On second-order sliding mode observers with residuals’ projection for switched systems

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Abstract—This paper outlines some results concerning the application of second-order sliding mode techniques to address the problem of simultaneous discrete and continuous state reconstruction for linear autonomous switched systems. A stack of dynamical observers produces suitable residual signals which are used for reconstructing the discrete state and, at the same time, for identifying the unique observer that provides an asymptotically converging estimate of the continuous state. Simple tuning formulas for the suggested schemes are constructively developed along the paper by means of appropriate Lyapunov analysis. The main result and most significant novelty of this work is a projection procedure for deriving residuals characterized by finite-time convergence. Simulation results confirm the effectiveness of the proposed approach.

I. INTRODUCTION

A problem of great interest is the reconstruction of the discrete and/or the continuous state through the observation of measurable system outputs. This is a key problem because both the active mode and the continuous state have to be estimated during a finite time interval. The techniques developed in this framework can be applied to several problems where the discrete events driving the mode switchings are unknown or even unobservable.

Many systems encountered in practice exhibit switchings between several subsystems, inherently by nature, such as when a physical plant has the capability of undergoing several operational modes, or as a result of the controller design, such as in switching supervisory control.

Observability notions for some classes of hybrid systems such as switched linear systems have been discussed and characterized in [2], [7], [10], [27], [24]. Several approaches have been proposed to estimate the discrete state in switched (and more general hybrid) dynamical systems through the observation of the measurable system states or outputs ([1], [3], [20]). Usually, the hybrid observer consists of two parts: an index estimator of the current active sub-model and a continuous observer that estimates, asymptotically in most cases, the continuous state of the hybrid system.

The possibility to have finite time estimate for that class of systems is clearly important, not to say crucial. Finite time observers can be designed using approaches arising from sliding mode concept (see e.g. [4], [9], [18], [21], [22]). Algebraic reconstruction, also providing finite time convergence, of unknown estimates is performed in [13] for the estimation of the index corresponding to the current active subsystem, and the state variable of this subsystem. Following a similar reasoning and using properties of the distribution theory, switching time estimation is given by an explicit formula, as a function of the integral of the output, in [25] for linear switched systems and in [5] for impulsive systems.

This paper proposes a methodology for reconstructing the discrete and continuous state of linear autonomous switched systems assuming that the switching signal is unknown. A stack of second-order sliding mode observers is implemented. Each observer of the stack delivers a residual signal on the basis of which a suitable discrete mode identification logic can identify the actual mode of operation and select, at the same time, the unique observer in the stack which is delivering the correctly reconstructed continuous state profiles. An analysis of the switched system “distinguishability” conditions guaranteeing that the discrete state can be correctly reconstructed is also developed. A projection procedure is used on order to recover residual signals with finite-time convergence.

The paper structure is as follows. Section 2 formulates the problems under analysis and describes the proposed observer stack. Section 3 presents the method for discrete and continuous state reconstruction. Section 4 illustrates some simulation result and Section 5 gives concluding remarks.

A. Notation

For a vector \( z = [z_1, z_2, \ldots, z_m] \in \mathbb{R}^m \) denote
\[
\text{sign}(z) = [\text{sign}(z_1), \text{sign}(z_2), \ldots, \text{sign}(z_m)]^T.
\] (1)

For a square symmetric matrix \( M \) of order \( m \), denote as
\[
\sigma_j(M), \quad j = 1, 2, \ldots, m
\] (2)
the set of the corresponding eigenvalues and let
\[
\sigma_{\text{max}}(M) = \sup_{j \in \{1, 2, \ldots, m\}} |\sigma_j(M)|
\] (3)

