A Purely Equational Formalism for Functorial Data Migration

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Abstract. In this paper we describe a simple equational formalism for expressing functorial data migration [8]. A graphical IDE and implementation of this formalism are available at categoricaldata.net/fql.html.

1 Introduction to Functorial Data Migration

In the functorial data model, which originated with Rosebrugh and others in the late 1990s [5], a database schema is a finitely presented category (a directed multigraph with path equations [3]), and a database instance on a schema $S$ is a functor from $S$ to the category of sets, $\text{Set}$. The database instances on a schema $S$ constitute a category (indeed, a topos [3]), denoted $S\mathchar`-\text{Inst}$, and a functor $F : S \to T$ between schemas $S$ and $T$ induces three adjoint data migration functors, $\Sigma_F : S\mathchar`-\text{Inst} \to T\mathchar`-\text{Inst}$, $\Pi_F : S\mathchar`-\text{Inst} \to T\mathchar`-\text{Inst}$, and $\Delta_F : T\mathchar`-\text{Inst} \to S\mathchar`-\text{Inst}$. These functors provide a category-theoretic alternative to traditional, set-theoretic operations for information integration such as the chase [4].

In their original work [5], Rosebrugh et al. identified two key issues preventing the functorial data model from being a practical foundation for data management. First, database instances must be considered up to isomorphism because the data migration functors only characterize results up to isomorphism. In other words, we must consider database instances to consist solely of “meaningless IDs” or “labelled nulls” [1]. Of course, to be practical databases instances must contain not just meaningless identifiers but also meaningful values such as strings and integers. Second, even though set-valued functors may be stored as tables, it was unclear how or if the data migration operations could be implemented using SQL and vice-versa.

In previous work [9] we solve the first problem by extending the functorial data model to define a schema as two categories and an instance to be two set-valued functors $I$ and $J$ and a natural transformation $I \Rightarrow J$. As a consequence, schemas can be understood as particular kinds of entity-relationship (ER) diagrams [1] – those in “categorical normal form” [8]. We solve the second problem by defining a simple algebraic query language, FQL, based on our extension, prove that FQL can implement the SPCU (select-project-product-union) fragment of SQL, and prove that the SPCU fragment of SQL extended with a globally unique ID generator can implement FQL.
Unfortunately, FQL is too weak to express certain operations, such as selection by a constant. Moreover, it lacks the ability to change the type of data, for example, by converting strings to integers via a length function. Finally, in practice it can be difficult to write FQL code because FQL is only a thin veneer over data migration functors. In this paper, we propose an alternative, more expressive formalism that does not suffer from FQL’s drawbacks. We call this formalism FPQL, for “functorial programming and query language”. In this paper we describe FPQL, but do not describe functorial data migration; see [9] and [8] for suitable background material. An implementation of FPQL and an associated graphical IDE are available at categoricaldata.net/fql.html.

Overview. Our formalism defines a typing, schema, and instance to each be a category presentation, and hence to each be an equational theory – technically, a variety in the sense of universal algebra [2]. For example, we can specify a typing with sorts Nat and String, constants 1, 2, . . . : Nat, function symbols reverse : String → String and length : String → Nat, and equations length(reverse(x)) = length(x) and reverse(reverse(x)) = x. A schema extends a typing with additional sorts for entities, such as Person, additional function symbols for attributes and foreign keys, such as name : Person → String and father : Person → Person, and additional equations for data integrity constraints, such as father(father(x)) = father(x) (to enforce that the fatherhood hierarchy must be at most two levels deep, for example). An instance extends a schema by adding constants for particular sorts and equations describing these constants, such as Bill, BillJr : Person and name(father(BillJr)) = name(Bill). Semantically, the meaning of a typing, schema, or instance is simply the category generated by the presentation; that is, the (potentially infinite) set of facts which can be derived from the equations. Hence, our typings, schemas, and instances can all be thought of as deductive, rather than extensional, databases [1]. Schema mappings and instance morphisms correspond to homomorphisms of these databases in the usual way (i.e., in sense of model theory [1]) and the ∆, Σ, Π data migration operations correspond to well-studied operations on equational theories [6]; Σ_F, for example, corresponds to substitution along F.

