q^{-\mu_R-\rho}) s_{R_2/R_4}(q^{-\mu_{R^t}-\rho}) Q_m^{\ell_{R_3}}(-1)^{\ell_{R_3}} \\ &\quad \Big(\sum_{R_1} s_{R_1/R_3}(-Q_m q^{-\mu_{R^t}-\rho}) s_{R_1^t/R_4}(q^{-\mu_R-\rho}) \Big) \end{aligned} \tag{5.24}$$

Using the identity ([71] page 93),

$$\sum_R s_{R/R_3}(x) s_{R^t/R_4}(y) = \prod_{i,j}(1+x_i y_j) \sum_{\widetilde{R}} s_{R_4^t/\widetilde{R}}(x) s_{R_4/\widetilde{R}}(y)\,. \tag{5.25}$$

we get

$$\begin{aligned} Z_R &= s_R(q^{-\rho})s_{R^t}(q^{-\rho}) \prod_{i,j}(1-Q_m q^{-h_R(i,j)}) \\ &\quad \sum_{R_2,\widetilde{R}} Q_1^{\ell_{R_2}} Q_m^{\ell_{\widetilde{R}}} (-1)^{\ell_{\widetilde{R}}+\ell_{R_2}} s_{R_2^t/\widetilde{R}}(q^{-\mu_R-\rho}, Q_m q^{\mu_R+\rho})\, s_{R_2/\widetilde{R}^t}(q^{-\mu_{R^t}-\rho}, Q_m q^{\mu_{R^t}+\rho})\,. \end{aligned} \tag{5.26}$$

Where

$$s_{R/R_1}(x,y) = \sum_{R_4} s_{R/R_4}(x) s_{R_4/R_1}(y)\,, \tag{5.27}$$

The sum over R_2 and $\widetilde{R}$ can be determined exactly using the following identity ([71], page 93),

$$\begin{aligned} &\sum_{A,B} Q_1^{\ell_A} Q_2^{\ell_B} (-1)^{\ell_A+\ell_B} s_{A/B^t}(x,y) s_{A^t/B}(z,w) = \\ &\prod_{k=1}^{\infty} \frac{\prod_{i,j}(1-Q_1^k Q_2^{k-1} x_i z_j)(1-Q_1^k Q_2^{k-1} x_i w_j)(1-Q_1^k Q_2^{k-1} y_i z_j)(1-Q_1^k Q_2^{k-1} y_i w_j)}{(1-Q_1^k Q_2^k)}\,. \end{aligned} \tag{5.28}$$

$$\begin{aligned} Z_R &= \frac{s_R(q^{-\rho})s_{R^t}(q^{-\rho})}{\prod_{k=1}^{\infty}(1-Q_1^k Q_2^k)} \prod_{i,j}(1-Q_m q^{-h_R(i,j)}) \\ &\quad \prod_{k=1}^{\infty}\prod_{i,j}(1-Q_1^k Q_m^{k-1} q^{-h_R(i,j)})(1-Q_1^k Q_m^{k+1} q^{h_R(i,j)})(1-Q_1^k Q_m^k q^{-\mu_{R,i}-\rho_i+\mu_{R^t,j}+\rho_j}) \\ &\quad (1-Q_1^k Q_m^k q^{-\mu_{R^t,i}-\rho_i+\mu_{R,j}+\rho_j})\,. \end{aligned} \tag{5.29}$$

The above expression can be simplified using

$$\prod_{i,j}(1-Q_1^kQ_m^k q^{-\mu_{R,i}-\rho_i+\mu_{R^t,j}+\rho_j})(1-Q_1^kQ_m^k q^{-\mu_{R^t,i}-\rho_i+\mu_{R,j}+\rho_j}) = \prod_{i,j}\frac{1}{(1-Q_1^kQ_m^kq^{-h(i,j)})(1-Q_1^kQ_m^kq^{h(i,j)})}\,. \tag{5.30}$$

to obtain

$$\begin{aligned} Z_R &= \frac{s_R(q^{-\rho})s_{R^t}(q^{-\rho})}{\prod_{k=1}^{\infty}(1-Q_1^kQ_2^k)}\prod_{i,j}(1-Q_mq^{-h_R(i,j)}) \\ &\prod_{k=1}^{\infty}\prod_{i,j}\frac{(1-Q_1^kQ_m^{k-1}\,q^{-h_R(i,j)})(1-Q_1^kQ_m^{k+1}q^{h_R(i,j)})}{(1-Q_1^kQ_m^kq^{-h_R(i,j)})(1-Q_1^kQ_m^kq^{h_R(i,j)})}\,, \qquad (5.31)\\ &= Z_{\bullet}(-1)^{\ell_R}\prod_{\square\in R}\frac{(1-Q_mq^{h(\square)})(1-Q_mq^{-h(\square)})}{(1-q^{h(\square)})(1-q^{-h(\square)})} \\ &\prod_{k=1}^{\infty}\frac{(1-Q_\rho^kQ_mq^{h(\square)})(1-Q_\rho^kQ_mq^{-h(\square)})(1-Q_\rho^kQ_m^{-1}q^{h(\square)})(1-Q_\rho^kQ_m^{-1}q^{-h(\square)})}{(1-Q_\rho^kq^{h(\square)})^2(1-Q_\rho^kq^{-h(\square)})^2}\,, \qquad (5.32)\end{aligned}$$

Where $Z_{\bullet}$ is the perturbative contribution to the partition function and as discussed before $Q_\rho = Q_1Q_m$,

$$Z_{\bullet} = \prod_{r=0}^{\infty}(1-Q_mq^{r+1})^{r+1}\Big(\prod_{k=1}^{\infty}\frac{(1-Q_\rho^kQ_m^{-1}q^{r+1})(1-Q_\rho^kQ_mq^{-r-1})}{(1-Q_\rho^kq^{r+1})(1-Q_\rho^kq^{-r-1})}\Big)^{r+1}\,. \tag{5.33}$$

Thus the instanton part of the partition function (which is zero for $Q=0$) is given by

$$\begin{aligned} Z_{inst} &= \sum_R (QQ_m)^{\ell_R}\Big(\prod_{\square\in R}\frac{(1-Q_mq^{h(\square)})(1-Q_m^{-1}q^{h(\square)})}{(1-q^{h(\square)})^2}\Big) \\ &\prod_{k=1}^{\infty}\frac{(1-Q_\rho^kQ_mq^{h(\square)})(1-Q_\rho^kQ_mq^{-h(\square)})(1-Q_\rho^kQ_m^{-1}q^{h(\square)})(1-Q_\rho^kQ_m^{-1}q^{-h(\square)})}{(1-Q_\rho^kq^{h(\square)})^2(1-Q_\rho^kq^{-h(\square)})^2}\,, \qquad (5.34)\end{aligned}$$

Note that for $Q_m = 1$ i.e., $m=0$ the full partition function is given by

$$Z = \prod_{k=0}^{\infty}\frac{(1-q^{k+1})^{k+1}}{(1-Q_\rho^{k+1})(1-Q_\tau^{k+1})}\,. \tag{5.35}$$

5.1.2 $N=2$

In this case the geometry is shown in Fig. 31. The partition function is given by

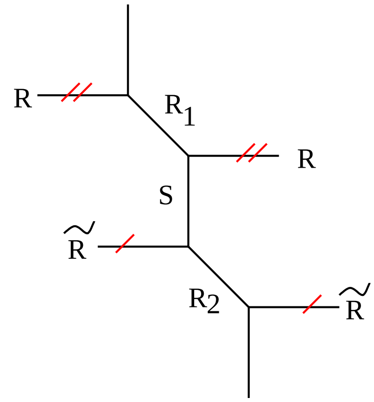

Figure 31: The web diagram of the 5-dimensional $U(2)$ adjoint theory.

$$
\begin{aligned}
Z :&= \sum_{R,\widetilde{R},R_{1,2},S} Q^{\ell_R+\ell_{\widetilde{R}}} Q_m^{\ell_{R_1}+\ell_{R_2}} Q_f^{\ell_S} (-1)^{\ell_R+\ell_{\widetilde{R}}+\ell_{R_1}+\ell_{R_2}+l_S}\, C_{R^t\,\bullet\,R_1}\, C_{S^t\,R_1^t\,R} C_{\widetilde{R}^t\,S\,R_2} C_{\bullet\,R_2^t\,\widetilde{R}}\,, \\
&= \sum_{R,\widetilde{R}} Q^{\ell_R+\ell_{\widetilde{R}}} (-1)^{\ell_R+\ell_{\widetilde{R}}} K_{R\widetilde{R}}(Q_m,Q_f)\,,
\end{aligned}
\tag{5.36}
$$

where

$$
K_{R\widetilde{R}}(Q_m,Q_f) = \sum_{R_{1,2},S} Q_m^{\ell_{R_1}+\ell_{R_2}} Q_f^{\ell_S} (-1)^{\ell_{R_1}+\ell_{R_2}+\ell_S}\, C_{R^t\,\bullet\,R_1}\, C_{S^t\,R_1^t\,R} C_{\widetilde{R}^t\,S\,R_2} C_{\bullet\,R_2^t\,\widetilde{R}}\,. \tag{5.37}
$$

Using the identities involving the skew-Schur functions the above partition function can be written as a sum over $R,\widetilde{R}$ of a product involving Q_m, Q_f. However, we will use a simpler method which makes use of the fact that the geometry has only a few holomorphic curves which can contribute.

To determine $K_{R,\widetilde{R}}(Q_m,Q_f)$ note that in the limit $Q_f \to 0$ it is clear from the geometry (Fig. 31) that we get two copies of the partition function of the $U(1)$ theory therefore we can write $K_{R,\widetilde{R}}$ as

$$
\begin{aligned}
K_{R\widetilde{R}}(Q_m,Q_f)) \;&=\; \Big(\prod_{k=0}^{\infty}(1-Q_m q^{k+1})^{2(k+1)}\Big)(-Q_m)^{\ell_R+\ell_{\widetilde{R}}} \prod_{(i,j)\in R,\widetilde{R}} \frac{(1-Q_m q^{h(i,j)})(1-Q_m^{-1}q^{h(i,j)})}{(1-q^{h(i,j)})^2} \\
&\exp\Big(\sum_{n=1}^{\infty}\frac{Q_f^n}{n}f_1(q^n)+\frac{(Q_fQ_m)^n}{n}f_2(q^n)+\frac{(Q_fQ_m^2)^n}{n}f_3(q^n)\Big)\,.
\end{aligned}
$$

Here, the three terms in the exponential correspond to the contribution of four holomorphic curves in the geometry. There are three terms rather than four since we are taking the area of the two exceptional curves to be equal to T_m. The contribution of the two curves which are locally like the conifold is given by the prefactor in the above equation. The coefficients $f_i(q)$ can be determined from Eq. (5.37) by expanding it to linear order in Q_f,

$$
\begin{aligned}
f_1(q) &= -\frac{C_{\square\bullet R}}{C_{\bullet\bullet R}}\frac{C_{\widetilde{R}^t\square\bullet}}{C_{\widetilde{R}^t\bullet\bullet}} = -\frac{W_{R\square}}{W_R}\frac{W_{\widetilde{R}^t\square}}{W_{\widetilde{R}^t}} = -\frac{q}{(1-q)^2} - f_{R,\widetilde{R}^t}\,, \\
f_2(q) &= -2f_1(q)\,, \\
f_3(q) &= f_1(q)\,,
\end{aligned}
\tag{5.38}
$$

where we used the identity

$$
C_{\bullet R_1 R_2} = W_{R_1 R_2^t} q^{\kappa_{R_2}/2}\,. \tag{5.39}
$$

$$
\begin{aligned}
K_{R\widetilde{R}}(Q_m,Q_f) &= K_{\bullet\bullet}\,(-Q_m)^{\ell_R+\ell_{\widetilde{R}}}\prod_{(i,j)\in R,\widetilde{R}}\frac{(1-Q_m q^{h(i,j)})(1-Q_m^{-1}q^{h(i,j)})}{(1-q^{h(i,j)})^2} \\
&\quad \prod_k\Big(\frac{(1-Q_f q^k)(1-Q_f Q_m^2 q^k)}{(1-Q_f Q_m q^k)^2}\Big)^{C_k(R,\widetilde{R}^t)}\,,
\end{aligned}
\tag{5.40}
$$

where $\sum_k C_k(R_1,R_2) = f_{R_1R_2}(q)$. $K_{\bullet\bullet}$ contributes to the perturbative part of the partition function,

$$
K_{\bullet\bullet} = \prod_{k=0}^{\infty}\Big((1-Q_m q^{k+1})^2\,(1-Q_f q^{k+1})(1-Q_f Q_m^2)(1-Q_f Q_m q^{k+1})^{-2}\Big)^{k+1}\,. \tag{5.41}
$$

Define $Q_F = Q_f Q_m = e^{-2a\beta}$, where a is the Coulomb branch moduli, then the full partition function is given by

$$
\begin{aligned}
Z &= K_{\bullet\bullet}\sum_{R,\widetilde{R}}(QQ_m)^{\ell_R+\ell_{\widetilde{R}}}\prod_{(i,j)\in R,\widetilde{R}}\frac{(1-Q_m q^{h(i,j)})(1-Q_m^{-1}q^{h(i,j)})}{(1-q^{h(i,j)})^2} \\
&\quad \prod_k\Big(\frac{(1-Q_F Q_m^{-1}q^k)(1-Q_F Q_m q^k)}{(1-Q_F q^k)^2}\Big)^{C_k(R,\widetilde{R}^t)}\,.
\end{aligned}
\tag{5.42}
$$

The above partition function agrees with the results of [57].