II. SLIDING MODE OBSERVER STACK FOR SWITCHED SYSTEMS

Consider the linear autonomous switched dynamics
\[
\begin{align*}
\dot{x}(t) &= A^j(t)x(t) \\
y(t) &= Cx(t)
\end{align*}
\] (4)

where \( x(t) \in \mathbb{R}^n \) represents the state vector, and \( y(t) \in \mathbb{R}^p \) represents the output vector. The so-called “discrete state” \( j(t) \in \{1, 2, \ldots, q\} \) determines the actual system dynamics among the possible \( q \) “operating modes” which are represented, for system (4), by the set of matrices \( (A^1, A^2, \ldots, A^q) \).
belonging to the “admissible matrix set”. Without loss of
generality, it is assumed that $C$ is a full rank matrix. It is
also assumed that the matrices $A_i^{(t)}$, $C$ of the $q$ modes above
are exactly known, while the discrete state $j(t)$ is uncertain
and needs to be estimated, along with the continuous state
vector $x(t)$, by relying on the availability for measurements
of the output vector $y(t)$, only. The case of a switched
output matrix $C^{(t)}$ is much more challenging and this issue
will be discussed in future works. Consider, without loss of
generality, the next expression for the discrete state
\[
j(t) = j_k, \quad t_{k-1} \leq t < t_k, \quad k = 1, 2, \ldots, \infty
\] (5)
where $t_0 = 0$ and $t_k$ ($k = 1, 2, \ldots$) are the “switching time
instants” at which the discrete state is changing.

**Assumption 1** The switching time instants fulfill the next
dwell-time restriction
\[
t_k - t_{k-1} \geq \Delta, \quad k = 1, 2, \ldots, \infty
\] (6)
for some known positive constant $\Delta$, and the resulting $x(t)$
trajectories evolve for all $t \geq 0$ inside an *a priori* known,
arbitrarily large, compact domain $D \subset \mathbb{R}^n$.

**Assumption 2** The matrix pairs $(A_i, C)$ are observable
for all $i = 1, 2, \ldots, q$.

As suggested in [12], consider the nonsingular coordinate
transformation such that the output vector $y$ is a subset of
the transformed state, i.e.
\[
\begin{bmatrix}
\xi \\
y
\end{bmatrix} = T_c x
\] (7)
with matrix
\[
T_c = \begin{bmatrix}
N_c^T & C
\end{bmatrix}
\] (8)
where the columns of $N_c \in \mathbb{R}^{n \times (n-p)}$ span the null space
of $C$. This transformation is always nonsingular, and the
switched system dynamics in the transformed coordinates are:
\[
\begin{align*}
\dot{\xi}(t) &= A_{11}^{(t)} \xi(t) + A_{12}^{(t)} y(t) \\
\dot{y}(t) &= A_{21}^{(t)} \xi(t) + A_{22}^{(t)} y(t)
\end{align*}
\] (9)
where $\xi(t) = N_c^T x \in \mathbb{R}^{n-p}$ and the submatrices in (9)
satisfy
\[
T_c A_i^{(t)} T_c^{-1} = \begin{bmatrix}
A_{11}^{(t)} & A_{12}^{(t)} \\
A_{21}^{(t)} & A_{22}^{(t)}
\end{bmatrix}
\] (10)
A stack of $q$ dynamical observers, each one associated to a
different mode of operation in the admitted set, is suggested
as follows:
\[
\begin{align*}
\hat{\xi}_i(t) &= A_{11}^{i(t)} \hat{\xi}_i(t) + A_{12}^{i(t)} \hat{y}_i(t) + L^i \nu_t(t), \quad i = 1, 2, \ldots, q, \\
\hat{y}_i(t) &= A_{21}^{i(t)} \hat{\xi}_i(t) + A_{22}^{i(t)} \hat{y}_i(t) - \nu_t(t)
\end{align*}
\] (11)
where $(\hat{\xi}_i, \hat{y}_i)$ is the state estimate provided by the $i$-th
observer, $L^i \in \mathbb{R}^{(n-p) \times p}$ are observer gain matrices and
$\nu_t \in \mathbb{R}^p$ represents the $i$-th observer injection input, yet to
be designed by appropriate discontinuous control techniques.
By introducing the error variables
\[
e^i_\xi = \hat{\xi}_i - \xi, \quad e^j_y = \hat{y}_j - y, \quad i, j = 1, 2, \ldots, q
\] (12)
the next error dynamics can be easily obtained
\[
\begin{align*}
\dot{e}^i_\xi(t) &= A_{11}^{i(t)} e^i_\xi(t) + A_{12}^{i(t)} e^i_y(t) + (A_{11}^{i(t)} - A_{11}^{j(t)}) \xi(t) \\
&\quad + (A_{12}^{i(t)} - A_{12}^{j(t)}) y(t) + L^i \nu_t(t) \\
\dot{e}^j_y(t) &= A_{21}^{j(t)} e^i_\xi(t) + A_{22}^{j(t)} e^i_y(t) + (A_{21}^{i(t)} - A_{21}^{j(t)}) \xi(t) \\
&\quad + (A_{22}^{i(t)} - A_{22}^{j(t)}) y(t) - \nu_t(t)
\end{align*}
\] (13)
By defining the next functions
\[
\Phi^i(e^i_\xi, e^i_y, \xi, y) = \begin{cases}
0 & \text{for } i = j(t) \\
-\phi^j_i(e^i_\xi, e^j_y, \xi, y) & \text{for } i \neq j(t)
\end{cases}
\] (14)
one can rewrite (14) as
\[
\dot{e}^j_y(t) = \Phi^i(e^i_\xi, e^i_y, \xi, y) - \nu_t(t)
\] (16)
The functions $\Phi^i$ are supposed to be smooth enough
according to the next:

