Symmetric Hilbert spaces arising from species of structures

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Abstract

Symmetric Hilbert spaces such as the bosonic and the fermionic Fock spaces over some 'one particle space' $\mathcal{K}$ are formed by certain symmetrization procedures performed on the full Fock space. We investigate alternative ways of symmetrization by building on Joyal's notion of a combinatorial species. Any such species F gives rise to an endofunctor Γ_F of the category of Hilbert spaces with contractions mapping a Hilbert space $\mathcal{K}$ to a symmetric Hilbert space $\Gamma_F(\mathcal{K})$ with the same symmetry as the species F. A general framework for annihilation and creation operators on these spaces is developed, and compared to the generalised Brownian motions of R. Speicher and M. Bożejko. As a corollary we find that the commutation relation $a_i a_j^* - a_j^* a_i = f(N)\delta_{ij}$ with $Na_i^* - a_i^* N = a_i^*$ admits a realization on a symmetric Hilbert space whenever f has a power series with infinite radius of convergence and positive coefficients.

1 Introduction

Symmetric Hilbert spaces play a role in physics as the state spaces of many particle systems. The type of particle dictates the type of symmetrization:

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bosons require complete symmetrization and fermions complete antisymmetrization.
More general ways of of symmetrization, although apparently not realized in nature, have been studied for their own sake: parastatistics [13] and interpolations by a parameter $q \in [-1, 1]$ between the above two cases [7, 6, 19, 4, 12].
All these constructions lead to quantum fields or generalized Brownian motions, each with their own generalized Gauss distributions [2, 3, 4, 5, 12]. One particularly important case is $q = 0$: the free Brownian motion, exhibiting the Wigner distribution. This case is related to free independence in the same way as the case $q = 1$ of complete symmetrization is related to ordinary commutative independence.
Although there are results [16] indicating that these two are the only notions of independence, more relaxed conditions such as the weak factorization property [11], or pyramidal independence [3] are satisfied in a variety of examples.
In this paper we study combinatorial ways of symmetrization. Our starting point is the following observation. The category E of finite sets has as its isomorphy classes the natural numbers $\mathbb{N}$, and for each object U in class $n \in \mathbb{N}$ there are $n!$ symmetries. This leads to the Fock space

$$l^2\left(\mathbb{N}, \frac{1}{n!}\right) =: \Gamma_E(\mathbb{C}).$$

Taking for an annihilation operator a the left shift on this space, we find that the field operator $X := a + a^*$ has distribution given by ([14])

$$\langle \delta_\emptyset, X^n \delta_\emptyset \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{1}{2}x^2}\, dx.$$

On the other hand, if we consider the category L of finite sequences (or linear orderings on a set), we obtain

$$\Gamma_L(\mathbb{C}) = l^2(\mathbb{N})$$

and, since a^* is now the right shift [18],

$$\langle \delta_\emptyset, X^n \delta_\emptyset \rangle = \frac{1}{\pi} \int_{-2}^{2} x^n \sqrt{4 - x^2}\, dx\ .$$

We conclude that the Gauss and Wigner distributions are produced by the concepts of 'set' and 'sequence'. Our program in this paper is to generalize the Fock space construction to such combinatorial concepts as 'tree', 'graph' and 'cycle'.

The proper framework for this undertaking turns out to be Joyal's notion of a *combinatorial species of structures* [9]. These are defined as functors from the category of finite sets with bijections to the category of finite sets with maps. Combinatorial species of structures can be viewed as coefficients of the Taylor expansion of analytic functors [10] and lead to Joyal's notion of a tensorial species, very close to the our $\Gamma_F(\mathcal{K})$. This circle of ideas is introduced in Sections 2 and 3.

A natural way to introduce an annihilation operator into this context is via the operation of removal of a point from a structure, which is called *differentiation* $F \mapsto F'$ by Joyal. We thus arrive at operators

$$a(k) : \Gamma_F(\mathcal{K}) \to \Gamma_{F'}(\mathcal{K}), \quad a^*(k) : \Gamma_{F'}(\mathcal{K}) \to \Gamma_F(\mathcal{K}), \quad (k \in \mathcal{K})$$

However, these operators can only be added, in order to yield field operators, if the species F and F' are the same. This holds in two cases: the species E of sets and the species $E_{\pm}$ of oriented sets, related to the Bose and Fermi symmetries. Natural as this may be, we cannot move any further if we do not modify the operation of removal of a point in some way.

Now in fact, already in the case of sequences it is required that *the last* point is removed. In the same way, we may require in the case of trees that only leaves may be picked off (points that leave the tree connected when removed). In the case of cycles we may require that the chain, coming from a broken, cycle must be connected up again. All of this leads to the study of suitable transformations between F and F', which are the subject of Section 4.

Our approach to symmetric Hilbert spaces and field operators provides a tool for creating new examples, and is particularly transparent due to the use of combinatorial objects which are easy to visualize. The two examples of "q-deformations" appearing in [3] and [4] are cast in the form of combinatorial Fock spaces for the species of ballots and the species of simple directed graphs respectively. In section 5 we point out the connection between this combinatorial approach and the one based on positive definite functions on pair partitions [3]. We describe how the operations between species can be extended to the weights, illustrating this by examples.

2 Species of Structures

This section is a brief introduction to the combinatorial theory of species of structures [1], [9], insofar as is needed here.

We are concerned with the different kinds — or 'species' — of structures that can be imposed on a set U. The basic idea is that such a species is characterized by the way it transforms under permutations of the set U.

It will be convenient to consequently adhere to von Neumann's construction of the natural numbers according to which $0 = \emptyset$ and $n + 1 = n \cup \{n\}$, so that the number n coincides with the set $\{0, 1, 2, \ldots, n-1\}$.

Definition. [1] A *species of structures* is a rule F which

(i) produces for each finite set U a finite set $F[U]$,

(ii) produces for each bijection $\sigma : U \to V$ a function $F[\sigma] : F[U] \to F[V]$.

The function $F[\sigma]$ should have the following functorial properties:

(a) for all bijections $\sigma : U \to V$, $\tau : V \to W$, we have $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$,

(b) for the identity map $\mathrm{Id}_U : U \to U$, $F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$.

The elements of $F[U]$ are called *F-structures* on U and the function $F[\sigma]$ describes the transport of F-structures along σ. Note that $F[\sigma]$ is a bijection by the functorial property of F.

We denote by H_s the stabilizer $\{\sigma \in \mathrm{S}(U) : F[\sigma](s) = s\}$ of the structure $s \in F[U]$.

Examples.

1. The species E of sets is given by

$$\begin{aligned} E[U] &= \{U\}. \\ E[\sigma] &: \quad U \mapsto V \quad \text{if} \quad \sigma : U \to V. \end{aligned}$$

Thus the only E-structure over U is the set U itself. The stabilizer of this structure coincides with the whole permutation group $H_s = \mathrm{S}(U)$.

2. The species L of linear orderings:

$$L[U] = \{f : |U| \to U \,;\, f \text{ bijective}\}$$

where $|U| = \{0, 1, 2, \ldots, |U|-1\}$ is the cardinality of U. The transport along the bijection $\sigma : U \to V$ is given by

$$L[\sigma](f) = \sigma \circ f.$$

The stabilizer of each linear ordering is trivial. The cardinality of the set of structures $L[U]$ is equal to that of the permutation group $\mathrm{S}(U)$.

3. The species $\mathcal{C}$ of cyclic permutations:

$$\begin{aligned} \mathcal{C}[U] &= \{\pi \in \mathrm{S}_U \mid \pi^k(u) \neq u \text{ for all } u \in U, k < |U|\}; \\ \mathcal{C}[\sigma] : \pi &\mapsto \sigma \circ \pi \circ \sigma^{-1} \ . \end{aligned}$$

Each structure $\pi \in \mathcal{C}[U]$ has a nontrivial stabilizer $H_\pi = \{\pi^k \mid k < |U|\}$, the number of structures is

$$|\mathcal{C}[U]| = \frac{|U|!}{|H_\sigma|} = (|U| - 1)!$$

Definition. A species of structures F is called *molecular* if the permutation group acts transitively on its structures. A molecular species can be characterized by the conjugacy class of the stabilizer of any of its structures. Indeed for $s, t \in F[U]$ and $s = F[\sigma](t)$ we have $H_s = \sigma \circ H_t \circ \sigma^{-1}$. By a well-known combinatorial lemma we have for each structure s:

$$|F[U]| \cdot |H_s| = |U|! \quad \text{for} \quad s \in F[U]$$

In general a species of structure may not be molecular, in which case it is a sum of species:

Definition. Let F, G be species of structures. Then their *sum* $F + G$ is the species defined by the disjoint union

$$(F + G)[U] = F[U] \cup G[U],$$

and the transport along the bijection $\sigma : U \to V$ is given by:

$$(F + G)[\sigma](s) = \begin{cases} F[\sigma](s) & \text{if } s \in F[U] \\ G[\sigma](s) & \text{if } s \in G[U] \end{cases}$$

The *canonical decomposition* of a species F is its decomposition as a sum $F = F_0 + F_1 + F_2 + \cdots$ where F_n denotes the $n - th$ *level* of F:

$$F_n[U] = \begin{cases} F[U] & \text{if } |U| = n \\ \emptyset & \text{if } |U| \neq n \end{cases}$$