Generalization to the case of $U(N)$ is simple using the corresponding web diagram discussed in section 3 and the A_{N-1} Weyl symmetry present in the geometry,

$$
\begin{aligned}
Z_{inst} &= \sum_{R_1,\cdots R_N}(QQ_m)^{\ell_1+\cdots+\ell_N}\prod_{(i,j)\in R_{1,2,\cdots,N}}\frac{(1-Q_m q^{h(i,j)})(1-Q_m^{-1}q^{h(i,j)})}{(1-q^{h(i,j)})^2} \\
&\quad \prod_{1\le i<j\le N}\prod_k\Big(\frac{(1-Q_{F_{ij}}Q_m^{-1}q^k)(1-Q_{F_{ij}}Q_m q^k)}{(1-Q_{F_{ij}}q^k)^2}\Big)^{C_k(R_i,R_j^t)}\,.
\end{aligned}
\tag{5.43}
$$

Where $Q_{F_{ij}} = e^{-\beta(a_i - a_j)}$.

Partition function of the 6-dimensional theory: The geometry giving rise to 6 dimensional theory is shown in Fig. 32. The partition function is given by

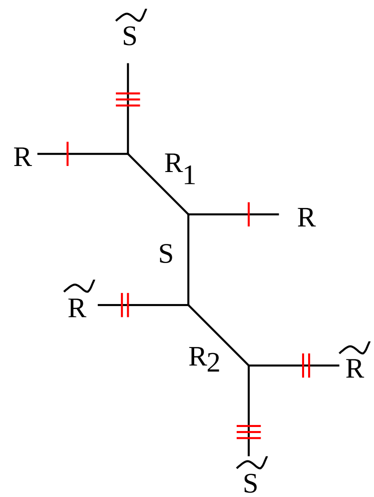

Figure 32: The web diagram of the 6-dimensional $U(2)$ adjoint theory.

$$\begin{aligned}Z := & \sum_{R,\widetilde{R},\widetilde{S},R_{1,2},S} Q^{\ell_R+\ell_{\widetilde{R}}} Q_1^{\ell_{\widetilde{S}}} Q_m^{\ell_{R_1}+\ell_{R_2}} Q_f^{\ell_S} (-1)^{\ell_R+\ell_{\widetilde{R}}+\ell_{R_1}+\ell_{R_2}+\ell_S+\ell_{\widetilde{S}}} C_{R^t\,\widetilde{S}\,R_1} C_{S^t\,R_1^t\,R} C_{\widetilde{R}^t\,S\,R_2} C_{\widetilde{S}^t\,R_2^t\,\widetilde{R}}\,, \\ = & \sum_{R,\widetilde{R}} Q^{\ell_R+\ell_{\widetilde{R}}} (-1)^{\ell_R+\ell_{\widetilde{R}}} \mathcal{K}_{R\widetilde{R}}(Q_m, Q_f, Q_1)\,,\end{aligned} \tag{5.44}$$

where

$$\begin{aligned}&\mathcal{K}_{R\widetilde{R}}(Q_m, Q_f, Q_1) \\ &= \sum_{R_{1,2},S,\widetilde{S}} Q_m^{\ell_{R_1}+\ell_{R_2}} Q_f^{\ell_S} Q_1^{\ell_{\widetilde{S}}} (-1)^{\ell_{R_1}+\ell_{R_2}+\ell_S+\ell_{\widetilde{S}}} C_{R^t\widetilde{S}R_1} C_{S^t\,R_1^t\,R} C_{\widetilde{R}^t\,S\,R_2} C_{\widetilde{S}^t\,R_2^t\,\widetilde{R}}\,.\end{aligned} \tag{5.45}$$

We can write $\mathcal{K}_{R\widetilde{R}}$ as

$$\mathcal{K}_{R\widetilde{R}}(Q_m, Q_f, Q_1) = \mathcal{K}_{R\widetilde{R}}(Q_m, Q_f, Q_1 = 0)\, \exp\Big(\sum_{n,k=1}^{\infty} \frac{(Q_1 Q_m^2 Q_f)^{nk}}{n} f_{n,k}(Q_m, Q_f, q)\Big)\,. \tag{5.46}$$

The coefficients $f_{n,k}$ can be determined easily by comparing the above expression with Eq. (5.45). It turns out that $f_{n,k}$ is independent of k and has the form

$$f_{n,k}(Q_m, Q_f, q) = f(Q_m^n, Q_f^n, q^n)\,. \tag{5.47}$$

The function $f(Q_m, Q_f, q)$ is given by

$$f(Q_m, Q_f, q) = 1 - Q_f f_{R\widetilde{R}^t} - Q_f^{-1} f_{R^t\widetilde{R}} + 2Q_f Q_m f_{R\widetilde{R}^t} \tag{5.48}$$
$$+2(QfQ_m)^{-1} f_{R^t\widetilde{R}} - Q_f Q_m^2 f_{R\widetilde{R}^t} - (Q_f Q_m^2)^{-1} f_{R^t\widetilde{R}} - (2 - Q_m - Q_m^{-1})(f_{R,R^t}(q) + f_{\widetilde{R},\widetilde{R}^t}(q))$$
$$-\frac{q}{(1-q)^2}\Big(Q_f + Q_f^{-1} + 2Q_m + \frac{2}{Q_m} - 2Q_f Q_m - \frac{2}{Q_f Q_m} + Q_f Q_m^2 + \frac{1}{Q_f Q_m^2} - 4\Big)\,.$$

It is easy to see from Eq(5.46) and Eq(5.48) that the partition function of the 6-dimensionaltheory is given by the following substitution in the corresponding 5-dimensional partition function,

$$(1 - z\,q^k) \mapsto (1 - z\,q^k)\prod_{r=1}^{\infty}\frac{(1 - Q_\rho z q^k)(1 - Q_\rho z^{-1} q^{-k})}{(1 - Q_\rho^k)^k}\,. \tag{5.49}$$

5.2 $U(N)$ with $N_f = 2N$

5.2.1 $N = 1$

We start by discussing the case of $U(1)$ theory with two hypermultiplets in the fundamental representation. The CY geometry which gives rise to this theory via geometric engineering [1] is well known and is blowup of $T^*\mathbf{P}^1 \times \mathbf{C}$ at two points. The toric geometry of this CY space is encoded in the toric web shown below (for more details about toric web see [39,54]). Since

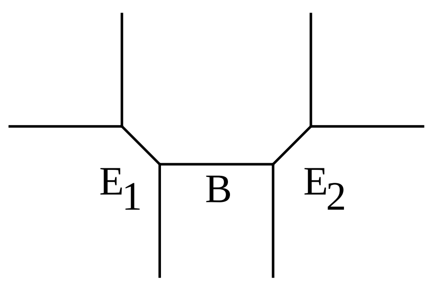

Figure 33: The web diagram of the 5-dimensional $U(1)$ theory with two hypermultiplets.

this geometry is so simple it is possible to write down the partition function almost without any calculation using expression for the free energy in terms of integer invariants [10, 11]. The only holomorphic curves in the geometry are $B, E_1, E_2, B - E_1, B - E_2, B - E_1 - E_2$ with integer invariants [54]

$$N_B^g = N_{B-E_1-E_2}^g = -\delta_{g,0}\,, \tag{5.50}$$
$$N_{E_i}^g = N_{B-E_i}^g = \delta_{g,0}\,.$$

It is easy to derive the above expression from the definition of the integer invariants. To see this note that the curves E_i and $B - E_i$ are rigid and therefore the corresponding moduli space is just a point. On the other hand the moduli space of B and $B-E_1-E_2$ is $\mathbf{C}$. Hence, since the genus zero integer invariant of a curve C with moduli space $\mathcal{M}$ is $(-1)^{\dim \mathcal{M}}\chi(\mathcal{M})$ therefore $N^0_B = N^0_{B-E_1-E_2} = -1$ and $N^0_{E_i} = N^0_{B-E_i} = 1$. Thus the instanton part of the free energy is given by

$$F = \sum_{n=1}^{\infty} \frac{Q_b^n - \lambda_1^{-n} - \lambda_2^{-n} - Q_b^n\lambda_1^{-n} - Q_b^n\lambda_2^{-n} + Q_b^n\lambda_1^{-n}\lambda_2^{-n}}{n(q^{n/2}-q^{-n/2})^2}\,, \tag{5.51}$$

where $Q_b = e^{-T_b}$, $\lambda_i = e^{T_{E_i}}$ and T_b, T_{E_i} are the area of curves B and E_i respectively.

The partition function $Z = e^F$ can be written easily using the multicovering structure of the free energy and is given by

$$Z \;\;=\;\; Z_{pert}\prod_{k=0}^{\infty}\frac{(1-\lambda_1^{-1}Q_b q^{k+1})^{k+1}(1-\lambda_2^{-1}Q_b q^{k+1})^{k+1}}{(1-Q_b q^{k+1})^{k+1}(1-Q_b\lambda_1^{-1}\lambda_2^{-1}q^{k+1})^{k+1}}\,, \tag{5.52}$$

where

$$Z_{pert} = \prod_{k=0}^{\infty}(1-\lambda_1^{-1}q^{k+1})^{k+1}(1-\lambda_2^{-1}q^{k+1})^{k+1}\,, \tag{5.53}$$

and gives the perturbative contribution to the prepotential in the 4D field theory limit because in this limit $\beta \to 0$ such that

$$Q_b = (\frac{\beta\Lambda}{2})^2\,,\;\; \lambda_i = e^{\beta m_i}\,,\;\; q^{-\beta\hbar}\,. \tag{5.54}$$

The partition function of the pure $5D$ $U(1)$ theory is recovered in the limit $\lambda_i \to \infty$.

For the $U(1)$ theory we are discussing we saw that the corresponding geometry is simple enough to allow us to write down the partition function directly. However, for $U(N)$ with $N > 1$ the corresponding geometries are such that the partition functions can not be derived so easily. For this reason we now derive the partition function using the open-closed duality via geometric transition [54] since this method can be extended to the case of geometries giving rise to $U(N)$ theories with $N_f = 2N$. The geometry shown in Fig. 34 has two exceptional curves $E_{1,2}$ with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ and therefore the geometry in the neighborhood of these curves in that of resolved conifold. Thus these curves can be shrunk and deformed into two three cycles which are topologically $\mathbf{S}^3$. The A-model partition function is then given by the partition function of $U(N_1)\times U(N_2)$ Chern-Simons theory on the two three cycles modified by the holomorphic maps between the two three cycles as shown in Fig. 34 below. The partition function in this case is given by

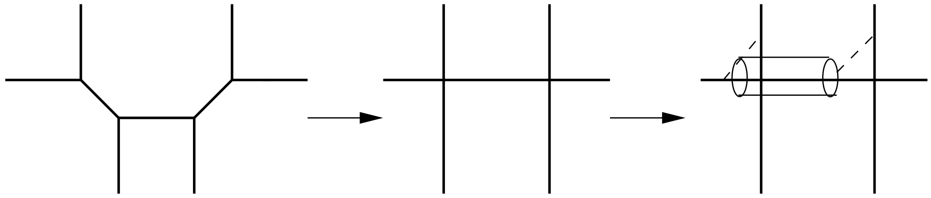

Figure 34: Transition from closed string geometry to open string geometry by a geometric transition.

$$Z = \int e^{S_{cs}(A_1)+S_{cs}(A_2)+\mathcal{O}(r)} \,. \tag{5.55}$$

Here, $A_{1,2}$ are the $U(N_{1,2})$ gauge fields on the two three cycles respectively and $\mathcal{O}(r)$ is the contribution from the annuli shown in Fig. 34 of length r,

$$e^{\mathcal{O}(r)} = \sum_R e^{-r\,\ell_R}\,\mathrm{Tr}_R U_1\,\mathrm{Tr}_R U_2\,, \tag{5.56}$$

where $U_{1,2}$ is the holonomy of $A_{1,2}$ around the two boundary components of the annuli. Thus the partition function is given by

$$\begin{aligned} Z &= \sum_R e^{-r\,\ell_R}\langle \mathrm{Tr}_R U_1\rangle\langle \mathrm{Tr}_R U_2\rangle\,, \qquad (5.57)\\ &= (S^{-1})_{00}(q,\lambda_1)\,(S^{-1})_{00}(q,\lambda_2)\,\sum_R e^{-r\,\ell_R}\,W_R(q,\lambda_1)\,W_R(q,\lambda_2)\,, \end{aligned}$$

where $\langle \mathrm{Tr}_R U_{1,2}\rangle = (S^{-1})_{0R}(q,\lambda_{1,2}) = (S^{-1})_{00}\,W_R(q,\lambda_{1,2})$ and

$$W_R(q,\lambda_i) = \prod_{(i,j)\in R}\frac{[j-i]_{\lambda_i}}{[h(i,j)]}\,,\;\; [x]_\lambda = q^{x/2}\lambda^{1/2} - q^{-x/2}\lambda^{-1/2}\,, \tag{5.58}$$

is the quantum dimension of the representation R with $\lambda_i = q^{N_i}$. Thus the partition function is given by

$$\begin{aligned} Z &= Z_{pert}\sum_R Q^{\ell_R}W_R(q,\lambda_1)\,W_R(q,\lambda_2)\,, \qquad (5.59)\\ &= Z_{pert}\,Z_{inst} \end{aligned}$$

where $Q = e^{-r}$ and $Z_{pert} = (S^{-1})_{00}(q,\lambda_1)(S^{-1})_{00}(q,\lambda_2)$ gives the perturbative contribution to the prepotential in the field theory limit such that $\lambda_i = e^{\beta m_i}$. In terms of the 4D gauge theory $\beta \to 0$ and

$$\begin{aligned} Q\sqrt{\lambda_1\lambda_2} &= (\frac{\beta\Lambda}{2})^2\,, \qquad (5.60)\\ q &= e^{-\beta\hbar}\,,\\ \lambda_i &= e^{\beta m_i}\,. \end{aligned}$$

The sum giving the partition function can be evaluated to get a product formula

$$Z_{pert} = \prod_{k=0}^{\infty}(1-\lambda_1^{-1}q^{k+1})^{k+1}(1-\lambda_2^{-1}q^{k+1})^{k+1}\,, \tag{5.61}$$

and

$$\begin{aligned} Z_{inst} &= \sum_R Q^{\ell_R} W_R(q,\lambda_1)\, W_R(q,\lambda_2)\,, \\ &= \exp\Big(\sum_{n\geq 1}\frac{Q^n}{n}\, W_{\square}(q^n,\lambda_1^n)W_{\square}(q^n,\lambda_2^n)\Big)\,, \\ &= \prod_{k=0}^{\infty}\frac{(1-Q\sqrt{\frac{\lambda_2}{\lambda_1}}q^{k+1})^{k+1}(1-Q\sqrt{\frac{\lambda_1}{\lambda_2}}q^{k+1})^{k+1}}{(1-Q\sqrt{\lambda_2\lambda_1}q^{k+1})^{k+1}(1-\frac{Q}{\sqrt{\lambda_2\lambda_1}}q^{k+1})^{k+1}}\,, \\ &= \prod_{k=0}^{\infty}\frac{(1-\lambda_1^{-1}Q_b q^{k+1})^{k+1}(1-\lambda_2^{-1}Q_b q^{k+1})^{k+1}}{(1-Q_b q^{k+1})^{k+1}(1-Q_b\lambda_1^{-1}\lambda_2^{-1}q^{k+1})^{k+1}}\,, \end{aligned} \tag{5.62}$$

where $Q_b = Q\sqrt{\lambda_1\lambda_2}$ and the partition function of the pure 5D $U(1)$ theory is recovered in the limit

$$\lambda_i \to \infty\,, \quad Q_b = \text{fixed}\,. \tag{5.63}$$

It is easy to see that the above partition function agrees with the one given by Nekrasov and with the one given in Eq. (5.52).