**Assumption 3** There is a constant $\Phi$ such that
\[
\left\| \frac{d}{dt} \Phi^i(e^i_\xi, e^i_y, \xi, y) \right\| \leq \Phi, \quad \forall i, j = 1, 2, \ldots, q
\] (17)
The next Theorem establishes some properties of the
proposed observer stack that will be instrumental in our next
developments.

**Theorem 1** Consider the linear switched system (4) sat-
sfying the Assumptions 1, 2 and 3, along with the observer
stack (11), (10), (8), and the observer injection terms set
according to
\[
\begin{align*}
\nu_t &= -(\nu_{i1} + \nu_{2i}) \\
\nu_{i1} &= -k_1 e^i_y(t) - k_2 |e^i_y(t)|^{1/2} \text{sign}(e^i_y(t)) \\
\nu_{2i} &= -k_3 \text{sign}(e^i_y(t))
\end{align*}
\] (18)
with the next tuning inequalities imposed on the tuning
coefficients:
\[
k_1 > 2 \Phi, \quad k_2 > 0, \quad k_3 > \Phi \sqrt{k_1}
\] (21)
Let $L^j$ be chosen such that, for all $j = 1, 2, \ldots, n-p$,
\[
\text{Re} \left\{ \sigma_j \left\{ A_{11}^j + L^j A_{22}^j \right\} \right\} \leq -\gamma, \quad \gamma > 0
\] (22)
Then, for sufficiently large $\Phi$ and $\gamma$, there exists $T^* < \Delta
such that for all $k = 1, 2, \ldots, \infty$ and for some $\alpha > 0$, on the
intervals $t \in [t_{k-1} + T^*, t_k]$, the next properties hold:
\[
e^j_y(t) = 0 \quad \forall i
\] (23)
\[
\nu_{2i}(t) = \begin{cases}
0 & \text{for } i = j(t) \\
-\varphi^j_i(e^i_\xi, e^j_y, \xi, y) & \text{for } i \neq j(t)
\end{cases}
\] (24)
\[
\|e^j_y(t)\| \leq \alpha e^{-\gamma(t-t_{k-1})}
\] (25)
Proof of Theorem 1
By (16), the dynamics of the output error are given by:
\[ \dot{e}_i^y(t) = \begin{cases} \varepsilon_i^j(t) & \text{for } i = j(t) \\ \varepsilon_i^j(t) - \nu_i(t) & \text{for } i \neq j(t) \end{cases} \] (26)

During the first interval \( t \in [0, t_1] \) the actual mode is \( j(t) = j_1 \). Then (26) specializes as
\[ \begin{align*}
\dot{e}_{y1}^i(t) &= -\nu_{i1}(t) \\
\dot{e}_{y1}(t) &= \varepsilon_{i1}(\varepsilon_i^1, \varepsilon_i^2, \xi, \gamma) - \nu_i(t), \ i \neq j_1
\end{align*} \] (27)