The simplest species having structures at only one level is the species of singletons X:

$$X[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Besides addition, there is a number of other operations between species by which to construct new species out of simpler ones. Following a standard

notation [1], we use the sum symbol to denote disjoint reunion.
Definition. Let F, G be two species of structures. We define the *product* species $F \cdot G$ as:

$$(F \cdot G)[U] = \sum_{(U_1,U_2)} F[U_1] \times G[U_2]$$

where the sum runs over all partitions of the set U into disjoint parts U_1 and U_2. The transport along the bijection $\sigma : U \to V$ of the structure $s = (f, g) \in (F \cdot G)[U]$ is:

$$(F \cdot G)[\sigma](s) = (F[\sigma_1](f), G[\sigma_2](g))$$

where $f \in F[U_1]$, $g \in G[U_2]$ and σ_1, σ_2 are the restrictions of σ to the sets U_1 and U_2 respectively.
The stabilizer of $s = (f, g)$ is $H_{(f,g)} = H_f \cdot H_g \subset \mathrm{S}(U_1) \cdot \mathrm{S}(U_2) \subset \mathrm{S}(U_1 + U_2)$.
As an example let us consider the n-th power of the species X of singletons:

$$X^n[U] = \begin{cases} \{(u_1, .., u_n) | u_i \in U,\ u_i \neq u_j \text{ for } i \neq j\} & \text{if } |U| = n \\ \emptyset & \text{otherwise} \end{cases}$$

It is clear that the species X^n and L_n are essentially the same. Indeed there exists a natural bijection between $X^n[U]$ and $L_n[U]$:

$$(u_1, .., u_n) \mapsto \ (u : n \to U : i \mapsto u_i) \ .$$

Remark: In the language of category theory, a species of structures F is a *functor* from the category $\mathbb{B}$ of finite sets with bijections to the category $\mathbb{E}$ of finite sets with functions.

Definition. A *morphism* from the species of structures F to the species G is a natural transformation of functors, that is a family of functions m_U : $F[U] \to G[U]$ such that:

$$G[\sigma] \circ m_U = m_V \circ F[\sigma] \quad \text{for all} \quad \sigma : U \to V.$$

An *isomorphism* is an invertible morphism.

Definition. The *cartesian product* $F \times G$ of two species of structures F and G is given by:

$$\begin{aligned} (F \times G)[U] &= F[U] \times G[U] \qquad (1) \\ (F \times G)[\sigma](f, g) &= (F[\sigma](f), G[\sigma](g)) \end{aligned}$$

The canonical decomposition of the cartesian product is:

$$F \times G = \sum_{n=0}^{\infty} F_n \times G_n. \tag{2}$$

A structure $(f, g) \in (F \times G)[U]$ has the stabilizer $H_{(f,g)} = H_f \cap H_g \subset \mathrm{S}(U)$.

An operation which will play an important role later is the derivation.

Definition. The *derivative* F' of a species F is a species whose set of structures over a finite set U is given by:

$$F'[U] = F[U \cup \{U\}]$$

and $F'[\sigma](s) = F[\sigma^+](s)$ where $\sigma^+ : U \cup \{U\} \to V \cup \{V\}$ is the extension of $\sigma : U \to V$:

$$\sigma^+(x) = \begin{cases} \sigma(x) & \text{if } x \in U, \\ V & \text{if } x = U. \end{cases}$$

Remark: The term $\{U\}$ in $U \cup \{U\}$ is just any additional point, not belonging to U. In particular for $U = n$ we have $U \cup \{U\} = n + 1$. If no confusion arises, we may write $U \cup \{U\}$ as $U + \{*\}$. The transport along bijections is the one inherited from the species F but it is restricted to those transformations that keep the point $*$ fixed. The stabilizer of a structure s when considered as a $F'-$structure is different from its stabilizer as a $F-$structure:

$$s \in F'[U] = F[U + \{*\}] \;\Rightarrow\; H_s^{F'} = H_s^F \cap \mathrm{S}(U).$$

As explained in the introduction, we wish to compare successive levels of a species, i.e. to compare F with F'. In this direction there is a small

Lemma 2.1 *There are only two species (up to multiplicity) which satisfy $F = F'$.*

Proof. Clearly, the species F must have the same number of structures at all levels. For $s \in F[n]$, the stabiliser H_s satisfies $|H_s| \geq \frac{n!}{|F[n]|}$ which for n big enough, reduces the possibilities to either the whole symmetric group $\mathrm{S}(n)$ or the subgroup $A(n)$ of even permutations. In the first case we obtain the species E of sets which has only one structure at each level, in the second we have the species $E^{\pm}$ of *oriented sets* with exactly two structures at each level

$$E^{\pm}[U] = \{U_+,\ U_-\}$$

the stabiliser of each structure being $H_{U_\pm} = A(U)$.

□

Besides these two ideal cases, we are interested in species F whose structure at successive levels "resemble" each other. That means that $F_n[U]$ and $F_{n+1}[U+\{*\}]$ should contain structures that behave similarly under permutations of U. Suppose that we are given a morphism m from a subspecies F_1 of F' to F $(F' = F_1 + F_2)$. Then the F-structures which belong to the image of this morphism are similar to their preimages in the sense that their stabilizers contain those of their preimages. The action of the morphism m can be encoded in a weight on the species $F \times F'$.

Definition. A *weighted species* (F, ω) consists of

1. a species of structures F
2. a family of functions $\omega_U : F[U] \to \mathbb{C}$ called *weights*,

such that for a bijection $\sigma : U \to V$ one has $\omega_V \circ F[\sigma] = \omega_U$.
The weight ω_m associated to the morphism $m : F_1 \to F$ is the indicator function of its graph:

$$\omega_{m,U}(f,g) = \begin{cases} \delta_{f,m(g)} & \text{if } g \in F_1[U]\ ,\ f \in F[U], \\ 0 & \text{if } g \notin F_1[U]. \end{cases}$$

One of the most interesting operations between species is the composition.

Definition. Let F and G two species of structures such that $G[\emptyset] = \emptyset$. The *composition* $F \circ G$ is a species whose structures on a set U are made in the following way:

1. make a partition π of the set U;
2. choose an F-structure over the set π: $f \in F[\pi]$;
3. for each $p \in \pi$ choose a structure $g_p \in G[p]$. Then the triple $(\pi, f, (g_p)_{p\in\pi})$ is a structure in $F \circ G[U]$. The transport along $\sigma : U \to V$ is the natural one.

In brief, an $F \circ G$ structure is an F-assembly of G-structures. As an example consider the following combinatorial equation.

$$\mathcal{A} = X \cdot E(\mathcal{A}) \tag{3}$$

This equation implicitly defines the species $\mathcal{A}$ of rooted trees. Here is an explicit definition:

$$\begin{array}{rcl}\mathcal{A}[U] & = & \{f : U \to U \mid \forall_{u \in U} : f^{\circ k}(u) \text{ is eventually constant}\}; \\ \mathcal{A}[\sigma] & : & f \mapsto \sigma \circ f \circ \sigma^{-1}.\end{array}$$

The constant is the root of the tree. The preimage of the root consists of roots of subtrees. One can thus consider the tree f as the pair $(\text{root}(f)\ ,\ \{f_a \mid a \in f^{-1}(\text{root}(f))\})$ with $f_a \in \mathcal{A}[U_a]$ the subtree of f with root a:

$$U_a = \{u \in U \mid \exists k \in \mathbb{N} \text{ such that } f^{\circ k} = a\} \quad , \quad f_a = f\restriction_{U_a}$$

We finally note

$$U = \{\text{root}(f)\} \cup \bigcup_{a \in f^{-1}(\text{root}(f))} U_a$$

thus completing the bijection between $\mathcal{A}[U]$ and $X \cdot E(\mathcal{A})[U]$.

3 Fock Spaces and Analytic Functors

In this section we will describe how one can associate to a species of structures an endofunctor of the category of Hilbert spaces with contractions. We call the images of this functor symmetric spaces associated to the species F and as we shall see in the following sections, they are suitable for constructing algebras of creation and annihilation operators, by exploiting the symmetry properties of the species F.

Following Joyal [10] we define a special class of endofunctors of the category of sets with maps.

Definition. Let $F[\cdot]$ be a species of structures. The *analytic functor* $F(\cdot)$ is an endofunctor of the category **Set** of sets with maps, defined by:

$$F(J) = \tilde{\sum_U} F[U] \times J^U \tag{4}$$

where $J^U = \{c | c : U \to J\}$ and the symbol $\tilde{\sum}_U$ means the set of equivalence classes under bijective transformations:

$$F[U] \times J^U \ni (s, c) \mapsto (F[\sigma](s), c \circ \sigma^{-1}) \in F[V] \times J^V$$

for $\sigma : U \to V$. We call the elements of J "colors". Thus, an element in $F(J)$ is an orbit of J-colored F-structures denoted by $[s, c]$. Alternatively

$$F(J) = \tilde{\sum_U} F[U] \times J^U = \sum_{n=0}^{\infty} F[n] \times J^n \ /\ \mathrm{S}(n).$$

Remark. This relation can be viewed as a Taylor expansion of the set $F(J)$, which explains the name "analytic functor" for $F(\cdot)$ [10].