Partition function of the 6-dimensional theory: To discuss the six dimensional case and the corresponding geometries we will have to make toric web diagrams periodic around one extra direction, as discussed in section 3. Consider the case of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to \mathbf{P}^1$ blown up at two points, *i.e.* the geometry we considered in the previous section. In this case we can glue the external lines to obtain the geometry shown in Fig. 35 below.

To obtain the partition function we divide the geometry in two parts as shown in Fig. 35(b). Then the partition function is given by

$$Z = \sum_R Q_b^{\ell_R}(-1)^{\ell_R}\mathcal{K}_R(Q_{m_1},Q_1)\,\mathcal{K}_{R^t}(Q_{m_2},Q_1)\,, \tag{5.64}$$

where

$$\begin{aligned} \mathcal{K}_R(Q_m,Q_1) &= \sum_{R_1,S} Q_1^{\ell_S}Q_m^{\ell_{R_1}}(-1)^{\ell_S+\ell_{R_1}} C_{\bullet\, S^t\, R_1^t}\, C_{R_1\, R^t\, S}\,, \\ &= (-1)^{\ell_R}s_R(q^{-\rho})\sum_{S,R_1}Q_1^{\ell_S}Q_m^{\ell_{R_1}}(-1)^{\ell_{R_1}+\ell_S} \\ &\quad \sum_{R_3,R_4} s_{S/R_3}(q^{-\rho})s_{S^t/R_4}(q^{-\mu_{R^t}-\rho})s_{R_1^t/R_3}(q^{-\rho})s_{R_1/R_4}(q^{-\mu_R-\rho})\,. \end{aligned} \tag{5.65}$$

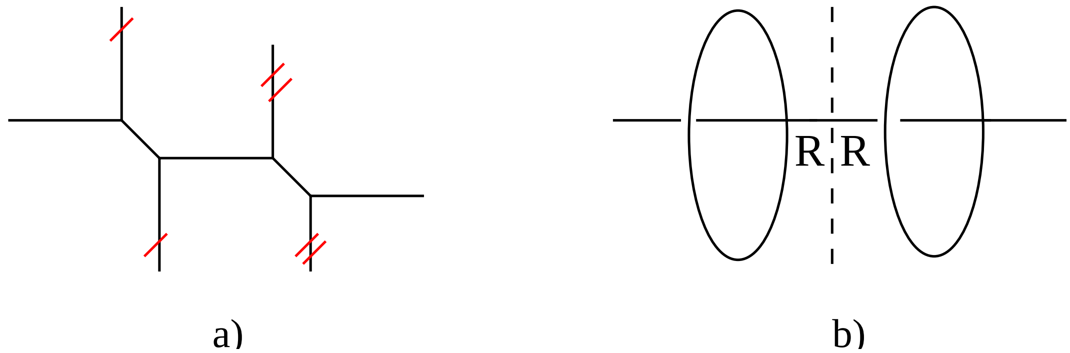

Figure 35: The web diagram of the 6-dimensional $U(1)$ theory with two fundamental hypermultiplets.

Summing over R_1 we get

$$\begin{aligned}\mathcal{K}_R &= (-1)^{\ell_R} s_R(q^{-\rho}) \prod_{i,j}(1-Q_m q^{-\mu_{R,i}-\rho_i-\rho_j}) \\ &\sum_{S,R_3,R_4,R_5} Q_1^{\ell_S} Q_m^{\ell_{R_4}} (-1)^{\ell_{R_4}+\ell_S} s_{S/R_3}(q^{-\rho}) s_{S^t/R_4}(q^{-\mu_{R^t}-\rho}) s_{R_3^t/R_5}(-Q_m q^{-\mu_R-\rho}) s_{R_4^t/R_5^t}(q^{-\rho})\,.\end{aligned} \tag{5.66}$$

Simplifying the above expression by summing over $R_{3,4}$ gives

$$\begin{aligned}\mathcal{K}_R &= (-1)^{\ell_R} s_R(q^{-\rho}) \prod_{i,j}(1-Q_m q^{-\mu_{R,i}-\rho_i-\rho_j}) \\ &\sum_{S,R_5} Q_1^{\ell_S} Q_m^{\ell_{R_5}} (-1)^{\ell_S+\ell_{R_5}} s_{S/R_5^t}(q^{-\rho}, Q_m q^{\mu_R+\rho}) s_{S^t/R_5}(q^{-\mu_{R^t}-\rho}, Q_m q^{-\rho})\,.\end{aligned} \tag{5.67}$$

Using the identity Eq(5.29) we get

$$\begin{aligned}&\mathcal{K}_R = (-1)^{\ell_R} s_R(q^{-\rho}) \prod_{i,j}(1-Q_m q^{-\mu_{R,i}-\rho_i-\rho_j}) \\ &\prod_{k=1}^{\infty} \frac{\prod_{i,j}(1-Q_\rho^k Q_m^{-1} q^{-\mu_{R^t,i}-\rho_i-\rho_j})(1-Q_\rho^k q^{-\rho_i+\rho_j})(1-Q_\rho^k q^{\mu_{R,i}+\rho_i-\mu_{R^t,j}-\rho_j})(1-Q_\rho^k Q_m q^{\mu_{R,i}+\rho_i+\rho_j})}{(1-Q_\rho^k)}\,.\end{aligned} \tag{5.68}$$

To simplify this expression note that

$$\begin{aligned}\prod_{i,j}(1-Q_\rho^k q^{-\rho_i+\rho_j}) &= \prod_{i,j}(1-Q_\rho^k q^{-\rho_i-\rho_j})^{-1}\,, \\ \prod_{i,j}(1-Q_\rho^k q^{\mu_{R,i}+\rho_i-\mu_{R^t,j}-\rho_j}) &= \prod_{i,j}(1-Q_\rho^k q^{-\mu_{R,i}-\rho_i-\mu_{R^t,j}-\rho_j})^{-1}\,.\end{aligned} \tag{5.69}$$

$$\mathcal{K}_R = (-1)^{\ell_R} s_R(q^{-\rho}) \prod_{i,j}(1 - Q_m q^{-\mu_{R,i}-\rho_i-\rho_j}) \tag{5.70}$$

$$\prod_{k=1}^{\infty} \frac{\prod_{i,j}(1 - Q_\rho^k Q_m^{-1} q^{-\mu_{R^t,i}-\rho_i-\rho_j})(1 - Q_\rho^k Q_m q^{\mu_{R,i}+\rho_i+\rho_j})}{(1 - Q_\rho^k)(1 - Q_\rho^k q^{-\rho_i-\rho_j})(1 - Q_\rho^k q^{-\mu_{R,i}-\rho_i-\mu_{R^t,j}-\rho_j})},$$

$$= \mathcal{K}_\bullet (-1)^{\ell_R} s_R(q^{-\rho}) \prod_{(i,j)\in R}(1 - Q_m q^{i-j}) \Big(\prod_{k=1}^{\infty} \frac{(1 - Q_\rho^k Q_m^{-1} q^{j-i})(1 - Q_\rho^k Q_m q^{i-j})}{(1 - Q_\rho^k q^{h_R(i,j)})(1 - Q_\rho^k q^{-h_R(i,j)})} .$$

Where we have used the identity

$$\sum_{i,j} q^{\mu_{R,i}+\rho_i+\rho_j} = \frac{q}{(1-q)^2} + \sum_{(i,j)\in R} q^{j-i} . \tag{5.71}$$

and $\mathcal{K}_\bullet$ is the perturbative contribution

$$\mathcal{K}_\bullet(Q_m) = \prod_{r=0}^{\infty}(1 - Q_m q^{r+1})^{r+1} \Big(\prod_{k=1}^{\infty} \frac{(1 - Q_\rho^k Q_m^{-1} q^{r+1})^{r+1}(1 - Q_\rho^k Q_m q^{-r-1})^{r+1}}{(1 - Q_\rho^k)(1 - Q_\rho^k q^{r+1})^{2r+2}} \Big) \tag{5.72}$$

Thus the full partition function is given by

$$Z = \mathcal{K}_\bullet(Q_{m_1}) \mathcal{K}_\bullet(Q_{m_2}) \sum_R Q_b^{\ell_R} \prod_{(i,j)\in R} \frac{(1 - Q_{m_1} q^{j-i})(1 - Q_{m_2} q^{i-j})}{(1 - q^{h_R(i,j)})(1 - q^{-h_R(i,j)})} \tag{5.73}$$

$$\prod_{k=1}^{\infty} \frac{(1 - Q_\rho^k Q_{m_1}^{-1} q^{j-i})(1 - Q_\rho^k Q_{m_1} q^{i-j})(1 - Q_\rho^k Q_{m_2}^{-1} q^{i-j})(1 - Q_\rho^k Q_{m_2} q^{j-i})}{(1 - Q_\rho^k q^{h_R(i,j)})^2 (1 - Q_\rho^k q^{-h_R(i,j)})^2} .$$

The 5D limit is given by $Q \to 0$ and the 4D limit is given by $\beta \to 0$.

5.2.2 $N = 2$

The Calabi-Yau geometry giving rise to this theory is shown in Fig. 36 below. The partition function for this case was calculated in [54] and is given by

$$Z = \sum_{R_{1,2,3,4}} Q_{B_1}^{\ell_1} Q_{B_2}^{\ell_3} Q_{F_1}^{\ell_2} Q_{F_2}^{\ell_4} W_{R_1R_4}(\lambda_4, q) W_{R_4R_3}(\lambda_3, q) W_{R_3R_2}(\lambda_2, q) W_{R_2R_1}(\lambda_1, q) ,$$

where

$$Q_{B_{1,2}} = e^{-T_{B_{1,2}}} , \;\; Q_{F_{1,2}} = e^{-T_{F_{1,2}}} , \;\; \lambda_{1,2,3,4} = e^{t_{1,2,3,4}} . \tag{5.74}$$

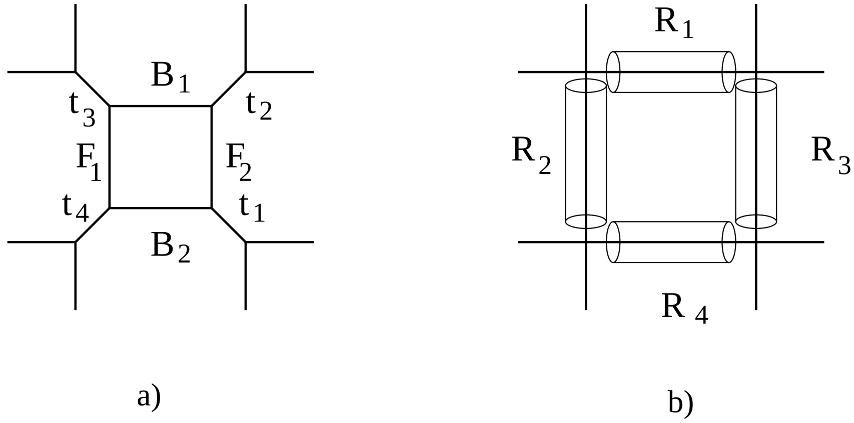

Figure 36: a) The web diagram of the 5-dimensional $U(2)$ theory with four fundamental hypermultiplets, b) the corresponding open string geometry obtained by geometric transition.

T_{B_i} and T_{F_i} are the lengths of the annuli shown in Fig. 36(b) and t_a are the area of the four exceptional curves and

$$W_{R_1R_2} = \sum_R N^R_{R_1R_2} q^{\frac{1}{2}(\kappa_R - \kappa_{R_1} - \kappa_{R_2})} W_R \,. \tag{5.75}$$

The partition function can be written as $Z = Z_{pert} Z_{inst}$ where Z_{pert} is the perturbative contribution to the partition function and Z_{inst} is the instanton contribution. From the discussion of section 3 it follows that the instanton contribution arises from the terms involving $Q_{B_{1,2}}$. In the following we will focus our attention on the instanton contribution only.