Considering (18)-(20) into (27) yields
\[ \begin{align*}
\dot{e}_{y1}^i(t) &= -k_1 \varepsilon_{i2} - k_2 \varepsilon_{i1}^{1/2} \text{sign}(\varepsilon_{i1}^1) + \nu_{i2}^i \quad (29) \\
\dot{\nu}_{i2} &= -k_3 \text{sign}(\varepsilon_{i2}^1) \quad (30)
\end{align*} \]
while for (28) one has:
\[ \begin{align*}
\dot{e}_{y2}(t) &= -k_1 \varepsilon_{y2} - k_2 \varepsilon_{y1}^{1/2} \text{sign}(\varepsilon_{y1}^1) + \nu_{y2} + \varepsilon_{y1}^1 \quad (31) \\
\dot{\nu}_{y2} &= -k_3 \text{sign}(\varepsilon_{y2}^1), \ i \neq j_1 \quad (32)
\end{align*} \]

By introducing the new coordinates
\[ z_i(t) = \nu_{i1}(t) + \text{sign}(z_i^1) \varepsilon_{i1}, \xi, y \quad (33) \]
one can augment and rewrite (31)-(32) as
\[ \begin{align*}
\dot{\xi}_i^1 &= -k_1 \varepsilon_{\xi i}^1 - k_2 \varepsilon_{\xi i}^{1/2} \text{sign}(\varepsilon_{\xi i}^1) + z_i \quad (34) \\
\dot{z}_i &= -k_3 \text{sign}(\varepsilon_{z i}^1) + \frac{d}{dt} \phi_i^1, \ i \neq j_1 \quad (35)
\end{align*} \]

To demonstrate the stability of (34)-(35) the next family of Lyapunov functions is considered
\[ V_{ip} = \zeta_{ip}^2 H \zeta_{ip}, \quad p = 1, 2, \ldots, n \quad (36) \]
\[ \zeta_{ip} = \begin{bmatrix} |e_{y2}^{1/2} \text{sign}(e_{yp}^1) | \\ e_{yp}^1 \\ z_{ip} \end{bmatrix} \quad (37) \]
\[ H = \begin{bmatrix} (4k_3 + k_2) & k_1k_2 & -k_2 \\ k_1k_2 & k_2^2 + k_1 & -k_1 \\ -k_2 & -k_1 & 2 \end{bmatrix} \quad (38) \]
where \( e_{yp}^1 \) and \( z_{ip} \) represent the \( p \)-th element of \( e_y^1 \) and \( z_i \), respectively. The same dynamics and Lyapunov function were considered in [17]. It was shown that if the tuning conditions (21) hold then the next estimation is in force for some \( \gamma_1, \gamma_2 > 0 \) and for all \( p = 1, 2, \ldots, n \)
\[ \dot{V}_{ip} \leq -\gamma_1 V_{ip} - \gamma_2 \sqrt{V_{ip}} \quad (39) \]
that guarantees the finite time convergence of \( e_y^1 \) and \( z_i \) to zero, starting from some finite time \( T^* > 0 \). Therefore on the interval \( t \in [T^*, t_1] \) property (23) is achieved and, by the definition (33) of \( z_{i1} \), (24) is achieved as well on the same interval. By iteration, they are proven on all the successive intervals.

The transient time \( T^* \) can be made as small as desired by sufficiently increasing the \( \Phi \) constant in the tuning formula (21), and in particular such that \( T^* < \Delta \).

We shall now analyze the state error variables \( e_{y1}^j(t) \) once the trajectories are restricted to the sliding manifold defined by \( \{ e_{y1}^j = z_i = 0 \} \). According to (15), (18) and (33), this implies
\[ \nu_i(t) = A_{y 1} e_{y1}^i(t) + (A_{y 1} - A_{y 2}^j) \xi(t) + (A_{y 2} - A_{y 2}^j) y(t) \quad (40) \]

By substituting (40) into the first of (13), it yields the next equivalent dynamics of the error variables \( e_i^j(t) \)
\[ \begin{align*}
\dot{e}_{y1}^j(t) &= \Delta A_{y 1} e_{y1}^j(t) + \Delta A_{y 2}^j y(t) \\
&= \Delta A_{y 1} e_{y1}^j(t) + \Delta A_{y 2}^j y(t) \quad (41)
\end{align*} \]
where
\[ \begin{align*}
\Delta A_{y 1} &= (A_{y 1} + L^i A_{y 1}^i) \\
\Delta A_{y 2}^j &= (A_{y 1} - A_{y 2}^j(t)) + L^i (A_{y 2} - A_{y 2}^j(t)) \quad (42)
\end{align*} \]

It is worth noting that along every time intervals \( t \in [t_k + 1 + T^*, t_{k+1}] \), with \( k = 1, 2, \ldots \), the error dynamics of the “correct” observer (i.e., that having the index \( i = j_k \) which matches the current mode of operation \( j(t) \)) are given by
\[ \dot{e}_{y1}^j(t) = (A_{y 1} + L^i A_{y 1}^i) e_{y1}^j(t) \quad (43) \]
which is asymptotically stable by (22). The solution of (43) also fulfills the trivial relation (25). Theorem 1 is proven.