Parallel to the functor $F(\cdot)$ we define another endofunctor, this time on the category **Hilb** of Hilbert spaces with contractions. For any Hilbert space $\mathcal{H}$ and a finite set U we denote by $\mathcal{H}^{\otimes U}$ the Hilbert space arising from the positive definite kernel on $\mathcal{H}^U$ given by

$$k\left(\bigotimes_{u\in U}\psi_u, \bigotimes_{u'\in U}\varphi_{u'}\right) = \prod_{u\in U}\langle\psi_u,\varphi_u\rangle\,.$$

For every bijection $\sigma : U \to V$ there is a unitary transformation $U(\sigma) : \mathcal{H}^{\otimes U} \to \mathcal{H}^{\otimes V}$ obtained by linear extension of:

$$U(\sigma) : \bigotimes_{u\in U}\psi_u \to \bigotimes_{v\in V}\psi_{\sigma^{-1}(v)}.$$

Definition. Let F be a species of structures. For each Hilbert space $\mathcal{K}$ we construct the *symmetric Hilbert space*

$$\Gamma_F(\mathcal{K}) := \bigoplus_{n=0}^{\infty}\frac{1}{n!}\;\ell^2_{\text{symm}}(F[n]\to\mathcal{K}^{\otimes n}) \tag{5}$$

where the subscript "symm" denotes the invariance under the natural action of the symmetric group $\mathrm{S}(n)$:

$$\Psi \mapsto U(\sigma)\Psi\circ F[\sigma^{-1}].$$

The factor $\frac{1}{n!}$ refers to the inner product on ℓ^2_{symm}.

Remark. There is an equivalent way of writing $\Gamma_F(\mathcal{K})$:

$$\Gamma_F(\mathcal{K}) = \bigoplus_{n=0}^{\infty}\frac{1}{n!}\;\ell^2(F[n])\otimes_{\mathrm{S}(n)}\mathcal{K}^{\otimes n} \tag{6}$$

where $\otimes_{\mathrm{S}(n)}$ means that we consider only the subspace of the tensor product whose vectors are invariant under the action of $\mathrm{S}(n)$.
Let us choose an orthonormal basis $(e_j)_{j\in J}$ for the Hilbert space $\mathcal{K}$. Let $(e_c)_{c\in J^n}$ be the basis of $\mathcal{K}^{\otimes n}$ given by $e_c := \otimes_{j\in n}e_{c(j)}$, and

$$\begin{aligned}\gamma_{F,J} : F(J) &\;\to\; [0,\infty)\\ \gamma_{F,J}([s,c]) &\;=\; |H_{(s,c)}|\end{aligned}$$

where $[s,c]$ denotes the orbit of the colored structure (s,c).

Lemma 3.1 *There is a unitary equivalence between* $\ell^2(F(J),\gamma_{F,J})$ *and* $\Gamma_F(\mathcal{K})$, *given by*

$$U\delta_{[s,c]} = \delta_s \otimes_{\mathrm{S}(n)} e_c := \sum_{\sigma\in \mathrm{S}(n)} \delta_{F[\sigma](s)} \otimes e_{c\circ\sigma^{-1}}$$

Proof. Considering $\mathcal{K}^{\otimes n}$ as $\ell^2(J^n)$ we may write

$$\begin{aligned} U\delta_{[s,c]} &= \sum_{\sigma\in \mathrm{S}(n)} \delta_{F[\sigma](s)} \otimes e_{c\circ\sigma^{-1}} \\ &= \sum_{\sigma\in \mathrm{S}(n)} \delta_{F[\sigma](s),c\circ\sigma^{-1}} = |H_{(s,c)}| \cdot \mathbf{1}_{[s,c]} \end{aligned}$$

It follows that

$$\begin{aligned} \|U\delta_{[s,c]}\|^2 &= \frac{1}{n!}|H_{(s,c)}|^2 \cdot |[s,c]| = |H_{(s,c)}| \\ &= \|\delta_{[s,c]}\|^2. \end{aligned}$$

Since the functions $\mathbf{1}_{[s,c]}$ span the space $\ell^2_{\mathrm{symm}}(F[n]\times J^n)$, the operator U is surjective and hence unitary.

□

Remark. For a constant coloring c we have $\|\delta_{[s,c]}\|^2 = |H_s|$, whereas for all colors different, $\|\delta_{[s,c]}\|^2 = 1$.

Certain operations with species of structures extend to analytic species [10] and to the symmetric spaces: addition, multiplication and substitution.

1) Addition. As $(F+G)(J)$ is the disjoint union of $F(J)$ and $G(J)$, we have

$$\Gamma_{F+G}(\mathcal{K}) = \Gamma_F(\mathcal{K}) \oplus \Gamma_G(\mathcal{K}).$$

2) Multiplication. Similarly, we have

$$\begin{aligned} &(F\cdot G)(J) = \tilde{\sum_U}\Big(\sum_{U_1+U_2=U} F[U_1]\times G[U_2]\Big)\times J^U \\ &= \tilde{\sum_U}\sum_{U_1+U_2=U} F[U_1]\times G[U_2]\times J^{U_1+U_2} = \tilde{\sum_{U_1,U_2}} F[U_1]\times G[U_2]\times J^{U_1+U_2} \\ &= F(J)\times G(J). \end{aligned}$$

which suggests the following unitary transformation from $\Gamma_{F\cdot G}(\mathcal{K})$ to $\Gamma_F(\mathcal{K})\otimes\Gamma_G(\mathcal{K})$:

$$T : \delta_{[(f,g),c]} \to \delta_{[f,c_1]} \otimes \delta_{[g,c_2]}$$

for $f \in F[n]$, $g \in G[m]$, $c \in J^{m+n}$ and c_1, c_2 the restrictions of c to n respectively m. Indeed the map preserves orthogonality and is isometric:

$$\begin{aligned} \|\delta_{[(f,g),c]}\|^2 &= |H_{((f,g),c)}| = |H_{(f,c_1)} \cdot H_{(g,c_2)}| = |H_{(f,c_1)}| \cdot |H_{(g,c_2)}| \\ &= \|\delta_{[f,c_1]}\|^2 \cdot \|\delta_{[g,c_2]}\|^2 \end{aligned}$$

From now on we will consider $\Gamma_{F \cdot G}(\mathcal{K})$ and $\Gamma_F(\mathcal{K}) \otimes \Gamma_G(\mathcal{K})$ as identical, without mentioning the unitary T.

3) Substitution. we start with the analytic functors:

$$\begin{aligned} (F \circ G)(J) &= \tilde{\sum_U}(F \circ G)[U] \times J^U \\ = \tilde{\sum_U}(\tilde{\sum_\pi} F[\pi] \times G^\pi[U]) \times J^U &= \tilde{\sum_\pi}(F[\pi] \times \tilde{\sum_U} G^\pi[U]) \times J^U \\ = \tilde{\sum_\pi} F[\pi] \times G^\pi(J) &= F(G(J)) \end{aligned}$$

where we have used $G^\pi(J) = G(J)^\pi$, which follows from the multiplication property. At the level of symmetric spaces we have the unitary transformation from $\Gamma_F(\Gamma_G(\mathcal{K}))$ to $\Gamma_{F \circ G}(\mathcal{K})$:

$$T : \delta_{[f,C]} \to \delta_{[f,(g_a)_{a \in \pi},c]}$$

with the following relations for the structures appearing above: $f \in F[\pi]$, $C : \pi \to G(J)$ such that $C(a) = [g_a, c_a]$, and $c \restriction_a = c_a$. Let us check the isometric property:

$$\begin{aligned} \|\delta_{[f,C]}\|^2 &= \prod_{a \in \pi} \|\delta_{C(a)}\|^2 \cdot |H_{(f,C)}| = \prod_{a \in \pi} \|\delta_{[g_a,c_a]}\|^2 \cdot |H_{f,C}| \\ &= \prod_{a \in \pi} |H_{(g_a,c_a)}| \cdot |H_{f,C}| = \|\delta_{[f,(g_a)_{a \in \pi},c]}\|^2 \end{aligned}$$

Symmetric Fock space. The symmetric Hilbert space associated to the species of sets is the well known symmetric Fock space:

$$\Gamma_E(\mathcal{K}) = \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ell^2_{\text{symm}}(E[n] \to \mathcal{K}^{\otimes n}) = \bigoplus_{n=0}^{\infty} \frac{1}{n!} \mathcal{K}^{\otimes_s n}.$$

Full Fock space. For the linear orders we obtain the full Fock space:

$$\Gamma_L(\mathcal{K}) = \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ell^2_{\text{symm}}(L[n] \to \mathcal{K}^{\otimes n}) = \bigoplus_{n=0}^{\infty} \mathcal{K}^{\otimes n}.$$

Antisymmetric Fock space. We recall from lemma 2.1 that the species $E^{\pm}$ of oriented sets has two structures at all levels

$$E^{\pm}[U] = \{U_+,\ U_-\}$$

which are mapped into each other by odd permutations and have as stabiliser the group $A(U)$ of even permutation. The representation of $\mathrm{S}(n)$ on $\ell^2(E^{\pm}[n])$ contains two one-dimensional irreducible sub-representations, the symmetric and the antisymmetric representation. Accordingly the symmetric Hilbert space associated to $E^{\pm}$ is the direct sum of the symmetric and antisymmetric Fock spaces:

$$\Gamma_{E^{\pm}}(\mathcal{K}) = \Gamma_s(\mathcal{K}) \oplus \Gamma_a(\mathcal{K}).$$

Remark. The set of species as defined in the previous section can be enlarged by defining [1] the *virtual species* as equivalence classes of pair of species of structures under the equivalence relation:

$$(F_1, G_1) \sim (F_2, G_2) \Leftrightarrow F_1 + G_2 = F_2 + G_1$$

One denotes the equivalence class of (F, G) by $F - G$. Thus we can say that the antisymmetric Fock space is associated to the virtual species $E^{\pm} - E$.