To determine the partition function note that it can be written as

$$Z = \sum_{R_1,R_3} Q_{B_1}^{\ell_{R_1}} Q_{B_2}^{\ell_{R_3}} G_{R_1R_3}(Q_{F_1}, \lambda_1, \lambda_2) G_{R_1R_2}(Q_{F_2}, \lambda_4, \lambda_3)\,, \tag{5.76}$$

where

$$G_{R_1R_2}(Q, \lambda_1, \lambda_2) \;\; = \;\; \sum_R W_{R_1R}(q, \lambda_1) W_{RR_2}(q, \lambda_2) Q^{\ell_R} \,. \tag{5.77}$$

As discussed in before the Hopf link invariants $W_{R_1R_2}(\lambda, q)$ are given by Schur functions, $s_R(x)$. Using the identity

$$\sum_R s_R(x) s_R(y) = \prod_{i,j} (1 - Q x_i y_j)^{-1} = \mathrm{Exp}\Big(\sum_{n=1}^{\infty} \frac{Q^n}{n} f(x^n, y^n) \Big)\,, \;\; f(x,y) = \sum_{i,j} x_i y_j \,. \tag{5.78}$$

we get

$$G_{R_1R_2}(Q, \lambda_1, \lambda_2) \;\; = \;\; W_{R_1}(q, \lambda_1) W_{R_2}(q, \lambda_2) \mathrm{Exp}\Big(\sum_{n=1}^{\infty} \frac{Q^n}{n} F^{R_1R_2}(q^n, \lambda_1^n, \lambda_2^n) \Big)\,, \tag{5.79}$$

The function $F^{R_1R_2}$ can be determined easily by expanding Eq(5.77) to first order in Q,

$$F^{R_1R_2} - F^{\bullet,\bullet} = \sqrt{\lambda_1\lambda_2} f_{R_1R_2} - \sqrt{\frac{\lambda_2}{\lambda_1}} f_{R_2} - \sqrt{\frac{\lambda_1}{\lambda_2}} f_{R_1} \tag{5.80}$$

Where the function f_R and $f_{R_1R_2}$ are given by

$$\begin{aligned} f_R(q) &= \sum_{(i,j)\in R} q^{j-i}, \\ f_{R_1R_2}(q) &:= \sum_k C_k(R_1,R_2)q^k = (q+q^{-1}-2)f_{R_1}f_{R_2} + f_{R_1} + f_{R_2}\,. \end{aligned} \tag{5.81}$$

Using the above definitions in Eq(5.77) we get

$$G_{R_1R_2}(Q,\lambda_1,\lambda_2) = G_{00}(Q,\lambda_1,\lambda_2)\frac{\prod_k(1-q^k\sqrt{\frac{\lambda_1}{\lambda_2}}Q)^{C_k(R_1,\bullet)}\prod_k(1-q^k\sqrt{\frac{\lambda_2}{\lambda_1}}Q)^{C_k(R_2,\bullet)}}{\prod_k(1-q^k\sqrt{\lambda_1\lambda_2}Q)^{C_k(R_1,R_2)}}\,.$$

The full partition function is given by

$$Z = Z_{pert}\sum_{R_{1,2}} Q_{B_1}^{\ell_1}Q_{B_2}^{\ell_2}W_{R_1}(q,\lambda_1)W_{R_2}(q,\lambda_2)W_{R_2}(q,\lambda_3)W_{R_1}(q,\lambda_4)$$
$$\frac{\prod_k(1-q^k\sqrt{\frac{\lambda_1}{\lambda_2}}Q_{F_1})^{C_k(R_1,\bullet)}(1-q^k\sqrt{\frac{\lambda_2}{\lambda_1}}Q_{F_1})^{C_k(R_2,\bullet)}(1-q^k\sqrt{\frac{\lambda_3}{\lambda_4}}Q_{F_2})^{C_k(R_2,\bullet)}(1-q^k\sqrt{\frac{\lambda_4}{\lambda_3}}Q_{F_2})^{C_k(R_1,\bullet)}}{\prod_k(1-q^k\sqrt{\lambda_1\lambda_2}Q_{F_1})^{C_k(R_1,R_2)}(1-q^k\sqrt{\lambda_3\lambda_4}Q_{F_2})^{C_k(R_1,R_2)}}\,, \tag{5.82}$$

Define the renormalized Kähler parameters $T_{b,f}$ of the base and the fiber $\mathbf{P}^1$,

$$\begin{aligned} T_{B_1} &= T_b - \frac{1}{2}(t_1+t_4)\,, \\ T_{B_2} &= T_b - \frac{1}{2}(t_3+t_4)\,, \\ T_{F_1} &= T_f - \frac{1}{2}(t_1+t_2)\,, \\ T_{F_2} &= T_f - \frac{1}{2}(t_3+t_4)\,. \end{aligned} \tag{5.83}$$

Then in terms of the renormalized parameters we get ($C_k(R) = C_k(R,\bullet)$)

$$Z_{inst} = \sum_{R_{1,2}} Q_b^{\ell_1+\ell_2} Z^{(0)}_{R_1,R_2}\prod_k (1-q^k\lambda_1^{-1})^{C_k(R_1)}(1-q^k\lambda_1^{-1}Q_f)^{C_k(R_2)}(1-q^{-k}\lambda_2^{-1})^{C_k(R_2)}(1-q^k\lambda_2^{-1}Q_f)^{C_k(R_1)}$$
$$(1-q^{-k}\lambda_3^{-1})^{C_k(R_2)}(1-q^k\lambda_3^{-1}Q_f)^{C_k(R_1)}(1-q^k\lambda_4^{-1})^{C_k(R_1)}(1-q^k\lambda_4^{-1}Q_f)^{C_k(R_2)}\,. \tag{5.84}$$

Where

$$Z^{(0)}_{R_1 R_2} = \frac{C_{R_1}(q)^2 C_{R_2}(q)^2}{\prod_k (1-q^k Q)^{C_k(R_1,R_2)}} \,. \tag{5.85}$$

The renormalized parameters are define such that in the limit $\lambda_i \to \infty$ we get the partition function of pure 5D gauge theory, *i.e.* the A-model partition function of local $\mathbf{P}^1 \times \mathbf{P}^1$.

To obtain the partition function of the 4-dimensional gauge theory we have to take the limit

$$Q_f = e^{-2a\beta}\,,\;\; \lambda_a = e^{-\beta m_a}\,,\;\; q = e^{-\beta\epsilon}\,,\;\; \beta \mapsto 0\,. \tag{5.86}$$

In this limit it is easy to show that Eq(5.85) agrees with the results of [5]. The case of $N_f = 0, 1$ was discussed recently in [68].

$U(N)$ **with** $N_f = 2N$**:** The Calabi-Yau geometry in this case is shown in Fig. 37 below. The

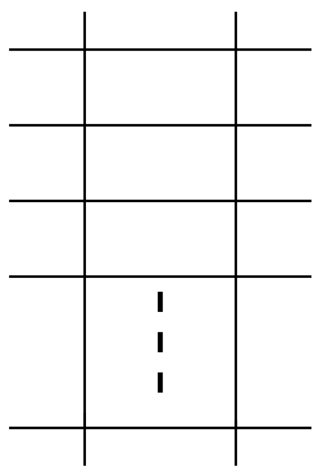

Figure 37: The web diagram of the 5-dimensional $U(N)$ theory with $2N$ fundamental hypermultiplets.

partition function can be calculated using either the topological vertex or the Chern-Simons theory. We just state the result which can also be obtained from the Weyl symmetry present in the geometry and the result of the $U(2)$ partition function calculated before. In this case the partition function is given by

$$Z := \sum_{R_{1,2,\dots,N}} Q_{B_1}^{\ell_1} Q_{B_2}^{\ell_2} \cdots Q_{B_N}^{\ell_N}\, K_{R_1\cdots R_N}(\lambda_{1,\dots,N}, Q_{F_1,\dots,F_N})\, K_{R_1\cdots R_N}(\lambda_{N+1,\dots,2N}, Q_{F_1,\dots,F_N}) \tag{5.87}$$

where

$$K_{R_1\cdots R_N}(\lambda_{1,\dots,N}, Q_{F_1,\dots,F_N}) = \prod_{1\le i<j\le N} G_{R_i,R_j}(Q_{ij}, \lambda_i, \lambda_j)\;, \tag{5.88}$$

where $Q_{ij} = \prod_{k=i}^{j-1} Q_{f_k}$. Define

$$Q_{ij} = Q_{F_{ij}}/\sqrt{\lambda_i \lambda_j}\,, \qquad (5.89)$$
$$Q_{B_i} = Q_b/\sqrt{\lambda_i \lambda_{N+i}}\,.$$

Then we get

$$Z = \sum_{R_{1,\dots,N}} Q_b^{\ell_1+\dots+\ell_N} Z^{(0)}_{R_1,\dots,R_N} \prod_{i=1}^{N}\prod_k (1-q^k\lambda_i^{-1})^{C_k(R_i)}(1-q^k\lambda_{i+N}^{-1})^{C_k(R_i)} \qquad (5.90)$$
$$\times \prod_{1\le i<j\le N}\prod_k (1-q^k\lambda_i^{-1}Q_{F_{ij}})^{C_k(R_j)}(1-q^k\lambda_j^{-1}Q_{F_{ij}})^{C_k(R_i)}$$
$$\times(1-q^k\lambda_{N+i}^{-1}Q_{F_{ij}})^{C_k(R_j)}(1-q^k\lambda_{N+j}^{-1}Q_{F_{ij}})^{C_k(R_i)}\,.$$

In the limit $\lambda_i \to \infty$ we get the partition function of the pure 5D $SU(N)$ gauge theory with zero Chern-Simons term.

6D case: In this case the partition function can be calculated from the geometry shown in Fig. 38(a) below. To calculate the partition function we will slice the geometry in two parts

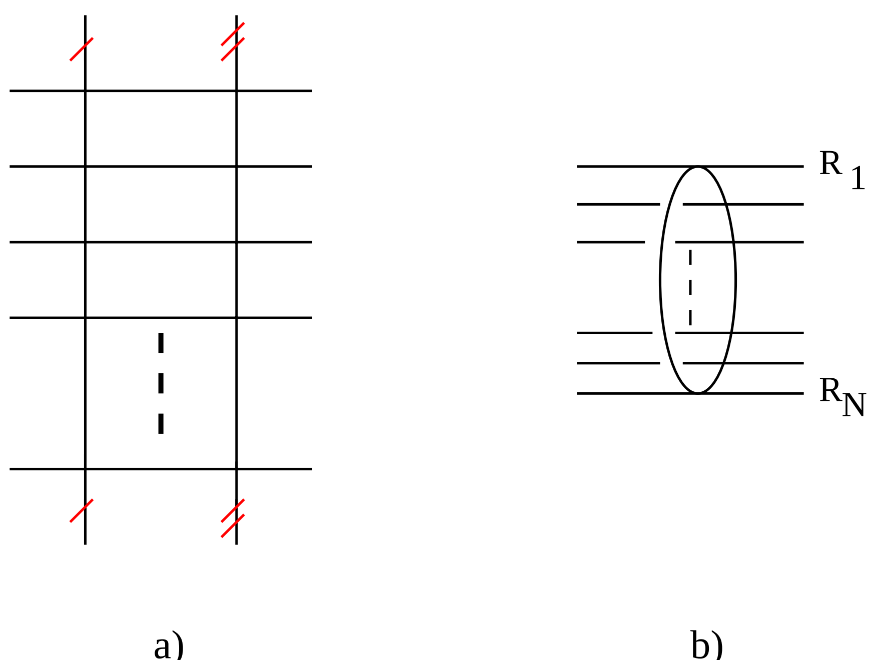

Figure 38: a) The web diagram of the 6D $U(N)$ theory with $2N$ hypermultiplets, b) the half of the web diagram used to calculate the partition function.

(shown in Fig. 38(b)) and after calculating the partition function of each part we will glue them together. If we denote the partition function of the geometry in Fig. 38(b) by $K_{R_1\cdots R_N}$ then the full partition function is given by

$$Z=\sum_{R_{1,\dots,N}} Q_{B_1}^{l_1}\cdots Q_{B_N}^{l_N}\,K_{R_1\cdots R_N}(Q_{F_{1,2,\dots,N}},\lambda_{1,\dots,N},q)\,K_{R_1\cdots R_N}(Q_{F_{1,2,\dots,N}},\lambda_{i+N},q)\,. \quad (5.91)$$

In calculating $K_{R_1\cdots R_N}$ we will have to take into account the contributions from annuli which start and end on the same three cycle and wind around the circle arbitrary number of times. Also contribution from annuli which start and end on different three cycles after winding around the circle arbitrary number of times have to be considered. It is easy to see that

$$\begin{aligned} K_{R_1\cdots R_N} &= \sum_{i=1}^{N}\sum_{R^{(i)}_{1,2,\dots}} Q_\rho^{\sum_{k\geq 1} k l_{R^{(i)}_k}}\, W_{\prod_k R^{(i)}_k\otimes R^{(i)}_k,R_i} \\ &\times \sum_{i<j,R^{(ij)}_{0,1,2,\dots}} Q_\tau^{\sum_{k\geq 0} k l_{R^{(ij)}_k}}\, Q_{ij}^{\sum_{k\geq 0} l_{R^{(ij)}_k}}\, W_{\prod_k R^{(ij)}_k,R_i}\, W_{\prod_k R^{(ij)}_k,R_j}\,. \end{aligned} \quad (5.92)$$

Using the above expression in Eq(5.91) we can evaluate the partition function as a series in Q_ρ.