2 A Simple Formalism for Functorial Data Migration

Our formalism defines typings, schemas, and instances to each be particular kinds of category presentations, and defines mappings (schema morphisms) and homomorphisms (instance morphisms) to be each be particular kinds of functor presentations. In the next section, we describe presentations, and in the following section, we define typings, schemas, and instances in terms of presentations. We then describe functor presentations, and then mappings and homomorphisms in terms of functor presentations.
2.1 Finitely Presented Categories

Let \((N, E)\) be a directed labelled multi-graph, with a set of nodes \(N\) and a set of edges \(E\). In this paper, we assume all multi-graphs have a finite number of nodes and edges. When there exists a node 1 and edges \(1 \rightarrow n\) for every \(n \in N\) we say that \((N, E)\) is pointed. For example, let 
\[ N := \{1, \text{String}, \text{Nat}\} \]
\[ E := \{\text{length}: \text{String} \rightarrow \text{Nat}, \text{reverse}: \text{String} \rightarrow \text{String}, \text{zero}: 1 \rightarrow \text{Nat}, \text{succ}: \text{Nat} \rightarrow \text{Nat}, \!_\text{String}: \text{String} \rightarrow 1, \!_\text{Nat}: \text{Nat} \rightarrow 1\} \]
which may be rendered graphically as

\[
\begin{array}{c}
\text{zero} \\
\downarrow \\
\text{Nat} \end{array}
\quad \begin{array}{c}
1 \\
\downarrow \\
\text{Nat} \\
\downarrow \\
\text{String} \\
\downarrow \\
\text{String} \\
\downarrow \\
\text{Nat} \\
\downarrow \\
\text{Nat}
\end{array}
\quad \begin{array}{c}
\text{succ} \\
\downarrow \\
\text{Nat} \end{array}
\quad \begin{array}{c}
\text{length} \\
\downarrow \\
\text{Nat} \end{array}
\quad \begin{array}{c}
\text{reverse} \\
\downarrow \\
\text{String} \end{array}
\]

Define a simply-typed language of paths \(P\) over \((N, E)\) as follows:
\[
P ::= \text{id}_n : n \rightarrow n \quad \text{identity} \\
| (P : n \rightarrow n').(e : n' \rightarrow n'') : n \rightarrow n'' \quad \text{composition}
\]
Let \(C\) be a set of equations between such paths in \((N, E)\), for example \(C := \{\text{reverse.reverse} = \text{id}, \text{reverse.length} = \text{length}\}\). We say that \(C\) is pointed when it proves the (finite) set of equations, for every well-typed \(e\):
\[
\text{id}_1 \quad (e : t \rightarrow 1) = \!_1 \quad (e : t \rightarrow t'), \!_t = \!_t
\]
Let \(\Gamma := (N, E, C)\) be such a pointed directed multigraph and pointed set of path equations. \(\Gamma\) is the presentation of a category, written \([\Gamma]\), with objects \(N\) and arrows equivalence classes of paths \(n \rightarrow n'\) modulo provability in \(C\). Because \(\Gamma\) is pointed, 1 is a terminal object in \([\Gamma]\). Provability in \(\Gamma\) can be semi-decided by Knuth-Bendix completion [7]. Note that if \(\Gamma\) contains a loop, the number of morphisms in \([\Gamma]\) may be infinite, but when \([\Gamma]\) is finite, any semi-decision procedure for equality can be used to construct \([\Gamma]\). Given a non-pointed multigraph and non-pointed set of path equations, it is always possible to complete them to be pointed; henceforth, all of the category presentations we consider in this paper will be pointed.

2.2 Typings, Schemas, and Instances

A typing \(\Gamma := (T, F, C)\) is a presentation of a category as defined above. The nodes \(T\) are understood as base types (e.g., \text{String}), the edges \(f : 1 \rightarrow t\) as constants (e.g., \text{blue}: \text{String} \rightarrow \text{String}), the edges \(f : t \rightarrow t'\) as functions (e.g., \text{reverse}: \text{String} \rightarrow \text{String}), and the equations \(C\) as an axiomatization of the intended denotation of these symbols (e.g., \text{reverse.reverse} = \text{id}).