4 Creation and Annihilation Operators

In this section we will describe a general framework for constructing *-algebras of operators on symmetric Hilbert spaces by giving the action of the generators of these algebras, the creation and annihilation operators. In particular in the case of the species of sets E and linear orderings L, we obtain the well known canonical commutation relations algebra (C.C.R.), respectively the algebra of creation/annihilation operators on the full Fock space.
The starting point is the observation that the operation of derivation of species of structures can be interpreted as removal of point $*$ from a structure. This makes it possible to define operators between the symmetric Hilbert spaces of a species of structure F and its derivative F'.
We will consider now "colored" F-structures. Let J be the set of "colors" and $i \in J$. We have the map

$$a^*(i) : F'[U] \times J^U \to F[U + \{*\}] \times J^{U+\{*\}}$$

such that

$$a^*(i) : (s, c) \to (s, c_i^+)$$

where $c_i^+ : U + \{*\} \to J$ is given by:

$$c_i^+(u) = \begin{cases} c(u) & \text{if } u \in U \\ i & \text{if } u = * \end{cases}$$

As we did in section 3, we pass to the set of orbits of J-colored F-structures. The map $a^*(i)$ projects to a well defined map from $F'(J)$ to $F(J)$:

$$\begin{aligned} a^*(i) : F'[U] \times J^U/\mathrm{S}(U) &\to F[U + \{*\}] \times J^{U+\{*\}}/\mathrm{S}(U + \{*\}) \\ a^*(i) : [s, c] &\to [s, c_i^+] \end{aligned}$$

But as $F(J)$ determines an orthogonal basis of the space $\Gamma_F(\mathcal{K})$ for $(e_j)_{j\in J}$ orthogonal basis in $\mathcal{K}$, we can extend $a^*(i)$ by linearity to an operator

$$a^*(i) : \Gamma_{F'}(\mathcal{K}) \to \Gamma_F(\mathcal{K}).$$

The adjoint of $a^*(i)$ acts in the opposite direction:

$$a(i) : \Gamma_F(\mathcal{K}) \to \Gamma_{F'}(\mathcal{K}).$$

The problem with this definition is that in general the species F and F' are distinct which means that one cannot take the "field operators" $a^*(i) + a(i)$ and only certain products of creation and annihilation operators are well defined. In section 2 we pointed out that the "similarity" of the structures of the species F and F' can be encoded in a weight on the cartesian product $F \times F'$. Let ω be such a weight. Then $\omega_U : F[U] \times F'[U] \to \mathbb{C}$ such that for all $s \in F[U],\ t \in F'[U]$ and $\sigma : U \to W$ we have:

$$\omega(s, t) = \omega(F[\sigma](s), F'[\sigma](t))$$

We will use this to define creation and annihilation operators which act on the same space $\Gamma_F(\mathcal{K})$. In the sequel we will refer to the pair $(\Gamma_F(\mathcal{K})\ ,\ \omega)$ as *combinatorial Fock space.*

Definition. a) The *annihilation operator* (before symmetrization) associated to the species F and weight ω is defined by:

$$\tilde{a}(h) : \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ell^2(F[n] \to \mathcal{K}^{\otimes n}) \to \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ell^2(F[n] \to \mathcal{K}^{\otimes n})$$

$$(\tilde{a}(h)\varphi)(f) = \sum_{g\in F[n+1]} \omega(f, g) \cdot \mathrm{inp}_n(h, \varphi(g))$$

where $f \in F[n]$, $h \in \mathcal{K}$ and $\text{inp}_k(h, \cdot)$ is the operator:

$$\text{inp}_k(h, \psi_0 \otimes .. \otimes \psi_n) = \langle h, \psi_k \rangle \, \psi_0 \otimes .. \otimes \psi_{k-1} \otimes \psi_{k+1} \otimes .. \otimes \psi_n$$

for $k \in \{0, 1, .., n\}$.
b) The *creation operator* (before symmetrization) is:

$$\tilde{a}^*(h) : \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ell^2(F[n] \to \mathcal{K}^{\otimes n}) \to \bigoplus_{n=0}^{\infty} \frac{1}{n!} \ell^2(F[n] \to \mathcal{K}^{\otimes n})$$

$$(\tilde{a}^*(h)\varphi)(f) = (n+1) \cdot \sum_{g \in F[n]} \overline{\omega(g, f)} \cdot \text{tens}_n(h, \varphi(g))$$

where $f \in F[n+1]$, $h \in \mathcal{K}$ and $\text{tens}_k(h, \cdot)$ is the operator:

$$\text{tens}_k(h, \psi_0 \otimes .. \otimes \psi_{n-1}) = \psi_0 \otimes .. \otimes \psi_{k-1} \otimes h \otimes \psi_k \otimes .. \otimes \psi_{n-1}$$

for $k \in \{0, 1, .., n\}$.
Remark. In order to avoid domain problems for $\tilde{a}, \tilde{a}^*$, we will restrict to weights which are bounded, $|\omega(t, s)| \leq C$ for all t, s. Then

$$\|a(h)\psi_n\| \leq n^{\frac{1}{2}} C \|h\| \|\psi_n\| \qquad \|a^*(h)\psi_n\| \leq (n+1)^{\frac{1}{2}} C \|h\| \|\psi_n\|$$

for $\psi_n \in \ell^2(F[n] \to \mathcal{K}^{\otimes n})$ thus $a(h), a^*(h)$ have well defined extensions to the domain $D(N^{\frac{1}{2}})$ $(N\psi_n = n\psi_n)$. As this will not play a major role here, we will omit specifying the domain, usually the the vectors considered should belong to $D(N^{\frac{1}{2}})$.
We consider now the symmetrized creation and annihilation operators which act on the symmetric Hilbert space and which are the main object of our investigation.

Lemma 4.1 *The unsymmetrized annihilation operator $\tilde{a}(h)$ restricts to a well defined operator $a(h)$ on the symmetric Hilbert space $\Gamma_F(\mathcal{K})$:*

Proof. Let $\varphi \in \Gamma_F(\mathcal{K})$. Then $\varphi(F[\sigma](f)) = U(\sigma)\varphi(f)$ for all $\sigma \in \mathrm{S}(n)$, $f \in F[n]$ and

$$\begin{aligned} (a(h)\varphi)(F[\sigma](f)) &= \sum_g \omega(F[\sigma](f), g) \cdot \text{inp}_n(h, \varphi(g)) \\ = \sum_{g'} \omega(f, g') \cdot \text{inp}_n(h, \varphi(F[\tilde{\sigma}]g')) &= \sum_{g'} \omega(f, g') \cdot U(\sigma)\text{inp}_n(h, \varphi(g')) \\ &= U(\sigma)(a(h)\varphi)(f) \end{aligned}$$

where $\tilde{\sigma} : n+1 \to n+1$ is given by

$$\tilde{\sigma}(i) = \begin{cases} \sigma(i) & \text{if } i \in n \\ n & \text{if } i = n \end{cases}$$

□

Lemma 4.2 *The operator $\tilde{a}^*(h)$ is the adjoint of $\tilde{a}(h)$ on the unsymmetrized space $\bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$.*

Proof. Let φ, ψ be two vectors in $\bigoplus_{n=0}^{\infty} \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$ and $\varphi_n, \psi_n \in \frac{1}{n!}\ell^2(F[n] \to \mathcal{K}^{\otimes n})$ their components on level n.
Then we have:

$$\begin{aligned} \langle \psi, \tilde{a}^*(h)\varphi \rangle &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{g} (n+1) \langle \psi_{n+1}(g), (\tilde{a}^*(h)\varphi_n)(g) \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{f,g} \left\langle \psi_{n+1}(g), \overline{\omega(f,g)} \cdot \varphi_n(f) \otimes h \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{f,g} \langle \omega(f,g) \cdot \mathrm{inp}_n(h, \psi_{n+1}(g)), \varphi_n(f) \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{f} \langle (\tilde{a}(h)\psi_{n+1})(f), \varphi_n(f) \rangle = \langle \tilde{a}(h)\psi, \varphi \rangle \end{aligned}$$

□

We will restrict our attention to the action of the creation and annihilation operators on the symmetric Hilbert space $\Gamma_F(\mathcal{K})$. From Lemma 4.1 the annihilation operator $a(h)$ is well defined on $\Gamma_F(\mathcal{K})$. We call its adjoint on the symmetric Hilbert space, the symmetrized creation operator. If P is the projection to $\Gamma_F(\mathcal{K})$ then the symmetrized creation operator is:

$$a^*(h)\varphi = P\tilde{a}^*(h)\varphi \text{ for } \varphi \in \Gamma_F(\mathcal{K})$$

Lemma 4.3 *Let $f \in F[n+1]$ and $\tau_{n,k} \in \mathrm{S}(n+1)$ the transposition of n and k. Then for any $\varphi \in \Gamma_F(\mathcal{K})$ the action of the symmetrized creation operator has the expression:*

$$(a^*(h)\varphi)(f) = \sum_{k=0}^{n} \sum_{g} \overline{\omega(g, F[\tau_{n,k}](f))} \cdot U(\tau_{n,k})(\varphi(g) \otimes h).$$