6 Instanton moduli spaces and partition functions

The most direct connection between geometric engineering and the gauge theory perspective appears in 5 dimensional theories with geometry $\mathbf{R}^4\times\mathbf{S}^1$, where we view $\mathbf{R}^4$ as space and $\mathbf{S}^1$ as the Euclidean time with radius β. In particular consider M-theory on $\mathbf{X}\times\mathbf{R}^4\times\mathbf{S}^1$ where $\mathbf{S}^1$ has radius β in M-theory units. Let T_i^M denote the Kähler moduli of CY in M-theory units. Assume $\mathbf{X}$ is such that it engineers an $\mathcal{N}=1$ supersymmetric $U(N)$ gauge theory in 5d. Moreover consider breaking $U(N)\to U(1)^N$ by going to a generic point on Coulomb branch given by Kähler moduli a_i^M. The Yang-Mills coupling constant is

$$1g_{YM}^2=T_B^M\ , \quad (6.1)$$

where T_B^M is the Kähler moduli of the base measured in M-theory units. Instantons are BPS particles of this theory and they can carry $U(1)$ charges. The BPS mass of such an instantons is given by

$$m=kT_B^M+n_i a_i^M\ , \quad (6.2)$$

where k denotes the instanton number and n_i denotes the $U(1)$ charges. Compactification on a circle of radius β lead to computations of the form

$$\mathrm{Tr}\,\exp(-\beta H)\ , \quad (6.3)$$

where for BPS states

$$\beta H = \beta m = \beta(kT_B^M + n_i a_i^M) \ . \tag{6.4}$$

From the perspective of 4d type IIA string on CY, this can be viewed as computing the partition function of topological string because the metric, or Kähler form, as measured in type IIA strings and M-theory differ by

$$\beta k_M = k_{II} \ , \tag{6.5}$$

thus $\beta m = \beta(kT_B^M + n_i a_i^M) = kT_B^{II} + n_i a_i^{II}$. This explains the fact that from type II string perspective the topological string partition function was related to $\mathrm{Tr}\exp(-T)$ where T measures the size of the cycles in type IIA units. Thus topological string partition function for type IIA strings can be viewed as an M-theory partition function on a circle, or a 5d gauge theory BPS partition function on a circle. Thus the second quantized partition function of BPS states we have computed in the context of topological string should be related to some suitable partition function of second quantized BPS states involving instantons of the gauge theory. This then makes contact with the work of Nekrasov [5] where he developed an instanton calculus precisely for such cases. The link between the topological string computations and the 5d gauge theory computation of Nekrasov has been proven in [64,63]. Our main aim in reviewing aspects of it here is twofold: First we want to generalize these to gauge theories in 5d involving adjoint fields. secondly, we wish to generalize these to 6d gauge theories compactified on $\mathbf{T}^2$.

Before describing the calculations in detail we first make some general comments about the meaning of the "instanton partition function". In the conventional setting charge k instanton effects are calculated in terms of a zero-dimensional (matrix) supersymmetric sigma model with the k-instanton moduli space of $U(N)$ $\mathfrak{M}_{k,N}$ as target (see [58]). The exact details will be somewhat different for the theory with an adjoint as opposed to fundamental hypermultiplets. The sigma models are coupled to various isometries of the target space. Firstly, to the abelian subgroup $U(1)^N \subset U(N)$ of the global gauge group which acts on $\mathfrak{M}_{k,N}$. This gives coupling which depend on N parameters which are identified with the VEVs a_i of the parent theory. These coupling imply that the integrals over $\mathfrak{M}_{k,N}$ localize over fixed points of global gauge transformations. In particular, the fixed-point set consists of the moduli space of point-like instantons [59,60]. This is still a complicated space to integrate over. The additional insight of [57] was that if one, in addition, coupled to isometries corresponding to the abelian parts of the Lorentz group, involving two parameters ϵ_1 and ϵ_2,[5] then the fixed-point set becomes discrete. The instanton partition function can then be expressed as a sum over these discrete points and each contribution is a ratio of the usual fermionic and bosonic fluctuation determinants.

[5] In the following we shall for the most part make the simplifying choice $\epsilon_1 = -\epsilon_2 = \epsilon$.

If we now lift the theory to five-dimensions compactified on a circle, instantons in four-dimensions are now solitons in five-dimensions whose world-lines can wrap around the circle. The instanton partition function now involves quantum mechanics on the instanton moduli space and due to the couplings to the isometries defines an equivariant generalization of an index on $\mathfrak{M}_{k,N}$. The localization techniques are still valid the only difference being that the fermionic and bosonic determinants now include a product over all the Kaluza-Klein modes of fluctuations.

We will also be interested in F-theory compactifications on elliptic threefolds times a $\mathbf{T}^2$. From the viewpoint of 6d theory, we now have a 1+1 dimensional sigma model from $\mathbf{T}^2$ to the instanton moduli space. It is now clear that whatever index one is computing will be replaced by the corresponding elliptic index, where the complex structure of $\mathbf{T}^2$ will enter the elliptic index. We will discuss this in more detail below in the context of our main example which is the mass deformed $(1,1)$ supersymmetric theory in 6D compactified on $\mathbf{T}^2$ (giving $\mathcal{N}=2^*$ in 4D). In this case the localization procedure naturally involves replacing the weights of the circle actions x_i by its elliptic generalization $\theta_1(\frac{\beta}{2i}x_i|\rho)$.[6]

6.1 Calculation of 5D partition functions

In this section we use the instanton calculus to compute the mass deformed $\mathcal{N}=2^*$ BPS partition function in 5D. As has been shown in [5] the relevant index computation involves the χ_y genus of the instanton moduli space.

For a closed complex manifold $\mathfrak{M}$, its χ_y genus is defined as

$$\begin{aligned}\chi_y(\mathfrak{M}) &= \sum_{p,q\geq 0} y^p(-1)^p \dim H^q(\mathfrak{M},\Lambda^pT^*\mathfrak{M}) = \sum_{p\geq 0} y^p\chi(\mathfrak{M},\Lambda^pT^*\mathfrak{M}) \\ &= \int_{\mathfrak{M}} \mathrm{ch}\,\Lambda_{-y}(T^*\mathfrak{M})\mathrm{Td}(\mathfrak{M}) = \int_{\mathfrak{M}}\prod_{j=1}^{d}(1-ye^{-x_j})\frac{x_j}{1-e^{-x_j}}\ ,\end{aligned} \tag{6.6}$$

where $\{x_1,\dots,x_d\}$ denote the Chern roots of $T\mathfrak{M}$, the tangent bundle. If $\mathfrak{M}$ has a torus action with isolated fixed points $\{p_1,\dots,p_n\}$ and weights $\{w_{i,1},\dots,w_{i,d}\}$ at p_i then it follows

[6] The fact that this depends on only ρ rather than its conjugate is a reflection of the fact that only the anti-holomorphic modes contribute: the ratio of determinants of the fermionic and bosonic holomorphic modes cancel.

from localization theorem

$$\begin{aligned}\chi_y(\mathfrak{M}) &= \sum_{i=1}^{n}\frac{\prod_{j=1}^{d}(1-ye^{-w_{i,j}})\frac{w_{i,j}}{1-e^{-w_{i,j}}}}{\prod_{j=1}^{d}w_{i,j}} \\ &= \sum_{i=1}^{n}\prod_{j=1}^{d}\frac{1-ye^{-w_{i,j}}}{1-e^{-w_{i,j}}}\,.\end{aligned} \tag{6.7}$$

For the case we are interested in, $\mathfrak{M}=\mathfrak{M}_{k,N}$, $y=e^{-\beta m}$, and the fixed points and weights at each fixed point of $\mathfrak{M}_{k,N}$ under the $U(1)^N\times U(1)\times U(1)$ action were calculated in [5, 65]. The group $U(1)^N\times U(1)\times U(1)$ mentioned above is the Cartan of the gauge group and the spacetime rotation group. The fixed points of $\mathfrak{M}_{k,N}$ are in one to one correspondence with the partitions of k into N colors *i.e.*, the fixed points are labelled by N representations R_α of $U(\infty)$ such that

$$k=\ell_{R_1}+\ell_{R_2}+\cdots+\ell_{R_N}\,. \tag{6.8}$$

Let us denote the corresponding Young diagrams by μ^i (and the transpose diagram by $\mu^{t,i}$) then given a fixed points of $\mathfrak{M}_{k,N}$ labelled by $(\mu^1,\ldots,\mu^N)$ the corresponding weights are given by [5, 65]

$$\sum_{i,j}e^{w_{i,j}}=\sum_{\alpha,\gamma=1}^{N}e^{\beta(a_\alpha-a_\gamma)}\Big(\sum_{(i,j)\in R_\alpha}q^{\mu_i^\alpha+\mu_j^{t,\gamma}-i-j+1}+\sum_{(i,j)\in R_\gamma}q^{-\mu_i^\gamma+\mu^{t,\alpha}-i-j+1}\Big)\,. \tag{6.9}$$

For the case of $N=1$ we see that the above expression simplifies to

$$\sum_{(i,j)\in R}(q^{h(i,j)}+q^{-h(i,j)})\,,\ h(i,j)=\mu_i+\mu_j^t-i-j+1 \tag{6.10}$$

Using the above weights for the $N=1$ case in Eq. (6.8) we get

$$\begin{aligned}\sum_k Q^k\chi(\mathfrak{M}_{k,1}) &= \sum_k Q^k\sum_{R,\ell_R=k}\prod_{(i,j)\in R}\frac{(1-yq^{h(i,j)})(1-yq^{-h(i,j)})}{(1-q^{h(i,j)})(1-q^{-h(i,j)})}\,, \\ &= \sum_R Q^{\ell_R}\prod_{(i,j)\in R}\frac{(1-yq^{h(i,j)})(1-yq^{-h(i,j)})}{(1-q^{h(i,j)})(1-q^{-h(i,j)})}\,.\end{aligned} \tag{6.11}$$

This agrees exactly with the Eq. (5.10) which was calculated using the topological vertex if we identify $y=Q_m$.

For $N>1$ using the weights given above in Eq. (6.9) in Eq. (6.8) we get

$$\begin{aligned}&\sum_k Q^k\chi(\mathfrak{M}_{k,N})= \\ &\sum_{R_{1,\ldots,N}}Q^{\sum_{i=1}^N\ell_{R_i}}\prod_{\alpha,\gamma=1}^{N}\prod_{(i,j)\in R_\alpha}\frac{(1-yq^{\mu_i^\alpha+\mu_j^{t,\gamma}-i-j+1})}{(1-q^{\mu_i^\alpha+\mu_j^{t,\gamma}-i-j+1})}\prod_{(i,j)\in R_\gamma}\frac{(1-yq^{-\mu_i^\gamma+\mu^{t,\alpha}-i-j+1})}{(1-q^{-\mu_i^\gamma+\mu^{t,\alpha}-i-j+1})}\,,\end{aligned} \tag{6.12}$$

which also agrees with the $U(2)$ case discussed in the last section for $N = 2$.

6.2 Calculation of 6D partition function

In this case, the instanton partition of the six-dimensional theory can be interpreted as the generating functional for an elliptic genus of the instanton moduli space,

$$Z = \sum_k Q^k \chi(\mathfrak{M}_{k,N}) \ . \tag{6.13}$$

The elliptic genus $\chi(\mathfrak{M})$ is defined as the partition function in the Ramond-Ramond sector of the $\mathcal{N} = 2$ two-dimensional sigma-model with $\mathfrak{M}$ as target on the torus $\mathbf{T}^2$ [69, 70]:

$$\chi(\mathfrak{M}) = \mathrm{Tr}\left((-1)^F \, y^{F_L} Q_\rho^{L_0} \bar{Q}_\rho^{\bar{L}_0} e^{\sum_{i=1}^N a_i J_i + \epsilon_1 K_1 + \epsilon_2 K_2} \right) \ , \tag{6.14}$$

where

$$Q_\rho = e^{2\pi i \rho} \ , \qquad y := e^{-\beta m} \ , \tag{6.15}$$

$F = F_L + F_R$, the sum of the left and right fermion numbers. The remaining terms correspond to coupling to abelian isometries of $\mathfrak{M}$. The charges J_i corresponding to the $U(1)^N \subset U(N)$ of the gauge group,[7] while $K_{1,2}$ are the charges corresponding to the abelian subgroup $U(1)^2 \subset SU(2)_L \times SU(2)_R$ of the Lorentz group of $\mathbf{R}^4$. Note that naively Z vanishes if we include the right-moving fermionic zero mode on $\mathbf{R}^4$ which is always a factor for $\mathfrak{M}_{k,N}$, and so this trace is meant with the zero mode deleted. This reflects the same condition for BPS partition function, namely we only include the lowest component for each BPS multiplet.

The simplest example is given by taking $N = 1$. The $U(1)$ theory does not have smooth instanton solutions. In fact one way to think about instantons in an abelian theory is to turn on spacetime non-commutativity. In that case, there are instanton solutions whose only moduli correspond to the positions of the individual instantons. A single instanton has a moduli space $\mathbf{R}^4$ which represents its position in Euclidean spacetime. For charge k, the moduli space is a smoothed version of the symmetric product

$$\mathfrak{M}_{k,1} \sim \mathrm{Sym}^k(\mathbf{R}^4) \ . \tag{6.16}$$

For the six-dimensional theory, we can work directly in terms of the symmetric product.

[7] Only $SU(N)$ acts non-trivially on $\mathfrak{M}$ so we can fix $\sum_{i=1}^N a_i = 0$.

The elliptic genus for one instanton, for which $\mathfrak{M}_{1,1} \simeq \mathbf{R}^4$, can be written down straightforwardly. Choosing $\epsilon_1 = -\epsilon_2 = \epsilon$,

$$\begin{aligned}\chi(\mathfrak{M}_{1,1}, Q_\rho, y, q) &= \prod_{n=1}^{\infty} \frac{(1-Q_\rho^n \alpha q)(1-Q_\rho^{n-1}y^{-1}q^{-1})(1-Q_\rho^n y^{-1}q)(1-Q_\rho^{n-1}yq^{-1})}{(1-Q_\rho^{n-1}q)^2(1-Q_\rho^n q^{-1})^2} \\ &= \frac{\theta_1\big(\frac{\beta}{2i}(\epsilon+m)\big|\rho\big)\theta_1\big(\frac{\beta}{2i}(\epsilon-m)\big|\rho\big)}{\theta_1^2\big(\frac{\beta}{2i}\epsilon\big|\rho\big)} ,\end{aligned} \tag{6.17}$$

where

$$y = e^{-\beta m} , \qquad q = e^{-\beta\epsilon} = e^{-\lambda_s} . \tag{6.18}$$

This expression as a function $\phi(\rho, z_1, z_2)$, $y = e^{2\pi i z_1}$, $q = e^{2\pi i z_2}$ is a weak Jacobi form of weight 0 and indices 2 and 0, for z_1 and z_2, respectively.