Remark 1: Assumption 2 implies that all the pairs \( (A_{y 1}, A_{y 2}^j) \) are observable, which motivates the tuning condition (22).

There is one observer in the stack that provides the asymptotic reconstruction of the continuous state of the switched system (4). The index of such “correct” observer is however unknown to the designer, hence the scheme needs to be complemented by a discrete node observer.

III. DISCRETE AND CONTINUOUS STATE ESTIMATION

In this section, a method is developed to identify the discrete mode by using the input injection \( \nu_i \) derived in accordance with (18)-(20) and “identifiability” requirements for the matrices of the switched system are given. By taking into account (15),(24) and (25) it turns out that along the time intervals \( [t_k + 1 + T^*, t_{k+1}] \) the norm of the injection term of the “correct” observer will be asymptotically vanishing in accordance with
\[ \| \nu_{j_k}^j(t) \| \leq A_{y 1}^M e^{-\gamma(t-t_{k+1}-T^*}) \rightarrow 0 \quad (44) \]
where
\[ A_{y 1}^M = \sup_{i \in \{1, 2, \ldots, q \}} \| A_{y 1}^i \| \quad (45) \]

The asymptotic nature is due to the error dynamics \( e_{y1}^j(t) \). Consequently, in order to obtain finite time convergence to zero of the “correct” residual signal, let us make the observer injection terms insensitive to the dynamics of \( e_{y1}^j(t) \).

Assumption 4 For all \( i = 1, 2, \ldots, q \), the submatrices \( A_{y 1}^i \) are not full row rank.

The major consequence of Assumption 4 is that is \( A_{y 1}^i \) has a non trivial left null space. Let us then consider a basis
matrix $U_i$ of the left null space of $A_{21}^i$ formed by the set of vectors $u_i \in \mathbb{R}^p$ such that $u_i^T A_{21}^i = 0$ and denote

$$\tilde{v}_i(t) = U_i^T v_i(t) \quad (46)$$

Note that $\tilde{v}_i \in \mathbb{R}^\bar{d}$ where $\bar{d}$ is the dimension of the left null space of $A_{21}^i$. Clearly, by (40), one has that

$$\tilde{v}_i(t) = U_i^T (A_{21}^i - A_{21}^{\tilde{j}(t)}) \xi(t) + U_i^T (A_{22}^i - A_{22}^{\tilde{j}(t)}) y(t) \quad (47)$$

We shall derive a structural condition on the system matrices guaranteeing that the projected input injection (46) associated to the “wrong” observers cannot be identically zero. For this purpose, consider the vector $\Gamma_i(t)$ obtained by stacking $\tilde{v}_i(t)$ and its successive derivatives up to a certain order $k \geq 1$, i.e.

$$\Gamma_i(t) = \begin{bmatrix} \tilde{v}_i(t) \\ \tilde{v}_i(t) \\ \vdots \\ d^k_{\tilde{v}_i} \tilde{v}_i(t) \end{bmatrix} \in \mathbb{R}^{(k+1)\bar{d}} \quad (48)$$