Proof. We have:

$$(P\tilde{a}^*(h)\varphi)(f) = \frac{1}{(n+1)!}\sum_{\sigma\in \mathrm{S}(n+1)} U(\sigma)(\tilde{a}^*(h)\varphi)(F[\sigma^{-1}]f)$$

$$= \frac{1}{n!}\sum_{\sigma\in \mathrm{S}(n+1)}\sum_{g}\overline{\omega(g, F[\sigma](f))}\cdot U(\sigma^{-1})(\varphi(g)\otimes h) \tag{7}$$

If $\sigma \in \mathrm{S}(n+1)$ and $\sigma^{-1}(n) = k$ then $\rho = \tau_{n,k}\circ\sigma^{-1} \in \mathrm{S}(n)$. Thus the sum over all permutations can be split into a sum over $k \in n+1$ and one over $\mathrm{S}(n)$. Moreover, from the definitions of $\Gamma_F(\mathcal{K})$ and that of a weight we know that

$$\begin{aligned} U(\rho)\varphi(g) &= \varphi(F[\rho](g)) \\ \omega(g, F[\rho^{-1}\circ\tau_{n,k}](f)) &= \omega(F[\rho](g), F[\tau_{n,k}](f)) \end{aligned}$$

which substituted into the sum (7) gives:

$$\frac{1}{n!}\sum_{k=0}^{n}\sum_{\rho\in \mathrm{S}(n)}\sum_{g}\overline{\omega(F[\rho](g), F[\tau_{n,k}](f))}\cdot U(\tau_{n,k})(\varphi(F[\rho](g))\otimes h)$$

$$= \sum_{k=0}^{n}\sum_{g'}\overline{\omega(g', F[\tau_{n,k}](f))}\cdot U(\tau_{n,k})(\varphi(g')\otimes h).$$

□

Sometimes algebras are defined by giving relations among generators as for example commutation relations. We will give next explicit formulas for the product of a creation and an annihilation operator.

Lemma 4.4 *Let $f \in F[n]$ and $\varphi \in \Gamma_F(\mathcal{K})$. Then*

$$(a^*(h_1)a(h_2)\varphi)(f) = \sum_{k=0}^{n-1}\sum_{f'}(\overline{\omega}\cdot\omega)_k(f, f')\cdot \mathit{tens}_k(h_1, \mathit{inp}_k(h_2, \varphi(f'))) \tag{8}$$

where we have made the notation

$$(\overline{\omega}\cdot\omega)_k(f, f') = \sum_{g}\overline{\omega(g, F[\tau_{n-1,k}](f))}\omega(g, F[\tau_{n-1,k}](f')) \tag{9}$$

Proof. By applying successively the definitions of $a^*(h_1)$ and $a(h_2)$ we have:

$$(a^*(h_1)a(h_2)\varphi)(f) = \sum_{k=0}^{n-1}\sum_{g}\overline{\omega(g, F[\tau_{n-1,k}](f))} \cdot U(\tau_{n-1,k})(a(h_2)\varphi)(g) \otimes h_1$$

$$= \sum_{k=0}^{n-1}\sum_{g,f'}\overline{\omega(g, F[\tau_{n-1,k}](f))}\omega(g, f') \cdot U(\tau_{n-1,k})(\mathrm{inp}_{n-1}(h_2, \varphi(f')) \otimes h_1)$$

$$= \sum_{k=0}^{n-1}\sum_{g,f'}\overline{\omega(g, F[\tau_{n-1,k}](f))}\omega(g, F[\tau_{n-1,k}]f') \cdot \mathrm{tens}_k(h_1, \mathrm{inp}_k(h_2, \varphi(f')))$$

$$= \sum_{k=0}^{n-1}\sum_{f'}(\overline{\omega} \cdot \omega)_k(f, f') \cdot \mathrm{tens}_k(h_1, \mathrm{inp}_k(h_2, \varphi(f'))).$$

□

Lemma 4.5 *Let $f \in F[n]$ and $\varphi \in \Gamma_F(\mathcal{K})$. Then:*

$$\begin{aligned}(a(h_1)a^*(h_2)\varphi)(f) \quad &= \quad \sum_{k=0}^{n-1}\sum_{f'}(\omega \cdot \overline{\omega})_k(f, f') \cdot \mathrm{tens}_k(h_2, \mathrm{inp}_k(h_1, \varphi(f'))) \\ &+ \quad \langle h_1, h_2\rangle \cdot \sum_{f'}(\omega \cdot \overline{\omega})_n(f, f')\varphi(f')\end{aligned}$$

where we have made the notation

$$(\omega \cdot \overline{\omega})_k(f, f') = \sum_{g}\omega(f, g)\overline{\omega(f', F[\tau_{n,k}](g))} \tag{10}$$

Proof. We use the definitions of $a(h_1)$ and $a^*(h_2)$:

$$(a(h_1)a^*(h_2)\varphi)(f) = \sum_{g}\omega(f, g) \cdot \mathrm{inp}_n(h_1, (a^*(h_2)\varphi)(g))$$

$$= \quad \sum_{g,f'}\sum_{k=0}^{n}\omega(f, g)\overline{\omega(f', F[\tau_{n,k}](g))} \cdot \mathrm{inp}_n(h_1, U(\tau_{n,k})(\varphi(f') \otimes h_2))$$

$$= \quad \sum_{k=0}^{n-1}\sum_{f'}(\omega \cdot \overline{\omega})_k(f, f') \cdot \mathrm{tens}_k(h_2, \mathrm{inp}_k(h_1, \varphi(f')))$$

$$+ \quad \langle h_1, h_2\rangle \cdot \sum_{f'}(\omega \cdot \overline{\omega})_n(f, f')\varphi(f')$$

□

4.1 Examples.

We will describe a few known operator algebras in the language developed so far and a new algebra based on the species $\mathcal{A}$ of rooted trees.

1) Sets: The combinatorial Fock space is (E, ω_E) with $E[U] = \{U\}$ and $\omega(\{U\}, \{U + \{*\}\}) = 1$. We use lemmas 4.4 and 4.5 to calculate the commutator of the creation and annihilation operator:

$$(a(h_1)a^*(h_2) - a^*(h_2)a(h_1))\varphi(f) = \langle h_1, h_2\rangle \cdot (\omega \cdot \overline{\omega})_n(f, f')\varphi(f')$$
$$+ \sum_{k=0}^{n-1}((\omega \cdot \overline{\omega})_k(f, f') - (\overline{\omega} \cdot \omega)_k(f, f')) \cdot \mathrm{tens}_k(h_2, \mathrm{inp}_k(h_1, \varphi(f'))$$

But $(\omega \cdot \overline{\omega})_k(f, f') = (\overline{\omega} \cdot \omega)_k(f, f') = (\omega \cdot \overline{\omega})_n(f, f') = \delta_{f,f'}$ for all $k \in n$ which implies the C.C.R.:

$$a(h_1)a^*(h_2) - a^*(h_2)a(h_1) = \langle h_1, h_2\rangle\, \mathbf{1}$$

In particular it is clear that $\Gamma_E(\mathcal{K})$ is the symmetric Fock space over the Hilbert space $\mathcal{K}$.

2) Linear Orders: Let (L, ω_L) be the combinatorial Fock space with

$$L[U] = \{f : U \to \{0, 1, .., |U| - 1\}\}$$

and

$$\omega_L(f, g) = \delta_{f,g\restriction_U} \text{ for } f \in L[U],\ g \in L[U + \{*\}]$$

where

$$\delta_{f,g\restriction_U} = \begin{cases} 1 & \text{if } f(u) = g(u) \text{ for } u \in U \\ 0 & \text{otherwise} \end{cases}$$

From (10) we have:

$$(\omega \cdot \overline{\omega})_k(f, f') = \sum_g \delta_{f,g\restriction_U} \cdot \delta_{f',L[\tau_{n,k}](g)\restriction_U} = \delta_{k,n} \cdot \delta_{f,f'}$$

Then by applying Lemma 4.5 we obtain

$$a(h_1)a^*(h_2) = \langle h_1, h_2\rangle\, \mathbf{1}$$

which characterizes the algebra of creation and annihilation operators on the full Fock space [18].