For higher instanton number we can apply the formula of [13] for the elliptic genus of a symmetric product. If

$$\chi(\mathfrak{M}) = \sum_{n\geq 0, p_1, p_2, \dots} c(n, p_1, p_2, \dots) Q_\rho^n y_1^{p_1} y_2^{p_2} \cdots , \tag{6.19}$$

then

$$\sum_{k=1}^{\infty} Q^k \chi(\mathrm{Sym}^k(\mathfrak{M})) = \prod_{k=1,n=0}^{\infty} \prod_{p_1,p_2,\dots} \frac{1}{(1-Q^k Q_\rho^n y_1^{p_1} y_2^{p_2}\cdots)^{c(nk,p_1,p_2,\dots)}} , \tag{6.20}$$

where we have allowed for coupling to an arbitrary number of conserved quantities though $y_1, y_2, \dots$. In the present case, we have coupling to two charges through α and q. The instanton partition function is then equal to the generating function (6.20) with the $c(n, p_1, p_2)$ extracted from (6.19). The fact that the relevant moduli space is that of a genus 2 curve, as we have found before, was already noted in [13] in the context of elliptic genus of symmetric products. The problem of interest there was related to computation of entropy of 5D black holes [14] where the relevant space is the moduli space of instantons on $K3$ or $\mathbf{T}^4$, as opposed to the case of interest here which is $\mathbf{R}^4$.

In the last section we saw that localization allows us to write the χ_y genus of $\mathfrak{M}$ as a sum of contributions from the fixed points. The elliptic genus of $\mathfrak{M}$ can similarly be written as the sum over the fixed points with weights $w_{i,j}$,

$$\chi(\mathfrak{M}) = \sum_{i=1}^{n} \prod_{j=1}^{d} \Big(\frac{1-ye^{-w_{i,j}}}{1-e^{-w_{i,j}}} \Big(\prod_{k=1}^{\infty} \frac{(1-Q_\rho^k\, y\, e^{-w_{i,j}})(1-Q_\rho^k\, y^{-1}\, e^{w_{i,j}})}{(1-Q_\rho^k e^{-w_{i,j}})(1-Q_\rho^k\, e^{w_{i,j}})} \Big)\Big) . \tag{6.21}$$

Let's first consider the case of $N = 1$. In this the weights are give by Eq. (6.10). Substituting then in in the above equation we get

$$\sum_k Q^k \chi(\mathfrak{M}_{k,1}) = \sum_R Q^{\ell_R} \prod_{(i,j)\in R} \frac{(1-yq^{h(i,j)})(1-yq^{-h(i,j)})}{(1-q^{h(i,j)})(1-q^{-h(i,j)})}$$
$$\prod_{n=1}^{\infty} \frac{(1-Q_\rho^n y q^{h(i,j)})(1-Q_\rho^n y^{-1} q^{-h(i,j)})(1-Q_\rho^n y q^{-h(i,j)})(1-Q_{rho}^n y^{-1} q^{h(i,j)})}{(1-Q_\rho^n q^{h(i,j)})^2(1-Q_\rho^n q^{-h(i,j)})^2} \, .$$

This agrees with the topological vertex computation of the last section, Eq. (5.35), once we use the identification $y = Q_m$. The above expression can also be written as

$$Z = \sum_R Q^{\ell_R} \prod_{(i,j)\in R} \frac{\theta_1\big(\frac{\beta}{2i}(h(i,j)\epsilon + m)\big|\rho\big)\theta_1\big(\frac{\beta}{2i}(h(i,j)\epsilon - m)\big|\rho\big)}{\theta_1\big(\frac{\beta}{2i}h(i,j)\epsilon\big|\rho\big)^2} \, . \tag{6.22}$$

One can easily check by hand that the two expressions Eq. (6.20) and Eq. (6.22) agree for the first few terms.

6.3 Extracting the Curves From Instantons

In this section, we show how the curves of our two six-dimensional theories compactified on a torus can be extracted by using instantons. In a sense this is already done in the previous section, when we showed that A-model topological string amplitudes agree with the instanton calculus computations. Since one knows how to extract the mirror curve in the A-model setup, this is a proof of how one can extract the curve from the instanton calculus. However, one can also do this directly as was done in [57].

The central quantity is the instanton partition function which is a given by a sum over instanton numbers of the form

$$Z = \sum_k Q^k Z_k \, , \qquad Q = e^{2\pi i \tau} \, , \tag{6.23}$$

where τ is the complexified coupling of the theory. The associated free energy has an expansion which includes the prepotential $\mathcal{F}_0$ as well as a whole series of gravitational couplings:

$$Z = \exp\Big(\sum \mathcal{F}_g \lambda_s^{2g-2}\Big) \, . \tag{6.24}$$

The curve appears in the limit of $\lambda_s \to 0$.

6.3.1 The theory with an adjoint

From our discussion of the last section where we computed the instanton partition functions from the topological vertex and the localization calculation we see that the instanton partition function can be expressed as [57]

$$
\begin{aligned}
Z(a_i) &= \exp\Big(\sum_{ij}\big(\gamma_\epsilon(a_i-a_j)-\gamma_\epsilon(a_i-a_j+m)\big)\Big)\\
&\times \sum_{R_1,\ldots,R_N} Q^{\ell_{R_1}+\cdots+\ell_{R_N}} \prod_{(i,p)\neq(j,q)} \frac{\sigma(a_i-a_j+\epsilon(\mu^i_p-\mu^j_q+q-p))\sigma(a_i-a_j+\epsilon(q-p)+m)}{\sigma(a_i-a_j+\epsilon(q-p))\sigma(a_i-a_j+\epsilon(\mu^i_p-\mu^j_q+q-p)+m)}\ .
\end{aligned}
\tag{6.25}
$$

The sum is over colored partitions of k, the instanton charge, as in (6.8). This is a partition of k into N Young Tableau $\{R_1,\ldots,R_N\}$ with a total of k boxes described by the data

$$
\mu^i_1\geq\mu^i_2\geq\cdots\geq\mu^i_{n_i} \qquad \text{with} \qquad \sum_{p=1}^{n_i}\mu^i_p=\ell_{R_i},\quad \sum_{i=1}^{N}\ell_{R_i}=N\ , \tag{6.26}
$$

for $i=1,\ldots,N$. The instanton partition function given above although looks different but is exactly the equal to the one calculated in the last section (for the case of $U(1)$ and $U(2)$) if we use the following two identities [64],

$$
\begin{aligned}
\prod_{i,j=1}^{\infty}\frac{\sigma(\mu_i-\mu_j+j-i)}{\sigma(j-i)} &= \prod_{(i,j)\in R}\frac{1}{\sigma^2(h(i,j))}\,, \qquad (6.27)\\
\prod_{\alpha\neq\gamma}\prod_{i,j=1}^{\infty}\frac{\sigma(a_{\alpha\gamma}+\epsilon(\mu^\alpha_i-\mu^\gamma_j+j-i))}{\sigma(a_{\alpha\gamma}+\epsilon(j-i))} &= \prod_k\frac{1}{(\sigma(a_{12}+\epsilon k))^{2C_k(R_1,R_2^t)}}\ ,\ \ \alpha,\gamma=1,2.
\end{aligned}
$$

The integers $C_k(R_1,R_2)$ were defined in Section 4, $\sum_k C_k(R_1,R_2)q^k=f_{R_1,R_2^t}$.

For the four, five and six-dimensional theory,

$$
\sigma_{4D}(x)=x\ ,\qquad \sigma_{5D}(x)=\sinh(\tfrac{\beta x}{2})\ ,\qquad \sigma_{6D}(x)=\theta_1\big(\tfrac{\beta x}{2i}\big|\rho\big) \tag{6.28}
$$

respectively. In addition, the kernel $\gamma_\hbar(x)$ is defined by the finite difference equation

$$
\gamma_\epsilon(x+\epsilon)+\gamma_\epsilon(x-\epsilon)-2\gamma_\epsilon(x)=\log\sigma(x)\ . \tag{6.29}
$$

In the four-dimensional case, the partition function can be simply written in terms of a sum over certain "paths" $f(x)$ which are associated to the colored partitions. In concrete terms, it is simpler to work on terms of the second derivative of $f''(x)$:

$$f''(x) = 2\sum_{i=1}^{N}\Big(\sum_{p=1}^{n_i}\big(\delta(x-a_i-\epsilon(\mu_p^i-p+1))-\delta(x-a_i-\epsilon(\mu_p^i-p))\big)+\delta(x-\epsilon n_i)\Big)\ . \tag{6.30}$$

In the limit $\epsilon \to 0$, $f''(x)$ becomes a positive density with N intervals of support along N open contours $\mathcal{C}_i$ with end-points

$$\mathcal{C}_i = [r_i, s_i] \qquad i = 1, \dots, N \tag{6.31}$$

located in the vicinity of a_i. This picture naturally extends by continuity to the six-dimensional theory where now $f''(x)$ is defined as a density in $\tilde{\mathbf{T}}^2$ with support along the N contours (6.31).

The following identities arise from (6.30). First of all, one has the normalization condition

$$\int_{\mathcal{C}_i} dx\, f''(x) = 2\ . \tag{6.32}$$

Secondly, the VEVs are recovered via

$$a_i = \tfrac{1}{2}\int_{\mathcal{C}_i} dx\, x\, f''(x)\ , \tag{6.33}$$

while the instanton charge is

$$k = -\frac{1}{2\epsilon^2}\sum_{i=1}^{N} a_i^2 + \frac{1}{4\epsilon^2}\int_{\mathcal{C}} dx\, x^2\, f''(x)\ , \tag{6.34}$$

where the union of all the contours is

$$\mathcal{C} = \bigcup_{i=1}^{N} \mathcal{C}_i\ . \tag{6.35}$$

In the $\epsilon \to 0$ limit, Z is dominated by a saddle-point determined by minimizing the functional $\mathcal{E}[f'']$

$$\begin{aligned}\mathcal{E}[f''] = -\tfrac{1}{4}\int_{\mathcal{C}} dx\, dy\, f''(x)f''(y)\big(\gamma_0(x-y) - \tfrac{1}{2}\gamma_0(x-y+m) - \tfrac{1}{2}\gamma_0(x-y-m)\big)\\ - \frac{i\pi\tau}{2}\int_{\mathcal{C}} dx\, x^2\, f''(x) + \sum_{i=1}^{N}\lambda_i\big(a_i - \tfrac{1}{2}\int_{\mathcal{C}_i} dx\, x\, f''(x)\big)\ .\end{aligned} \tag{6.36}$$

in which case

$$Z \sim \exp\Big(- \epsilon^{-2} \mathcal{E}[f''] \Big) \ . \tag{6.37}$$

In the above we, have included Lagrange multipliers λ_i to enforce the fact that the a_i are fixed. The kernel $\gamma_0(x)$ is the first term in the small ϵ expansion

$$\gamma_\epsilon(x) = \sum_{g=0}^{\infty} \gamma_g(x) \epsilon^{2g-2} \ . \tag{6.38}$$

It follows from (6.29) that

$$\gamma_0''(x) = \log \sigma(x) \ . \tag{6.39}$$

We can now calculate Z in this limit by finding the saddle-point. In fact rather than calculate Z we shall see that the Seiberg-Witten curve arises in the description of the critical density $f''(x)$. To start with, the saddle-point equation is, for $x \in \mathcal{C}_i$,

$$\int_{\mathcal{C}} dy\, f''(y) \big(\gamma_0(x-y) - \tfrac{1}{2}\gamma_0(x-y+m) - \tfrac{1}{2}\gamma_0(x-y-m) \big) + i\pi\tau x^2 + \lambda_i x = 0 \ . \tag{6.40}$$

In order to solve this equation, it is convenient to introduce a resolvent

$$\tilde{\omega}(x) = \int_{\mathcal{C}} dy\, f''(y)\, \partial_x \log \theta_1\big(\tfrac{\beta}{2i}(x-y) \big| \rho \big) \ . \tag{6.41}$$

This is a multi-valued analytic function on $\tilde{\mathbf{T}}^2$, since it picks up an additive piece under continuation around the B-cycle,

$$\tilde{\omega}(x + 2\pi i/\beta) = \tilde{\omega}(x) \ , \qquad \tilde{\omega}(x + 2\pi i \rho/\beta) = \tilde{\omega}(x) - 2\beta^{-1} \ , \tag{6.42}$$

with N branch cuts $\mathcal{C}_i$. As usual with a resolvent, the discontinuity across a cut is proportional to the density:

$$\tilde{\omega}(x+\epsilon) - \tilde{\omega}(x-\epsilon) = 2\pi i f''(x) \ , \qquad x \in \mathcal{C} \ , \tag{6.43}$$

where ϵ is a suitable infinitesimal chosen so that $x \pm \epsilon$ lie infinitesimally above and below the cut at x. The third derivative of (6.40) with respect to x can then be written

$$\tilde{\omega}(x+\epsilon) + \tilde{\omega}(x-\epsilon) - \tilde{\omega}(x+m) - \tilde{\omega}(x-m) = 0 \ , \qquad x \in \mathcal{C} \ . \tag{6.44}$$

This equation is identical the equation for the resolvent in the matrix model (2.23), except that there is no potential on the right-hand side. However, the similarity suggests that we define the function

$$\tilde{G}(x) = \tilde{\omega}(x + \tfrac{m}{2}) - \tilde{\omega}(x - \tfrac{m}{2}) \ , \tag{6.45}$$

to match (2.24). This function is now an analytic function on $\tilde{\mathbf{T}}^2$ with N pairs of branch cuts

$$\mathcal{C}_i^{\pm} = \mathcal{C}_i \pm \tfrac{m}{2} \ . \tag{6.46}$$

The equation (6.44) then becomes a gluing condition

$$\tilde{G}(x + \tfrac{m}{2} \pm \epsilon) = \tilde{G}(x - \tfrac{m}{2} \mp \epsilon) \qquad x \in \mathcal{C} \ . \tag{6.47}$$

Pictorially, the top/bottom of $\mathcal{C}_i^+$ is glued to the bottom/top of $\mathcal{C}_i^-$. So $\tilde{G}$ is single-valued on a Riemann surface of genus $N+1$ just as in the matrix model.