A schema \(S := (N, E, A, C')\) over \(\Gamma\) is an extension of \(\Gamma\) that includes:
– a set of fresh nodes $N$, called *entities*, such as

$$N := \{ \text{Dept}, \text{Emp} \}$$

– a set of fresh edges from entities to types, $A$, called *attributes*, such as

$$A := \{ \text{age} : \text{Emp} \rightarrow \text{Nat}, \text{title} : \text{Dept} \rightarrow \text{String} \}$$

– a set of fresh edges from entities to entities, $E$, called *foreign keys*, such as

$$E := \{ \text{manager} : \text{Emp} \rightarrow \text{Emp}, \text{worksIn} : \text{Emp} \rightarrow \text{Dept}, \text{admin} : \text{Dept} \rightarrow \text{Emp} \}$$

– a set of path equations, $C'$, over $(N \cup T, E \cup F \cup A)$, such as

$$C' := \{ \text{manager.manager} = \text{manager}, \text{manager.worksIn} = \text{worksIn}, \text{admin.worksIn} = id \}$$

Hence, $[[S]]$ is a category extending $[[S]]$. We may write $S$ graphically:

An *instance* $I := (V, C'')$ over schema $S$ is an extension of $S$ that includes
– a set of fresh edges $V$, originating at 1, called *variables*, such as

$$V := \{ \text{bill} : 1 \rightarrow \text{Emp}, \text{infinity} : 1 \rightarrow \text{Nat} \}$$

– a set of path equations, $C''$, over $(N \cup T, V \cup E \cup F \cup A)$, such as

$$C'' := \{ \text{bill.age} = \text{zero}, \text{bill.worksIn.admin.manager} = \text{bill} \}$$

Hence, $[[I]]$ is a category extending $[[S]]$. We may write $[[I]]$ as a set of tables:

<table>
<thead>
<tr>
<th>Emp</th>
</tr>
</thead>
<tbody>
<tr>
<td>ID</td>
</tr>
<tr>
<td>bill</td>
</tr>
<tr>
<td>bill.worksIn.admin</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dept</th>
</tr>
</thead>
<tbody>
<tr>
<td>ID</td>
</tr>
<tr>
<td>bill.worksIn</td>
</tr>
</tbody>
</table>
In the tables above, some IDs, such as bill, are *generators*; others, such as bill.worksIn.admin are *generated*; similarly, some attribute values, such as bill.age are equal to constants (in this case, zero), and others are not, such as bill.name. In relational database theory, these generated values are variously called *labelled nulls* or *skolem variables* [1]. When an instance has no labelled nulls, we say that it is *ground*; when an instance presentation is isomorphic to its tables (i.e., having one generator for each entry in the tables), we say that it is *saturated*. In the concrete syntax of FPQL, our complete example is written:

Nat : type
type Nat
succ : Nat -> Nat

String : type
reverse : String -> String
length : String -> Nat

eq1 : reverse.reverse = String
eq2 : length = reverse.length

S = schema {
  nodes Emp, Dept;
  edges age : Emp -> Nat, name : Emp -> String, title : Dept -> String,
    worksIn : Emp -> Dept, admin : Dept -> Emp, manager : Emp -> Emp;
  equations manager.manager = manager,
    Emp.manager.worksIn = Emp.worksIn, Dept.admin.worksIn = Dept;
}

I = instance {
  variables bill : Emp, infinity : Nat;
  equations bill.age = zero, bill.worksIn.admin.manager=bill;
} : S

**Remark on inconsistency.** Let \( \Gamma := (N, E, C) \) be a typing and \( c, c' : 1 \to n \) be distinct constants in \( E \). It may or may not be the case that \( C \vdash c = c' \). Moreover, \( C \vdash c = c' \) is not a contradiction in the sense of implying \( C \vdash p = q \) for every \( p, q \). However, if \( \| \Gamma \| \) is finite, it is possible to decide if there exists \( c, c' \) such that \( C \vdash c = c' \), and then to add (the finite set of equations) \( p = q \) for every \( p, q \). Whether or not \( C \vdash c = c' \) should be considered a contradiction typically depends on what \( \Gamma \) is being used for.