3) Oriented Sets: We refer to the previous sections for the definition of the species $E^{\pm}$ of oriented sets. The weight $\omega_{E^{\pm}}$ is given by

$$\begin{aligned} \omega_{E^{\pm}}(U_+, U_+^*) &= \omega_{E^{\pm}}(U_-, U_-^*) = 1, \\ \omega_{E^{\pm}}(U_+, U_-^*) &= \omega_{E^{\pm}}(U_-, U_+^*) = 0 \end{aligned} \tag{11}$$

where $U^* = U + \{*\}$. With the help of the "switching" sign operator

$$\mathbf{g}\varphi(\pm) = \varphi(\mp)$$

we obtain the $\mathbf{g}$-commutation relations:

$$a(h_1)a^*(h_2) - \mathbf{g}a^*(h_2)a(h_1) = \langle h_1, h_2 \rangle \, \mathbf{1} \tag{12}$$

As we saw in the previous section, the space $\Gamma_{E^{\pm}}(\mathcal{K})$ is isomorphic to the direct sum of the symmetric and antisymmetric Fock space over $\mathcal{K}$:

$$\Gamma_{E^{\pm}}(\mathcal{K}) = \Gamma_s(\mathcal{K}) \oplus \Gamma_a(\mathcal{K})$$

through the transformation:

$$\varphi_s = \varphi(+) + \varphi(-) \quad \text{and} \quad \varphi_a = \varphi(+) - \varphi(-)$$

then the $\mathbf{g}$-commutation relations can be written equivalently as:

$$(a(h_1)a^*(h_2) - a^*(h_2)a(h_1))\varphi_s = \langle h_1, h_2 \rangle \, \varphi_s$$

and

$$(a(h_1)a^*(h_2) + a^*(h_2)a(h_1))\varphi_a = \langle h_1, h_2 \rangle \, \varphi_a$$

4) Rooted Trees: We recall the definition of the species $\mathcal{A}$:

$$\mathcal{A}[U] = \{f : U \to U \mid f^{\circ k}(u) = \text{root}(f) \in U \text{ for } k \geq |U|, u \in U\}$$

with the transport along σ: $\mathcal{A}[\sigma](f) = \sigma \circ f \circ \sigma^{-1}$. We note that $\mathcal{A}[\emptyset] = \emptyset$. We consider a natural weight which can be described as follows: it takes value 1 on those pairs of trees for which the second is obtained by adding a leaf to the first one, and takes value 0 for the rest. Thus for $t_1 \in \mathcal{A}[U]$ and $t_2 \in \mathcal{A}[U + \{*\}]$ the weight is:

$$\omega_{\mathcal{A}}(t_1, t_2) = \left\{ \begin{array}{ll} 1 & \text{if } t_1(u) = t_2(u) \text{ for } u \in U \\ 0 & \text{otherwise} \end{array} \right\} := \delta_{t_1, t_2 \restriction_U}$$

We will compute the commutator of $a(h_2)$ with $a^*(h_1)$. For this we need to obtain the expressions of $(\overline{\omega} \cdot \omega)_k(\cdot, \cdot)$ and $(\omega \cdot \overline{\omega})_k(\cdot, \cdot)$. We start with

$$(\omega\cdot\overline{\omega})_n(f,f') = \sum_g \omega_{\mathcal{A}}(f,g)\cdot\overline{\omega_{\mathcal{A}}(f',g)} = \sum_g \delta_{f,g\restriction_n}\cdot\delta_{f',g\restriction_n}$$
$$= \delta_{f,f'}\sum_g \delta_{f,g\restriction_n} = n\delta_{f,f'} \tag{13}$$

The factor n appears because there are n possible way of attaching a leaf to the tree f each one giving a tree g such that $g\restriction_n = f$. For $k<n$ we have

$$(\omega\cdot\overline{\omega})_k(t,t') = \sum_g \omega_{\mathcal{A}}(t,g)\cdot\overline{\omega_{\mathcal{A}}(t',\mathcal{A}[\tau_{n,k}](g))} = \sum_g \delta_{t,g\restriction_n}\cdot\delta_{t',\mathcal{A}[\tau_{n,k}](g)\restriction_n}.$$

At most one term in this sum is different from zero, for the tree g satisfying:

$$\begin{cases} g(i)=t(i) & \text{if } i\in n\\ g(j)=t'(j) & \text{if } j\in n\setminus\{k\}\\ g(n)=t'(k) & \end{cases} \tag{14}$$

On the other hand

$$(\overline{\omega}\cdot\omega)_k(t,t') = \sum_{g'} \overline{\omega_{\mathcal{A}}(g',\mathcal{A}[\tau_{n-1,k}](t))}\cdot\omega_{\mathcal{A}}(g',\mathcal{A}[\tau_{n-1,k}](t'))$$

$$= \sum_g \delta_{g',\mathcal{A}[\tau_{n-1,k}](t)\restriction_{n-1}}\cdot\delta_{g',\mathcal{A}[\tau_{n-1,k}](t')\restriction_{n-1}} = \delta_{\mathcal{A}[\tau_{n-1,k}](t)\restriction_{n-1},\ \mathcal{A}[\tau_{n-1,k}](t')\restriction_{n-1}}$$

$$= \begin{cases} 1 & \text{if } t(i)=t'(i) \text{ for all } i\in n, i\neq k\\ 0 & \text{otherwise}\end{cases} \tag{15}$$

Finally from (14), (15) we conclude that $(\overline{\omega}\cdot\omega)_k(t,t') = (\omega\cdot\overline{\omega})_k(t,t')$ for $k\in\{0,1,..n-1\}$.
Let us define the "vertex number" operator N by

$$(N\varphi)(t) = n\varphi(t)$$

for $t\in\mathcal{A}[n]$. The usual commutation relations between N and the creation operator hold:

$$[N,a^*(h)] = a^*(h)$$

By using Lemmas 4.4 and 4.5 we obtain the following:

Theorem 4.1 *The following commutation relations hold on the combinatorial Fock space $(\mathcal{A},\omega_{\mathcal{A}})$:*

$$a(h_1)a^*(h_2) - a^*(h_2)a(h_1) = N\langle h_1,h_2\rangle \tag{16}$$

Remark: Notice that the vacuum of $\Gamma_{\mathcal{A}}(\mathcal{K})$ is an eigenvector of N with eigenvalue 1.

5) Simple Directed Graphs: Let us define a species whose structures are directed graphs for which any pair of vertices is connected by at most one edge:

$$\mathcal{D}_s[U] = \{g \in U \times U \mid (u,v) \in g \Rightarrow (v,u) \notin g\} \tag{17}$$

where the transport along σ is given by $\sigma \times \sigma$.
Let $g_1 \in \mathcal{D}_s[U]$ and $g_2 \in \mathcal{D}_s[U + \{*\}]$. Then $\omega(g_1, g_2) \neq 0$ if and only if g_2 contains g_1 as a subset and all edges of g_2 connecting the vertex $*$ with vertices in U are oriented from $*$ to U. We make the following convenient notation for the set of edges going out of a vertex a of $g \in \mathcal{D}_s[V]$:

$$v_a(g) = \{(a,v) \mid (a,v) \in g\} = \{a\} \times e_a(g)$$

The weight $\omega^{\mathcal{D}_s,q}$ depends on the real parameter $0 \leq q \leq 1$ and is defined by:

$$\omega^{\mathcal{D}_s,q}(g_1,g_2) = \delta_{g_2\ ,\ g_1 + v_*(g_2)} \cdot (q^{|U| - |v_*(g_2)|} \cdot (1-q)^{|v_*(g_2)|})^{\frac{1}{2}}$$

In the rest of this section we prove that $(\mathcal{D}_s, \omega^{\mathcal{D}_s,q})$ is a realization of the q-commutation relations ([7, 6, 19, 4, 12]).

Theorem 4.2 *On* $(\mathcal{D}_s, \omega^{\mathcal{D}_s,q})$ *we have:*

$$a(h_1)a^*(h_2) - q \cdot a^*(h_2)a(h_1) = \langle h_1, h_2 \rangle$$

Proof. We employ lemmas 4.5 and 4.4. First:

$$\begin{aligned}(\omega \cdot \overline{\omega})_n(f,f') &= \sum_g \omega^{\mathcal{D}_s,q}(f,g) \cdot \overline{\omega^{\mathcal{D}_s,q,}(f',g)} \\ &= \sum_g \delta_{g,f+v_n(g)} \cdot \delta_{g,f'+v_n(g)} \cdot q^{n-|v_n(g)|} \cdot (1-q)^{|v_n(g)|} \\ &= \delta_{f,f'} \cdot \sum_{v_n \subset n} q^{n-|v_n|} \cdot (1-q)^{|v_n|} = \delta_{f,f'}\end{aligned} \tag{18}$$

It remains to be proved that $(\omega \cdot \overline{\omega})_k(f,f') = q \cdot (\overline{\omega} \cdot \omega)_k(f,f')$ for $k \in \{0, 1, ..n-1\}$ and $f, f' \in \mathcal{D}_s[n]$.
In the sum

$$(\omega \cdot \overline{\omega})_k(f,f') = \sum_g \omega^{\mathcal{D}_s,q}(f,g) \cdot \overline{\omega^{\mathcal{D}_s,q,}(f', \mathcal{D}_s[\tau_{n,k}](g))}$$

the only nonzero contribution comes from $g \in \mathcal{D}_s[U + \{*\}]$ such that:

$$g = f + \{n\} \times e_n(g) = (f \setminus v_k(f)) + v_k(f) + \{n\} \times e_n(g)$$

and

$$\begin{aligned} \mathcal{D}_s[\tau_{n,k}](g) &= (f \setminus v_k(f)) + \{n\} \times e_k(f) + \{k\} \times e_n(g) \\ &= f' + \{n\} \times e_k(f) \end{aligned}$$

which together imply

$$(f \setminus v_k(f)) + \{k\} \times e_n(g) = f'.$$

But this means that $e_n(g) = e_k(f')$ and $f \setminus v_k(f) = f' \setminus v_k(f')$. Then we have the expression

$$(\omega \cdot \overline{\omega})_k(f, f') = \delta_{f \setminus v_k(f), f' \setminus v_k(f')} \cdot q^{n - \frac{|v_k(f)| + |v_k(f')|}{2}} \cdot (1-q)^{\frac{|v_k(f)| + |v_k(f')|}{2}} \quad (19)$$

On the other hand in $(\overline{\omega} \cdot \omega)_k(f, f')$ we get only the contribution from those g for which:

$$g' = \mathcal{D}_s[\tau_{n-1,k}](f) \restriction_{n-1} = \mathcal{D}_s[\tau_{n-1,k}](f') \restriction_{n-1}$$

Thus we obtain

$$(\overline{\omega} \cdot \omega)_k(f, f') = \delta_{f \setminus v_k(f), f' \setminus v_k(f')} \cdot q^{n-1-\frac{|v_k(f)| + |v_k(f')|}{2}} \cdot (1-q)^{\frac{|v_k(f)| + |v_k(f')|}{2}}. \quad (20)$$

Finally from (19) and (20) we have the desired expression:

$$(\omega \cdot \overline{\omega})_k(f, f') = q \cdot (\overline{\omega} \cdot \omega)_k(f, f')$$

□

5 Fock States and Operations with Combinatorial Fock Spaces

The operations between species of structures described in Section 2 are helpful in understanding the action of creation and annihilation operators in terms of elementary ones. The guiding example is Green's representation of the operators appearing in parastatistics, as sums of bosonic (fermionic) operators with the "wrong" commutation relations [8]. Similar ideas appear in [15] where the author considers macroscopic fields as linear combinations of basic bosonic fields with various commutation relations.