Let us set $\bar{k}$ such that the dimension $\bar{k}+\bar{d}$ of vector $\Gamma_i(t)$ is, at least, $n$. This can be guaranteed by choosing the smallest $\bar{k}$ such that $\bar{k} \geq \frac{n}{\bar{d}} - 1$. By successively differentiating (47), and considering (9), it follows that the vector $\Gamma_{ij} \in \mathbb{R}^{(k+1)d}$ can be statically related to the state variables $\xi(t)$ and $y(t)$ as

$$\Gamma_{ij}(t) = M_{ij} \xi(t) + N_{ij} y(t) \quad (49)$$

where

$$M_{ij} = \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_k \end{bmatrix} \in \mathbb{R}^{(k+1)d \times (n-p)} \quad (50)$$

$$N_{ij} = \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_k \end{bmatrix} \in \mathbb{R}^{(k+1)d \times p} \quad (51)$$

whose submatrices $M_k \in \mathbb{R}^{d \times (n-p)}$ and $N_k \in \mathbb{R}^{d \times p}$ can be easily computed iteratively as:

$$M_0 = U_i^T (A_{21}^i - A_{21}^{\tilde{j}(t)}) \quad (52)$$

$$M_0 = U_i^T (A_{22}^i - A_{22}^{\tilde{j}(t)}) \quad (53)$$

$$M_k = M_{k-1} A_{11}^{\tilde{j}(t)} + N_{k-1} A_{21}^{\tilde{j}(t)} \quad \forall k = 1, \ldots, \bar{k} (54)$$

$$N_k = M_{k-1} A_{12}^{\tilde{j}(t)} + N_{k-1} A_{22}^{\tilde{j}(t)} \quad \forall k = 1, \ldots, \bar{k} (55)$$

**Assumption 5** The matrices $[M_{ij}, N_{ij}]$ are full column rank for all values of $(i, j) \in \{1, 2, \ldots, q\} \times \{1, 2, \ldots, q\}$ with $i \neq j$.

**Assumption 5** implies that the null spaces of matrices $[M_{ij}, N_{ij}]$ are trivial, and, in turns, this assumption guarantees that $\tilde{v}_i(t)$ cannot be identically zero on a time interval unless the index $i$ is matching the current mode of operation of the switched system in the considered interval.

**Theorem 2:** Consider the linear switched system (4) and the observer stack described in the Theorem 1, fulfilling Assumptions 1, 2 and 3. Let Assumptions 4 and 5 be satisfied as well. Then, the continuous state estimation given by

$$\dot{x} = T^{-1} \begin{bmatrix} \hat{z}_{jk} \\ \hat{y}_{jk} \end{bmatrix} \quad (56)$$

will be such that

$$\parallel \dot{x}(t) - x(t) \parallel \leq a e^{-\gamma(t-t_{k-1}-T^* \rangle} \forall t \in [t_{k-1} + T^*, t_k) \quad (57)$$

for some $T^* \ll \Delta$, and the discrete state estimation

$$\hat{j}(t) = \arg \min_j R_i(t), \quad R_i(t) = \int_0^t \parallel \tilde{v}_i(\tau) \parallel d\tau \quad (58)$$

where $\tilde{v}_i = U_i^T v_i$, (with $v_i$ computed according to (18)-(20)), will be such that

$$\hat{j}(t) = j(t), \quad t_{k-1} + T^* \leq t \leq t_k, \quad k = 1, 2, \ldots \quad (59)$$

**Proof of Theorem 2**

The first part of the Theorem can be easily proven by considering the coordinate transformation (7) and Theorem 1, which imply (57). Then, by considering (47) which, specified for the correct observer ($i = j(t)$) guarantees that

$$\tilde{v}_{jk}(t) = 0, \quad t_{k-1} + T^* \leq t \leq t_k \quad (60)$$

along with the Assumption 5, whose main consequence is that $\tilde{v}_i(t)$ cannot be identically zero when $i \neq j(t)$, it follows that it is always possible to find a threshold $\eta$ such that for the residual signals $R_i(t)$ in (58) one has

$$R_i(t) \geq \eta, \quad t_{k-1} + T^* \leq t < t_k, \quad i \neq j \quad (61)$$

$$R_{jk}(t) < \eta, \quad t_{k-1} + T^* \leq t < t_k \quad (62)$$

Thus one easily get that

$$R_{jk}(t) \leq R_i(t), \quad t_{k-1} + T^* \leq t < t_k, \quad i \neq j_k \quad (63)$$

and, therefore, the residual-based estimation logic (58) provides the reconstruction of the discrete state after the finite time $T^*$ (according to (23)) successive to any switching time instant, i.e.

$$\hat{j}(t) = j_k, \quad t_{k-1} + T^* \leq t < t_k \quad (64)$$

**Theorem 2** is proven. $\Box$

Note that the design of continuous and discrete state estimators usually raises a tuning compromise: fast error dynamics (i.e., big values of $\gamma$ in equation (22)) implies a smaller difference between the correct and the wrong residuals, thereby reducing the reliability of the discrete mode identification and its robustness to noise. Such a drawback makes our method very interesting, since it provides residuals independent of the asymptotic error dynamics $e_{\xi_i}$ allowing a choice of $L_i$ in (22) such that the rate of convergence of the continuous state estimation can be chosen without affecting the reliability of the discrete mode identification.