We now prove that this curve is the Seiberg-Witten curve Σ. In order to do this, we prove that the period matrix has the form (2.40). This itself follows from the existence of the multi-valued function z with the monodromies (2.42). In the present context, we will identify

$$z(P) = \frac{1}{4\pi i}\int_{P_0}^{P} \tilde{G}(x)dx \ . \tag{6.48}$$

It follows from (6.43) that

$$\oint_{A_j} dz = \frac{1}{4\pi i}\int_{\mathcal{C}_j} dx\, f''(x) = 1 \ , \tag{6.49}$$

where A_j is a cycle encircling the top cut $\mathcal{C}_j^+$ as in Fig. 2. Now we consider the integral of dz over the conjugate cycle B_j which goes from a point on the lower cut $x - \frac{m}{2} \in \mathcal{C}_j^-$ to the point $x + \frac{m}{2}$ on the upper cut $\mathcal{C}_j^+$. For $x \in \mathcal{C}_j$,

$$\begin{aligned}\oint_{B_j} dz &= \int_{x-\frac{im}{2}}^{x+\frac{im}{2}} \tilde{G}(x')dx' \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} dy\, f''(y)\big(\gamma_0''(x-y) - \tfrac{1}{2}\gamma_0''(x-y+m) - \tfrac{1}{2}\gamma_0''(x-y-m)\big) = \tau\end{aligned} \tag{6.50}$$

independent of x, where the last equality follows from taking the second derivative of the saddle-point equation (6.40) for $x \in \mathcal{C}_j$ and using the relation (6.39).

From (6.49), it follows that $dz = \sum_{i=1}^{N} \omega_i$ and therefore that the first N rows and columns of the period matrix satisfy[8]

$$\sum_{j=1}^{N} \Pi_{ij} = \tau \qquad \forall i = 1, \ldots, N \ . \tag{6.51}$$

[8]In principal, one could have $dz = \sum_{i=1}^{N} \omega_i + \lambda\omega_{N+1}$, for arbitrary λ. The only effect of this is to re-define the coupling τ and so we choose $\lambda = 0$.

These are precisely the conditions on the period matrix (2.40). We can view these N equations as N conditions on the moduli $\{r_i, s_i\}$ (the ends of the contours $\mathcal{C}_i$). However, there are N additional conditions that arise from the constraints

$$a_j = \tfrac{1}{2}\int_{\mathcal{C}_j} dx\, x f''(x) = \oint_{A_j} x\, dz\ . \tag{6.52}$$

We can therefore think of the a_j as the moduli of the curve Σ and notice that $x\,dz$ is the Seiberg-Witten differential.

So we have shown that the Seiberg-Witten geometry that we engineered out of the matrix model in Section 2 also describes the $\epsilon \to 0$ limit of the instanton partition function.

6.3.2 The theory with fundamentals

In this section, we follow the same procedure using instantons to extract the Seiberg-Witten curve for the six-dimensional $\mathcal{N} = (1,0)$ theory with fundamental hypermultiplets compactified on the torus $\mathbf{T}^2$.

The instanton partition function with N_f fundamental hypermultiplets is [5],

$$\begin{aligned} Z(a_i) &= \exp\Big(\sum_{ij}\gamma_\epsilon(a_i - a_j) + \sum_{if}\gamma_\epsilon(a_i - m_f)\Big) \\ &\times \sum_{R_1,\dots,R_N} Q^{\ell_{R_1}+\cdots+\ell_{R_N}} \prod_{(i,p)\neq(j,q)} \frac{\sigma(a_i - a_j + \epsilon(\mu^i_p - \mu^j_q + q - p))}{\sigma(a_i - a_j + \epsilon(q-p))} \prod_{ipqf} \frac{\sigma(a_i - m_f + \epsilon(\mu^i_p + q - p))}{\sigma(a_i - m_f + \epsilon(q-p))}\ . \end{aligned} \tag{6.53}$$

In order to take the $\epsilon \to 0$, we follow exactly the same steps as for the adjoint theory. We assume that in the limit, $f''(x)$ is a density with N intervals of support along the contours (6.31) in $\tilde{\mathbf{T}}^2$. The conditions (6.32)-(6.34) continue to hold. The functional to be extremized, replacing (6.36), is

$$\begin{aligned} \mathcal{E}[f''] = &-\tfrac{1}{4}\int_{\mathcal{C}} dx\, dy\, f''(x) f''(y)\gamma_0(x-y) + \tfrac{1}{2}\sum_{f=1}^{N_f}\int_{\mathcal{C}} dx\, f''(x)\gamma_0(x - m_f) \\ &- \frac{i\pi\tau}{2}\int_{\mathcal{C}} dx\, x^2 f''(x) + \sum_{i=1}^{N}\lambda_i\big(a_i - \tfrac{1}{2}\int_{\mathcal{C}_i} dx\, x f''(x)\big)\ . \end{aligned} \tag{6.54}$$

This yields the saddle-point equation

$$\int_{\mathcal{C}} dy\, f''(y)\gamma_0(x-y) - \sum_{f=1}^{N_f}\gamma_0(x - m_f) + i\pi\tau x^2 + \lambda_i x = 0 \qquad x \in \mathcal{C}_i\ . \tag{6.55}$$

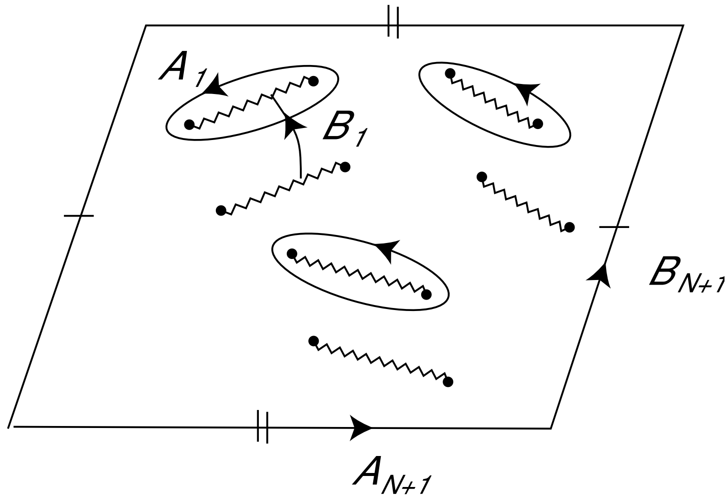

Figure 39: The cut $\tilde{\mathbf{T}}^2$ on which the resolvent $\tilde{\omega}(x)$ is defined. The contours A_j, $j = 1, \ldots, N$ encircle the $j^{\rm th}$ cut, while the contours B_j, $j = 1, \ldots, N-1$ join the $j^{\rm th}$ and $j+1^{rmth}$ cuts and return on the lower sheet.

As in the $\mathcal{N} = 2^*$ case, it is convenient to introduce a resolvent defined by

$$\tilde{\omega}(x) = \int_{\mathcal{C}} dy\, f''(y)\, \partial_x \log \theta_1\big(\tfrac{\beta}{2i}(x-y)\big|\rho\big) - \sum_{f=1}^{N_f} \partial_x \log \theta_1\big(\tfrac{\beta}{2i}(x-m_f)\big|\rho\big) \ , \tag{6.56}$$

in which case the third derivative of the saddle-point equation has the form

$$\tilde{\omega}(x+\epsilon) + \tilde{\omega}(x-\epsilon) = 0 \ , \qquad x \in \mathcal{C} \ . \tag{6.57}$$

Notice that the resolvent (6.56) is only well-defined on the torus $\tilde{\mathbf{T}}^2$ if $N_f = 2N$, otherwise it picks up an additive ambiguity around the B-cycle of the $\tilde{\mathbf{T}}^2$ torus. This is presumably related to the anomaly of the six-dimensional theory unless $N_f = 2N$.

The normalization condition (6.32) requires

$$\oint_{A_j} \tilde{\omega}(x)dx = -2\pi i \int_{\mathcal{C}_j} f''(x)dx = -4\pi i \ , \tag{6.58}$$

where A_j is a cycle that encircles the $j^{\rm th}$ cut, as illustrated in Fig. 39. In addition, for $x_j \in \mathcal{C}_j$ consider the integral

$$\int_{x_j}^{x_{j+1}} \tilde{\omega}(x)dx = \int_{\mathcal{C}} dy\, f''(y)\big(\log \theta_1\big(\tfrac{\beta}{2i}(x_{j+1}-y)\big|\rho\big) - \log \theta_1\big(\tfrac{\beta}{2i}(x_j-y)\big|\rho\big)\big) = 0 \ , \tag{6.59}$$

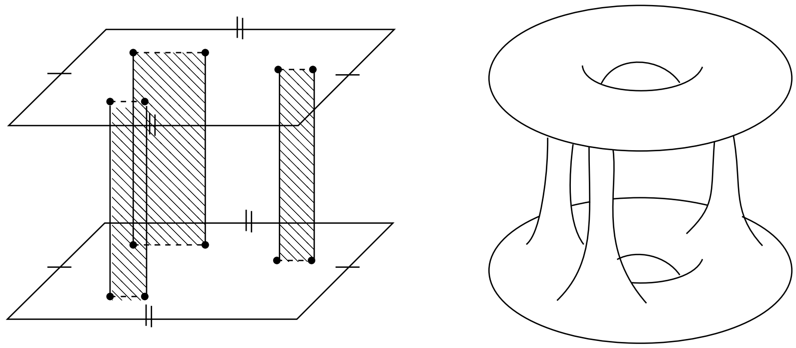

Figure 40: The solution involves a double cover of the $\tilde{\mathbf{T}}^2$ torus joined by N branch cuts to create a surface of genus $N+1$.

by the second derivative of (6.55).

The solution of these conditions naturally leads to a curve which is the double cover of the torus $\tilde{\mathbf{T}}^2$ for which the $\mathcal{C}_i$ are N square-root branch cuts joining the 2 sheets. This geometry is illustrated in Fig. 40. There is a natural involution which exchanges the two sheets. In particular, we can trivially solve (6.57) if $\tilde{\omega}(x)$ is a meromorphic function which is odd under the involution and which has, in view of (6.56), simple poles at $x = m_f$ of the form

$$\tilde{\omega}(x) = \mp \frac{1}{x - m_f} + \mathcal{O}(1) \tag{6.60}$$

on the top and bottom sheets, respectively. So $\tilde{\omega}(x)dx$ is a 1-form on Σ whose only singularities are simple poles at m_f, with residues $\mp 2\pi i$, on the bottom and top sheet, respectively, and whose integrals around the cycles A_j, $j = 1, \ldots, N$ and B_j, $j = 1, \ldots, N-1$ are

$$\oint_{A_j} \tilde{\omega}(x)dx = -4\pi i \ , \qquad \oint_{B_j} \tilde{\omega}(x)dx = 0 \ , \tag{6.61}$$

where B_j is the cycle illustrated in Fig. 39 and the latter integral follows from (6.59).

Since the contour $\cup_{j=1}^N A_j$ can be pulled off the back of the top sheet, at the expense of picking up residues at the simple poles $x = m_f$, we have

$$-4\pi i N = \sum_{j=1}^{N} \oint_{A_j} \tilde{\omega}(x)dx = -\sum_{f=1}^{N_f} \oint_{m_f} \tilde{\omega}(x)dx = -2\pi i N_f \ , \tag{6.62}$$

by (6.60). Hence, for consistency we find

$$N_f = 2N \ , \tag{6.63}$$

as we noted previously.

We claim that the unique solution to these conditions is

$$\tilde{\omega}(x)dx = -2d\log t \ , \tag{6.64}$$

where t is the function

$$t = \frac{P(x)}{\sqrt{Q(x)}} + \sqrt{\frac{P(x)^2}{Q(x)} - c} \tag{6.65}$$

with

$$P(x) = \prod_{i=1}^{N} \theta_1\big(\tfrac{\beta}{2i}(x-\zeta_i)\big|\rho\big) \ , \qquad Q(x) = \prod_{f=1}^{2N} \theta_1\big(\tfrac{\beta}{2i}(x-m_f)\big|\rho\big) \ . \tag{6.66}$$

Hence,

$$\tilde{\omega}(x) = -\frac{2}{y}\Big(P'(x) - \frac{P(x)Q'(x)}{2Q(x)}\Big) \ , \tag{6.67}$$

where

$$y^2 = P(x)^2 - cQ(x) \ . \tag{6.68}$$

Notice that (6.68) is precisely the curve we found from the web diagram in (3.27). Notice that we must also choose our x-origin so that

$$\sum_{i=1}^{N} \zeta_i = \sum_{f=1}^{2N} m_f \tag{6.69}$$

in order that $\tilde{\omega}(x)$ is valued on $\tilde{\mathbf{T}}^2$. From this solution, the density $f''(x)$ is determined by (6.43) once the cuts $\mathcal{C}_j$ are identified and where c a constant which is fixed in terms of the coupling constant τ by substituting $f''(x)$ in the second derivative of the saddle-point equation (6.55). The cuts are identified as follows. In the limit of weak coupling $c \to 0$ and the roots of $y = 0$ come in pairs on $\tilde{\mathbf{T}}^2$ located in the vicinity of each $x = \zeta_j$. The roots near ζ_j are the ends of the cut $\mathcal{C}_j$.