Thus, the first question we address in this section is the following: given two combinatorial Fock spaces (F, ω_F) and (G, ω_G), is there a natural natural weight associated to the species $F + G$, $F \cdot G$, $F \times G$, $F \circ G$? The second question is related to the notion of positive definite functions on pair partitions. A general theory of such functions has been introduced in [3] in connection with the so called generalized Brownian motion.

5.1 Fock States

We will start with the latter question by introducing the necessary definitions.

Definition. Let S be a finite ordered set. We denote by $\mathcal{P}_2(S)$ is the set of pair partitions of S, that is $\mathcal{V} \in \mathcal{P}_2(S)$ if $\mathcal{V} = \{V_1, .., V_r\}$ where each V_i is an ordered set containing two elements $V_i = (k_i, l_i)$ with $k_i, l_i \in S$, $k_i < l_i$ and $\{V_1, .., V_r\}$ is a partition of S ($V_i \bigcap V_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^r V_i = S$). The set of all pair partitions is

$$\mathcal{P}_2(\infty) = \bigcup_{r=1}^{\infty} \mathcal{P}_2(2r).$$

Let $\mathcal{K}$ be a Hilbert space. We denote by $\mathcal{C}_{\mathcal{K}}$ the *-algebra obtained from the free algebra with generators $c(f)$ and $c^*(f)$, $(f \in \mathcal{K})$ divided by the relations:

$$c^*(\lambda f_1 + \mu f_2) = \lambda c^*(f_1) + \mu c^*(f_2),\ \lambda, \mu \in \mathbb{C},\ f_1, f_2 \in \mathcal{K},$$

and

$$c^*(f) = (c(f))^*.$$

We are interested in a particular type of positive functionals on $\mathcal{C}_{\mathcal{K}}$, called *Fock states* [3] which have the following expression on monomials of creation and annihilation operators:

$$\rho_t(c^{\sharp_1}(f_1) \cdot .. \cdot c^{\sharp_n}(f_n)) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum_{\mathcal{V}=\{V_1,..,V_{\frac{n}{2}}\}} \rho_t[V_1] \cdot .. \cdot \rho_t[V_{\frac{n}{2}}] \cdot t(\mathcal{V}) & \text{if } n \text{ even} \end{cases} \tag{21}$$

the sum running over all pair partitions $\mathcal{V}$ in $\mathcal{P}_2(2r)$, and the symbols $\sharp_i$ standing for creation or annihilation. For $V = (k, l) \in \mathcal{V}$

$$\rho_t[V] = \langle f_k, f_l \rangle \cdot Q(\sharp_k, \sharp_l)$$

with the 2 by 2 covariance matrix

$$Q = \begin{pmatrix} \rho(c_i c_i) & \rho(c_i c_i^*) \\ \rho(c_i^* c_i) & \rho(c_i^* c_i^*) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

where $c_i = c(e_i)$ and e_i is an arbitrary normalized vector in $\mathcal{K}$.
Let us consider a real subspace $\mathcal{K}_{\mathbb{R}}$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}_{\mathbb{R}} \oplus i\mathcal{K}_{\mathbb{R}}$. The sub-algebra of $\mathcal{C}_{\mathcal{K}}$ generated by the "field operators" $\omega(f) = c(f) + c^*(f)$ with $f \in \mathcal{K}_{\mathbb{R}}$ is denoted by $\mathcal{A}_{\mathcal{K}}$. If the restriction of the functional ρ_t to the algebra $\mathcal{A}_{\mathcal{K}} \subset \mathcal{C}_{\mathcal{K}}$ is a state, then we call the function

$$t : \mathcal{P}_2(\infty) \to \mathbb{C}.$$

positive definite [3]. In particular if ρ_t is a state on $\mathcal{C}_{\mathcal{K}}$ then t is positive definite. The converse is not true in general. We will show first that the vacuum state of a symmetric Hilbert space is an example of Fock state.

Proposition 5.1 *Let (F, ω_F) be a combinatorial Fock space with $|F[\emptyset]| = 1$. Let Ω_F be the vacuum vector of $\Gamma_F(\mathcal{K})$, $\mathcal{K}$ a Hilbert space. Then the functional $\rho_F(\cdot) = \langle \Omega_F, \cdot \Omega_F \rangle$ is a Fock state on $\mathcal{C}_{\mathcal{K}}$.*

In order to prove this proposition we need to introduce one more tool.
Definition. Let $\mathcal{K}$ be a Hilbert space, (F, ω_F) a combinatorial Fock space and $A \in \mathcal{B}(\mathcal{K})$ a bounded operator on $\mathcal{K}$. The *second quantization* of A is defined by

$$\begin{aligned} d\Gamma_F(A) : \Gamma_F(\mathcal{K}) &\to \Gamma_F(\mathcal{K}) \\ (d\Gamma_F(A)\varphi)(g) &= d\Gamma(A)(\varphi(g)) \end{aligned}$$

where the meaning of $d\Gamma(A)$ on the right side is

$$d\Gamma(A) : \mathcal{K}^{\otimes n} \to \mathcal{K}^{\otimes n}$$

$$d\Gamma(A) : \psi_0 \otimes .. \otimes \psi_{n-1} \to \sum_{k=0}^{n-1} \psi_0 \otimes .. \otimes A\psi_k \otimes .. \otimes \psi_{n-1}$$

Remark. The second quantization operator is a well defined operator on $\Gamma_F(\mathcal{K})$. Indeed let $\varphi \in \Gamma_F(\mathcal{K})$ and $\tau \in \mathrm{S}(n)$, then

$$\begin{aligned} (d\Gamma_F(A)\varphi)(F[\tau](g)) &= d\Gamma(A)\varphi(F[\tau](g)) \\ = d\Gamma(A) \cdot U(\tau)\varphi(g) &= U(\tau)(d\Gamma_F(A)\varphi)(g) \end{aligned}$$

and thus $d\Gamma_F(A)\varphi \in \Gamma_F(\mathcal{K})$. We have used the invariance of $d\Gamma(A)$ under permutations:

$$d\Gamma(A)U(\tau) = U(\tau)d\Gamma(A).$$

Lemma 5.1 *We have the following commutation relations:*

$$[a(h), d\Gamma_F(A)] \quad = \quad a(A^* h) \tag{22}$$

$$[d\Gamma_F(A), d\Gamma_F(B)] \quad = \quad d\Gamma_F([A,B]) \tag{23}$$

Proof. Let $\varphi \in \Gamma_F(\mathcal{K}),\ f \in F[n],\ g \in F[n+1]$. Then

$$\begin{aligned}
(a(h)d\Gamma(A)\varphi)(f) &= \sum_g \omega(f,g) \cdot \mathrm{inp}_n(h, (d\Gamma(A)\varphi)(g)) \\
&= \sum_g \omega(f,g) \cdot \mathrm{inp}_n(h, \sum_{k=0}^{n} \mathbf{1} \otimes .. \otimes A \otimes .. \otimes \mathbf{1}\varphi(g)) \\
&= \sum_g \omega(f,g) \cdot \sum_{k=0}^{n-1} \mathbf{1} \otimes .. \otimes A \otimes .. \otimes \mathbf{1}\ \mathrm{inp}_n(h, \varphi(g)) \\
&\quad + \sum_g \omega(f,g) \cdot \mathrm{inp}_n(h, \mathbf{1} \otimes .. \otimes A\varphi(g)) \\
&= \left(\sum_{k=0}^{n-1} \mathbf{1} \otimes .. \otimes A \otimes .. \otimes \mathbf{1} \right) \sum_g \omega(f,g) \cdot \mathrm{inp}_n(h, \varphi(g)) \\
&\quad + \sum_g \omega(f,g) \cdot \mathrm{inp}_n(A^* h, \varphi(g)) \\
&= d\Gamma_F(A) a(h) + a(A^* h)
\end{aligned}$$

which proves (22). The other commutator follows directly from the definition of the second quantization operator.

□

Lemma 5.2 *Let $\{e_j\}_{j \in J}$ be an orthonormal basis of $\mathcal{K}$ and denote $a_j^\sharp = a(e_j)^\sharp$. Then the following equation holds:*

$$a_i \prod_{k=1}^{n} a_{i_k}^{\sharp_k} \Omega = \sum_{k=1}^{n} \delta_{i,i_k} \cdot \delta_{\sharp_k,*} \cdot a_{i_0} \prod_{p=1}^{k-1} a_{i_p}^{\sharp_p} \cdot a_{i_0}^* \cdot \prod_{q=k+1}^{n} a_{i_q}^{\sharp_q} \Omega \tag{24}$$

where the colors $(i_k)_{k=0,..,n}$ satisfy the property $i_k \neq i_0$ for all $k = 1,..n$.