**Remark 2:** Obviously, if Assumption 4 is not satisfied the projection (46) cannot be applied. Nevertheless discrete and
continuous state can be estimated with the difference that the transient time $T^*$ is slightly bigger than that in Theorem 2 because of the vanishing transient of the error variable $e_{\xi i}$ ([14]).

IV. SIMULATION RESULTS

In this section, we present an example to show the effectiveness of our method. Consider a switched linear system (4) with $q = 3$ modes described by the matrices

$$A^1 = \begin{bmatrix} 0.1 & 0.6 & -0.4 \\ -0.5 & -0.8 & 1 \\ 0.1 & 0.4 & -0.7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -0.2 & -0.4 & 0.8 \\ 1 & 0.6 & -0.3 \\ -0.8 & -0.5 & 0.2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -0.5 & -0.1 & -0.5 \\ -0.3 & -0.2 & 0.3 \end{bmatrix}$$

(65)

The system starts from the initial condition $x(0) = \begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix}^T$ and evolves switching in accordance with the modes sequence $\{1, 2, 1, 3, 2\}$ at the time instants $\{t_k\} = \{8, 14, 20, 24\}$ (Fig. 1). The output matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(66)

and the state transformation

$$T_e = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(67)

according to (8) gives the system in the proper form to apply our estimation procedure.

Assumption 2 (observability of $(A^1, C)$, $(A^2, C)$ and $(A^3, C)$) is satisfied. A stack of three observers of type (11) is used with the gain matrices such that the eigenvalues of the matrices $A^1_{11}$, $A^2_{11}$ and $A^3_{11}$ governing the error dynamics (41) are all located at $-4$. The input injections are designed in accordance with (18)-(20) with the parameters

$$k_1 = 10, \quad k_2 = 4, \quad k_3 = 5$$

(68)

As it can be seen in the graph of Fig. 2, after each switching instant, the unmeasurable state variable $x_3$ is properly estimated by the correct observer with a rate of convergence which can be chosen by tuning the gain matrices $L^i$.

Since Assumption 4 is fulfilled, the left null space $U_i$ of $A^i_{1j}$ for $i = 1, 2, 3$ can be found. Then, the projected input injection signals $\tilde{\nu}_i = U_i^T \nu_i$ are utilized in order to detect the active discrete mode, and consequently to identify, at each time instant, which one of the whole set of observers is providing the correct estimation of the continuous state. The evolutions of the projected input injections is depicted in Fig. 3, which show how, after a finite time transient starting at any switching, their components are identically zero for the correct observer, and separated from zero for the wrong observers.

In order to guarantee that the signals associated to the wrong observer cannot be identically zero on a time interval, the vector (48) up to the second derivative of $\tilde{\nu}_i(t)$ can be considered. By looking at the corresponding matrices $[M_{ij} \quad N_{ij}]$ for $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$ with $i \neq j$, it is straightforward to check that Assumption 5 is satisfied.

Finally, by making use of the residuals $R_i(t)$ defined in (58) and reported in the graph of Fig. 4, the discrete mode sequence can be reconstructed, and consequently the continuous state can be estimated as it is shown in Fig. 5.

V. CONCLUSIONS

The problem of simultaneous continuous and discrete state reconstruction has been tackled for a class of linear, autonomous, switched systems. The main ingredient of the proposed approach is an appropriate stack of second-order sliding mode observers used, both, as continuous
state observers and as residual generators for discrete mode identification. As a novelty, a procedure has been devised to algebraically process the residuals in such a way to make them “finite time” converging, and additionally conditions ensuring the identifiability of the system modes are derived. Identifiability issues in terms of the original pair of matrices $A^{j(t)}$ and $C$ are under active investigation.

REFERENCES