The geometry (3.27), (6.68), is the Seiberg-Witten curve of the $U(N)$ six-dimensional $\mathcal{N} = (1,0)$ theory with $N_f = 2N$ hypermultiplets. The ζ_j are moduli which are determined in terms of the a_j's by the conditions (6.33)

$$a_j = \tfrac{1}{2}\int_{\mathcal{C}_j} x f''(x)\,dx = -\tfrac{1}{4\pi i}\oint_{A_j} x\tilde{\omega}(x)dx = \tfrac{1}{2\pi i}\oint_{A_j} x\frac{dt}{t} \ , \tag{6.70}$$

from which we deduce that $\lambda = xdt/(2\pi i t)$ is the Seiberg-Witten differential.

Notice that the resulting curve Σ is embedded holomorphically in $\mathbf{T}^3 \times \mathbf{R}$ defined by the coordinates $(x, z = \frac{1}{2\pi i}\log t)$. In the M-theory picture, x and z are identified with the spacetime coordinates as in (2.57) and the M5-brane is wrapped on the curve.

7 6D SYM and the 5-brane

We have seen that 6d $(1,1)$ supersymmetric gauge theory compactified on $\mathbf{T}^2$ and mass deformed by the mass parameter m has an interesting moduli space. The moduli space is three dimensional, given by the complex structure of $\mathbf{T}^2$, ρ, the Kähler class of $\mathbf{T}^2$, τ, and the mass parameter m. The two natural $SL(2,\mathbf{Z})$ symmetries of τ, ρ are combined to an $Sp(4,\mathbf{Z})$ symmetry when $m \neq 0$. Note that this is a mass deformed NS 5 brane of type IIB compactified on $\mathbf{T}^2$. By a T-duality on one of the circles of $\mathbf{T}^2$ this can be viewed as NS5-brane of type IIA compactified on $\mathbf{T}^2$ with complex structure τ and Kähler structure ρ. Or, lifted up to M-theory, this can be viewed as a mass deformed M5 brane wrapped on a $\mathbf{T}^2$. The dual description we have found can also be given an M5 brane description: namely, we have given the dual description as an M5 brane wrapped on a genus 2 curve embedded in $\mathbf{T}^4$, where the τ and ρ are both complex moduli of this genus 2 curve. This is an amusing duality involving M5 brane where Kähler and complex structure on one side are mapped to complex parameters on the other side.

It is also noteworthy that we have found a triality symmetry between $(\hat{\tau}, \hat{\rho}, \frac{\beta m}{2\pi i}) = (\tau - \frac{\beta m}{2\pi i}, \rho - \frac{\beta m}{2\pi i}, \frac{\beta m}{2\pi i})$. The interpretation of this triality symmetry for the M5 brane theory wrapped on a $\mathbf{T}^2$ would be interesting to understand directly.

Acknowledgments

We would like to thank N. Nekrasov for valuable discussions. We would also like to thank the Simons Workshop on Mathematics and Physics which resulted in this work. TH would also like to thank H. Braden, K. Ohta, N. Dorey and P. Kumar for valuable discussions.

The research of AI and CV is supported in part by NSF grant DMS-0074329. CV is additionally supported by NSF grant PHY-9802709.

References

[1] S. Katz, A. Klemm and C. Vafa, "Geometric engineering of quantum field theories," *Nucl. Phys.* **B497**, (1997) 173–195, `hep-th/9609239`.

[2] S. Katz, P. Mayr and C. Vafa, "Mirror symmetry and exact solution of 4D N = 2 gauge theories. I," Adv. Theor. Math. Phys. **1**, 53 (1998) `hep-th/9706110`.

[3] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644** (2002) 3, hep-th/0206255.

[4] F. Cachazo, K. A. Intriligator and C. Vafa, "A large N duality via a geometric transition," Nucl. Phys. B **603**, 3 (2001) hep-th/0103067.

[5] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," hep-th/0206161.

[6] C. Vafa, "Evidence for F-Theory," *Nucl. Phys.* **B469**, (1996) 403–418, hep-th/9602022.

[7] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – I," Nucl. Phys. B **473**, 74 (1996) hep-th/9602114.

[8] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – II," Nucl. Phys. B **476**, 437 (1996) hep-th/9603161.

[9] R. Dijkgraaf and C. Vafa, "N = 1 supersymmetry, deconstruction, and bosonic gauge theories," hep-th/0302011.

[10] R. Gopakumar and C. Vafa, "M-theory and topological strings. I," hep-th/9809187.

[11] R. Gopakumar and C. Vafa, "M-theory and topological strings. II," hep-th/9812127.

[12] M. Aganagic, A. Klemm, M. Marino, C. Vafa, "The Topological Vertex," hep-th/0305132.

[13] R. Dijkgraaf, G. W. Moore, E. Verlinde and H. Verlinde, "Elliptic genera of symmetric products and second quantized strings," Commun. Math. Phys. **185** (1997) 197 hep-th/9608096.

[14] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," Phys. Lett. B **379**, 99 (1996) hep-th/9601029.

[15] S. Katz, A. Klemm and C. Vafa, "M-theory, topological strings and spinning black holes," Adv. Theor. Math. Phys. **3**, 1445 (1999) [arXiv:hep-th/9910181].

[16] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048.

[17] F. Cachazo and C. Vafa, "N = 1 and N = 2 geometry from fluxes," arXiv:hep-th/0206017.

[18] T. J. Hollowood, "Five-dimensional gauge theories and quantum mechanical matrix models," JHEP **0303** (2003) 039 [arXiv:hep-th/0302165].

[19] N. Dorey, T. J. Hollowood, S. P. Kumar and A. Sinkovics, "Exact superpotentials from matrix models," JHEP **0211** (2002) 039 `hep-th/020908`.

[20] N. Dorey, T. J. Hollowood, S. P. Kumar and A. Sinkovics, "Massive vacua of N = 1* theory and S-duality from matrix models," JHEP **0211** (2002) 040 `hep-th/0209099`.

[21] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644** (2002) 21, `hep-th/0207106]`.

[22] P. Griffiths and J. Harris, "Principles of Algebraic Geometry," Wiley 1978.

[23] H. Lange and Ch. Birkenhake, "Complex Abelian Varieties," Springer-Verlag 1992.

[24] H. W. Braden and T. J. Hollowood, *to appear*.

[25] O. J. Ganor, A. Y. Mikhailov and N. Saulina, "Constructions of non-commutative instantons on T(4) and K(3)," Nucl. Phys. B **591** (2000) 547 `hep-th/0007236`.

[26] Y. K. Cheung, O. J. Ganor, M. Krogh and A. Y. Mikhailov, "Instantons on a non-commutative T(4) from twisted (2,0) and little-string theories," Nucl. Phys. B **564** (2000) 259 `hep-th/9812172`.

[27] N. Dorey, T. J. Hollowood and S. Prem Kumar, "An exact elliptic superpotential for N = 1* deformations of finite N = 2 gauge theories," Nucl. Phys. B **624** (2002) 95 `hep-th/0108221`.

[28] T. J. Hollowood, "Critical points of glueball superpotentials and equilibria of integrable systems," `hep-th/0305023`.

[29] V. Fock, A. Gorsky, N. Nekrasov and V. Rubtsov, "Duality in integrable systems and gauge theories," JHEP **0007** (2000) 028 `hep-th/9906235`.

[30] H. W. Braden, A. Gorsky, A. Odessky and V. Rubtsov, "Double-elliptic dynamical systems from generalized Mukai-Sklyanin algebras," Nucl. Phys. B **633** (2002) 414 `hep-th/0111066`.

[31] R. Donagi and E. Witten, "Supersymmetric Yang-Mills Theory And Integrable Systems," Nucl. Phys. B **460** (1996) 299 `hep-th/9510101`.

[32] E. J. Martinec, "Integrable Structures in Supersymmetric Gauge and String Theory," Phys. Lett. B **367** (1996) 91 `hep-th/9510204`.

[33] E. D'Hoker and D. H. Phong, "Calogero-Moser systems in SU(N) Seiberg-Witten theory," Nucl. Phys. B **513** (1998) 405 `hep-th/9709053`.

[34] N. Nekrasov, "Five dimensional gauge theories and relativistic integrable systems," Nucl. Phys. **B531** (1998) 323-344, hep-th/9609219.

[35] E. Witten, "Solutions of four-dimensional field theories via M-theory", *Nucl. Phys.* **B500**, (1997) 3–42, hep-th/9703166.

[36] M. R. Douglas, S. Katz and C. Vafa, "Small instantons, del Pezzo surfaces and type I' theory," Nucl. Phys. B **497**, 155 (1997) hep-th/9609071.

[37] D. R. Morrison and N. Seiberg, "Extremal transitions and five-dimensional supersymmetric field theories," Nucl. Phys. B **483**, 229 (1997) hep-th/9609070].

[38] K. A. Intriligator, D. R. Morrison and N. Seiberg, "Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces," Nucl. Phys. B **497**, 56 (1997) hep-th/9702198.

[39] N. C. Leung and C. Vafa, "Branes and toric geometry," *Adv. Theor. Math. Phys.* **2** (1998) 91–118, hep-th/9711013.

[40] B. Kol, "5d field theories and M theory," *JHEP* **9911**, 026 (1999),hep-th/9705031,
O. Aharony, A. Hanany and B. Kol, "Webs of (p,q) 5-branes, five dimensional field theories and grid diagrams," *JHEP* **9801**, 002 (1998),hep-th/9710116,
B. Kol and J. Rahmfeld, "BPS spectrum of 5 dimensional field theories, (p,q) webs and curve counting," *JHEP* **9808**, 006 (1998), hep-th/9801067.

[41] H. Ooguri and C. Vafa, "Two-Dimensional Black Hole and Singularities of CY Manifolds," Nucl. Phys. B **463**, 55 (1996) hep-th/9511164.

[42] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. P. Warner, "Self-Dual Strings and N=2 Supersymmetric Field Theory," Nucl. Phys. B **477**, 746 (1996) hep-th/9604034

[43] E. Witten, "Phases of $\mathcal{N} = 2$ theories in two dimensions," *Nucl. Phys.* **B403** (1993) 159–222, hep-th/9301042.

[44] K. Hori, C. Vafa, "Mirror Symmetry", hep-th/0002222.

[45] K. Hori, A Iqbal, C. Vafa, "D-branes and Mirror Symmetry", hep-th/0205xxx.

[46] M. Bershadsky and C. Vafa, "Global anomalies and geometric engineering of critical theories in six dimensions," hep-th/9703167.

[47] I. Brunner and A. Karch, "Branes at orbifolds versus Hanany Witten in six dimensions," JHEP **9803**, 003 (1998) hep-th/9712143.

[48] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, "Topological amplitudes in string theory," Nucl. Phys. B **413**, 162 (1994) hep-th/9307158.

[49] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) hep-th/9309140.

[50] J. A. Harvey and G. W. Moore, "On the algebras of BPS states," Commun. Math. Phys. **197**, 489 (1998) hep-th/9609017.

[51] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs," hep-th/0012041.

[52] M. Aganagic, A. Klemm and C. Vafa, "Disk instantons, mirror symmetry and the duality web," Z. Naturforsch. A **57**, 1 (2002) hep-th/0105045.

[53] R. Gopakumar and C. Vafa, "On the gauge theory/geometry correspondence," Adv. Theor. Math. Phys. **3**, 1415 (1999) hep-th/9811131.

[54] M. Aganagic, M. Marino, C. Vafa, "All Loop Topological String Amplitudes from Chern-Simons Theory," hep-th/0206164.

[55] D. E. Diaconescu, B. Florea and A. Grassi, "Geometric transitions, del Pezzo surfaces and open string instantons," Adv. Theor. Math. Phys. **6**, 643 (2003) hep-th/0206163.

[56] A. Okounkov, N. Reshetikhin and C. Vafa, "Quantum Calabi-Yau and classical crystals," hep-th/0309208.

[57] A. Okounkov, N. Nekrasov, "Seiberg-Witten theory and random partitions," hep-th/0306238.

[58] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "The calculus of many instantons," Phys. Rept. **371** (2002) 231 hep-th/0206063.

[59] T. J. Hollowood, "Testing Seiberg-Witten theory to all orders in the instanton expansion," Nucl. Phys. B **639** (2002) 66 hep-th/0202197.

[60] T. J. Hollowood, "Calculating the prepotential by localization on the moduli space of instantons," JHEP **0203** (2002) 038 hep-th/0201075.

[61] A. Gorsky and A. Mironov, "Integrable many-body systems and gauge theories," hep-th/0011197.

[62] A. Iqbal, "All genus topological string amplitudes and 5-brane webs as Feynman diagrams," hep-th/0207114.

[63] A. Iqbal, A. -Kian. Kashani-Poor, "$SU(N)$ geometries and topological string amplitudes," hep-th/0306032.

[64] A. Iqbal, A. -Kian. Kashani-Poor, "Instanton counting and Chern-Simons theory," hep-th/0212279.

[65] H. Nakajima, K. Yoshioka, "Instanton counting on blowup, I," math.AG/0306198.

[66] N. Dorey, T. J. Hollowood, S. P. Kumar and A. Sinkovics, "Exact superpotentials from matrix models," JHEP **0211** (2002) 039 hep-th/0209089.

[67] N. Dorey, T. J. Hollowood, S. P. Kumar and A. Sinkovics, "Massive vacua of N = 1* theory and S-duality from matrix models," JHEP **0211** (2002) 040 hep-th/0209099.

[68] T. Eguchi, H. Kanno, "Topological Strings and Nekrasov's formulas," hep-th/0310235.

[69] , A. N. Schellekens, N. P. Warner, "Anomalies, Characters and Strings," Nucl. Phys. **B287**: 317, 1987.

[70] E. Witten, "The Index Of The Dirac Operator In Loop Space," PUPT-1050 , *Proc. of Conf. on Elliptic Curves and Modular Forms in Algebraic Topology, Princeton, N.J., Sep 1986.*

[71] I. G. Macdonald, "Symmetric functions and Hall polynomials," (second edition, 1995), Oxford Mathematical Monographs, Oxford Science Publications.