### 2.3 Finitely Presented Functors

Let \( S := (N, E, C) \) and \( T := (N', E', C') \) be category presentations. A *functor presentation* is a function \( F : S \to T \) taking nodes to nodes and edges \( e : n \to n' \) to paths \( F(e) : F(n) \to F(n') \) such that
- $F$ preserves composition: $F(id_n) = id_{F(n)}$ and $F(f.g) = F(f).F(g)$.
- $F$ preserves the terminal object: $F(1) = 1$ (it follows that $F(!_n) = !_F(n)$).
- $F$ preserves equations: if $C \vdash p = q$, then $C' \vdash F(p) = F(q)$.

The functor presentation $F: S \to T$ denotes a functor $\llbracket F \rrbracket: \llbracket S \rrbracket \to \llbracket T \rrbracket$.

### 2.4 Mappings and Transforms

Let $\Gamma$ be a typing and let $S, T$ be schemas on $\Gamma$. A (schema) mapping $F: S \to T$ is a functor presentation from $S$ to $T$ that is the identity on $\Gamma$.

**Theorem 1.** The schemas and schema mappings on a fixed typing form a category which has all colimits.

Let $I, J$ be instances on $S$. A homomorphism $h: I \Rightarrow J$ is a functor presentation from $I$ to $J$ that is the identity on $S$.

**Theorem 2.** The instances and homomorphisms on a fixed schema $S$ form a category, $S\text{-Inst}$, which has all colimits. The instances $I$ on $S$ such that $\llbracket I \rrbracket$ is finite form a subcategory of $S\text{-Inst}$, denoted $S\text{-InstFin}$.

Associated with a mapping $F: S \to T$ are three data migration functors:

- $\Sigma_F: S\text{-Inst} \to T\text{-Inst}$ is defined to be substitution on presentations: if $e: x \to y \in I$, then $e: F(x) \to F(y) \in \Sigma_F(I)$, and similarly for equations. Intuitively, $\Sigma$ is similar to union (and also the chase [4]).
- $\Delta_F: T\text{-InstFin} \to S\text{-InstFin}$ is right adjoint to $\Sigma_F$, and is like projection.
- $\Pi_F: S\text{-InstFin} \to T\text{-InstFin}$ is right adjoint to $\Delta_F$, and is similar to join.

**Theorem 3.** Let $F: S \to T$ be a mapping. Then $\Sigma_F$ is left adjoint to $\Delta_F$, which is left adjoint to $\Pi_F$.

### 2.5 FPQL

The language FPQL consists of the following components:

- Typings, schemas, instances, and homomorphisms, written explicitly.
- Binary products and co-products of instances, as well as the associated projection and injection homomorphisms.
- Initial and terminal instances, and associated homomorphism for each.
- The data migration functors $\Delta_F$, $\Sigma_F$, $\Pi_F$, which can be applied to both instances and homomorphisms.
- The units and co-units for the data migrations, as homomorphisms.
- The relationalization operation [9] that equates IDs which are “observationally equivalent”.

The $\Pi$, $\Delta$, relationalize, and product operations require input instances to have a finite denotation (and hence a finite schema and typing).
3 FPQL and SQL

Entity-relationship (ER) diagrams in categorical normal form [8] and instances on them can be translated into FPQL, for example:

![Diagram](image)

<table>
<thead>
<tr>
<th>Department</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dept ID 101</td>
<td>CS</td>
</tr>
<tr>
<td>Dept ID 102</td>
<td>Math</td>
</tr>
</tbody>
</table>

S = schema {
nodes Emp, Dept;
edges worksIn : Emp -> Dept,
    admin : Dept -> Emp,
    manager : Emp -> Emp,
    first : Emp -> dom,
    last : Emp -> dom,
    name : Dept -> dom;
equations;
}

I = instance {
variables 101:Emp, 102:Emp, 103:Emp, q10:Dept, x02:Dept;
equations 101.manager=103, 101.worksIn=q10,
    102.manager=102, 102.worksIn=x02,
    103.manager=103, 103.worksIn=q10,
    101.first=Al, 101.last=Akin,
    102.first=Bob, 102.last=Bo,
    103.first=Carl, 103.last=Cork,
    q10.admin=102, q10.name=CS,
    x02.admin=101, x02.name=Math;
} : S
In practice it can be difficult to write data migration functors by hand. Hence, FPQL provides special syntax, in the guise of “select/from/where” (SFW) expressions, for writing data migration operations of the form $\Sigma_F \circ \Delta_G \circ \Pi_H$, where $F$ is a discrete op-fibration [3]. Syntactically, these expressions are similar to SQL’s SFW expressions in which WHERE clauses are conjunctions of equalities. However, FPQL’s SFW expressions have three key advantages over SQL: 1) foreign-key awareness, 2) invariance under change of ID, and 3) “database at a time” query capability:

1. FPQL’s SFW expressions are “foreign-key aware” and hence are less verbose and require fewer joins than corresponding SQL queries. Consider the Emp table from the previous subsection. If $e$ is an Emp, then in FPQL, $e$.manager.manager denotes $e$’s manager’s manager. Because SQL SFW expressions are not “foreign-key aware”, to compute $e$’s manager’s manager in SQL requires joins. A join-less FPQL example finding all “self-manager-managers” is shown on the left, and equivalent SQL containing a join is shown on the right:

```
FPQL                      SQL
select e.first          select e.first
from Emp as e           from Emp as e, Emp as f
where e.manager.manager = e
```

2. In FPQL, it is impossible to “observe” an ID by returning an ID from a SFW query or comparing an ID with anything besides another ID from the same entity. For example, when we find the “self-manager-managers” above, we return the first names of the managers. Because of this restriction, FPQL SFW queries cannot depend on the actual values of what should be meaningless identifiers in the input database.

3. In both SQL and FPQL, an entire target database can be populated by unioning a set of SFW expression per target table. SQL provides no special support for such “database at a time” queries, but FPQL provides special syntax for such queries, which (unlike SQL) statically guarantees that the resulting database will respect the target schema’s foreign-key and path-equality constraints. An FPQL example setting each employee’s manager to their manager’s manager is:
```
EmpQuery = { from Emp as e
    attributes first=e.first, last=e.last
    edges manager = {e=e.manager.manager} : EmpQuery
    worksIn = {d=e.worksIn} : DeptQuery
} : Emp

DeptQuery = { from Dept as d
    attributes name=d.name
    edges admin = {e=d.admin} : EmpQuery
} : Dept
```

Here `EmpQuery` populates the `Emp` table and `DeptQuery` populates the `Dept` table. `DeptQuery` is the identity query: for each department \( d \), it returns a new department with name \( d.name \) (specified in the `attributes` block), and with admin \( d.admin \) (specified in the `edges` block). This employee, \( d.admin \), is declared to be created by `EmpQuery`, and FQL checks if `EmpQuery` will in fact create such an employee. `EmpQuery` is defined similarly, except that the manager of each employee \( e \) is set to \( e.manager.manager \).

Although it is possible to implement (up to isomorphism) FPQL’s SFW expressions by translation to data migrations of the form \( \Sigma \circ \Delta \circ \Pi \), in practice we have found it is actually faster to execute SFW expressions directly in a manner similar to how RDBMSs implement SQL. The reason is that it is possible to take advantage of techniques originally developed for SQL [1], such as filtering by the `WHERE` clauses as early as possible, and re-ordering the `FROM` clauses to minimize intermediate results. One important difference from SQL execution is that to evaluate an FPQL `WHERE` conjunct \( p = q \) in an instance \( I \) it is not enough to check that \( p \) and \( q \) are syntactically equal, instead the entailment \( I \vdash p = q \) must be decided. Because FPQL SFW expressions require finite instances as input, such entailments are always decidable. However, FPQL SFW expressions cannot be evaluated by translation to SQL, because doing so would require saturating the input instance, and saturation is not expressible in SQL.

**Theorem 4.** Data migrations of the form \( \Sigma_F \circ \Delta_G \circ \Pi_H \), where \( F \) is a discrete op-fibration [3], are closed under composition.

**Remark on adjoints.** In addition to the evaluation functor \( S-\text{InstFin} \rightarrow T-\text{InstFin} \) of a SFW expression \( S \rightarrow T \), in the case where there is only one subquery per target table (meaning that the \( \Sigma \) part of the associated \( \Sigma \circ \Delta \circ \Pi \) data migration is the identity), there is a left adjoint functor \( T-\text{InstFin} \rightarrow S-\text{InstFin} \). However, in general this functor cannot be expressed as a SFW expression \( T \rightarrow S \).
References