Proof. For simplicity we denote $\Psi = \prod_{k=1}^{n} a_{i_k}^{\sharp_k} \Omega$. We notice that $a_{i_0}\Psi = 0$ due to the assumption that $i_k \neq i_0$ for all $k = 1,..n$. Then using (22) we get

$$a_i \Psi = [a_{i_0}, d\Gamma(|i_0\rangle\langle i|)]\Psi = a_{i_0} d\Gamma(|i_0\rangle\langle i|)\Psi \tag{25}$$

By successively applying the following commutator

$$[d\Gamma(|i_0\rangle\langle i|), a_{i_k}^{\sharp_k}] = \delta_{i_k,i} \cdot \delta_{\sharp_k,*} \cdot a_{i_0}^*$$

we obtain the sum in (24).

□

Proof of Proposition 5.1. It is clear that ρ_F is a positive linear functional on $\mathcal{C}_\mathcal{K}$. We need to prove that it has the expression (21). From linearity of the creation operators and anti-linearity of the annihilation operators we conclude that it is sufficient to consider the vectors f_i in (21) belonging to the chosen orthogonal basis. From

$$\rho_F(\prod_{k=1}^{n} a_{i_k}^{\sharp_k}) = \left\langle \Omega, \prod_{k=1}^{n} a_{i_k}^{\sharp_k}\Omega \right\rangle$$

and considering the fact that the creation operator increases the level by one while the annihilation operator decreases it by one, we deduce that nonzero expectations can appear only if n is even and the number of creators in the monomial $\prod_{k=1}^{n} a_{i_k}^{\sharp_k}$ is equal to that of annihilators. Furthermore $a_{i_1}^{\sharp_1}$ must be an annihilator and $a_{i_n}^{\sharp_n}$, a creator. We will thus consider that this is the case.
We put the monomial in the form $a_{i_1}\prod_{k=2}^{n} a_{i_k}^{\sharp_k}$ and apply lemma 5.2. We obtain a sum over all pairs $(a_{i_1}, a_{i_k}^*)$ of the same color ($i_1 = i_k$) and replace i_1 by a new color i_0. We go now to the next annihilator in each term of the sum and repeat the procedure, the new color which we add this time being different from all the colors used previously. After $\frac{n}{2}$ steps we obtain a sum containing all possible pairings of annihilators and creators of the same color in $\prod_{k=1}^{n} a_{i_k}^{\sharp_k}$:

$$\rho_F(\prod_{k=1}^{n} a_{i_k}^{\sharp_k}) = \sum_{\mathcal{V}=\{V_1,..,V_{\frac{n}{2}}\}} \prod_{p=1}^{\frac{n}{2}} \delta_{i_{k_p},i_{l_p}} \cdot Q(\sharp_{k_p}, \sharp_{l_p}) \cdot t(\mathcal{V})$$

with $V_p = (k_p, l_p)$ and $t(\mathcal{V})$ is given by

$$t(\mathcal{V}) = \rho_F(\prod_{k=1}^{n} a_{j_k}^{\sharp_k})$$

such that $j_{k_p} = j_{l_{p'}}$ if and only if $p = p'$, for $p, p' \in \{1, .., \frac{n}{2}\}$, $\sharp_{k_p}$ is annihilator and $\sharp_{l_p}$ is creator.

□

Thus for each combinatorial Fock space (F, ω_F) (which has a vacuum), the vacuum state is described by a positive definite function t_F on $\mathcal{P}_2(\infty)$.

Remark. We observe that the result can be generalized to a larger range of states and monomials. Let us partition the index set J of the orthonormal basis of the Hilbert space $\mathcal{K}$

$$J = J_1 + J_2$$

and choose a state $\rho_\Phi(\cdot) = \langle \Phi, \cdot \Phi \rangle$ and monomials $\prod_{k=1}^{n} a_{j_k}^{\sharp_k}$ such that $j_k \in J_1$ and $\Phi \in \Gamma_F(\mathcal{K}_2) \subset \Gamma_F(\mathcal{K})$ is a normalized vector where $\mathcal{K}_2$ is the subspace of $\mathcal{K}$ with the orthogonal basis $\{e_j\}_{j \in J_2}$. Then it is easy to see that the argument used in the above proof still holds and ρ_Φ is a Fock state for $\mathcal{C}_{\mathcal{K}_1}$. In general ρ_Φ and ρ_F do not coincide. When they do coincide we say that ρ_F has the *pyramidal independence* property [3].

5.2 Operations with symmetric Hilbert Spaces

We pass now to the first question which we have posed in the beginning of this section. The various operations between species offer the opportunity of creating new symmetric Hilbert spaces which sometimes give rise to interesting interpolations between the two members. For the definitions of the operations we refer back to Section 2.

1) Sums. Let (F, ω_F) and (G, ω_G) be two combinatorial Fock spaces. From Section 3 we know that

$$\Gamma_{F+G} = \Gamma_F \oplus \Gamma_G.$$

Note that the vacuum of Γ_{F+G} has dimension ≥ 2 if $F[\emptyset] \neq \emptyset \neq G[\emptyset]$. The natural weight on $F + G$ is

$$\omega_{F+G}(t_1, t_2) = \omega_F(t_1, t_2) + \omega_G(t_1, t_2)$$

which gives rise to operators

$$a_{F+G}(h) = a_F(h) \oplus 0 + 0 \oplus a_G(h).$$

We consider a linear combination of the two vacua (for $|F[\emptyset]| = |G[\emptyset]| = 1$)

$$\Omega_\lambda = \sqrt{\lambda}\Omega_F + \sqrt{1-\lambda}\Omega_G.$$

The corresponding state $\rho_{F+G,\lambda}(\cdot) = \langle \Omega_\lambda, \cdot\Omega_\lambda \rangle$ interpolates linearly between ρ_F and ρ_G

$$\rho_{F+G,\lambda} = \lambda\rho_F + (1-\lambda)\rho_G$$

and the same is true for the positive definite functions

$$t_{F+G,\lambda} = \lambda t_F + (1-\lambda)t_G. \tag{26}$$

1) Products. Let (F, ω_F) and (G, ω_G) be two combinatorial Fock spaces. We consider the product species $F \cdot G$. As we have proved in Section 3, there is the following isomorphism

$$\Gamma_{F\cdot G}(\mathcal{K}) = \Gamma_F(\mathcal{K}) \otimes \Gamma_G(\mathcal{K}). \tag{27}$$

Again there is a natural weight for the species $F \cdot G$. For $f \in F[U_1], g \in G[U_2], f' \in F'[U_1], g' \in G'[U_2]$:

$$\begin{aligned} \omega_{F\cdot G,\lambda}((f,g),(f,g')) &= \sqrt{\lambda}\omega_G(g,g') \\ \omega_{F\cdot G,\lambda}((f,g),(f',g)) &= \sqrt{1-\lambda}\omega_F(f,f') \end{aligned}$$

all other values of $\omega_{F\cdot G,\lambda}$ being 0.
From (27) and the expression of $\omega_{F\cdot G}$ we obtain

$$a^\sharp_{F\cdot G}(h) = \sqrt{\lambda}a^\sharp_F(h) \otimes \mathbf{1} + \sqrt{1-\lambda}\mathbf{1} \otimes a^\sharp_G(h)$$

If $|F[\emptyset]| = |G[\emptyset]| = 1$ then the state $\rho_{F\cdot G}(\cdot) = \langle \Omega_F \otimes \Omega_G, \cdot\Omega_F \otimes \Omega_G \rangle$ generates the positive definite function:

$$t_{F\cdot G}(\mathcal{V}) = \sum_{\mathcal{V}_1,\mathcal{V}_2} \lambda^{|\mathcal{V}_1|}(1-\lambda)^{|\mathcal{V}_2|} t_F(\mathcal{V}_1) \cdot t_G(\mathcal{V}_2)$$

where the sum runs over all partitions of $\mathcal{V}$ in two sets, $\mathcal{V}_1$ and $\mathcal{V}_2$.

Example: The Green representation [8] of the (Fermi) parastatistics of order p is an example of application of the product of species. We consider the p-th power $(E_\pm)^p$ of the species of oriented sets $E_\pm$. Then the annihilation operators are

$$a(h) = \frac{1}{\sqrt{p}} \sum_{k=1}^{p} a^{(k)}(h)$$

and the vacuum state is $\rho(\cdot) = \langle \Omega, \cdot\Omega \rangle$ where $a^{(k)}$ is the term corresponding to the k-th term in the product and

$$\Omega = \Omega_a^{(1)} \otimes .. \otimes \Omega_a^{(p)}$$

is the tensor product of the antisymmetric vacua of each of the species $E_{\pm}^{(k)}$.

3) Cartesian Products. Let (F, ω_F) and (G, ω_G) be two combinatorial Fock spaces. We consider the cartesian product species $F \times G$. The corresponding weight has the expression:

$$\omega_{F\times G}((f,g),(f',g')) = \omega_F(f,f') \cdot \omega_G(g,g')$$

We note that $\omega_{F\times G}$ satisfies the invariance condition stated in the definition of the weight.

Proposition 5.2 *Let (F, ω_F) and (G, ω_G) be two combinatorial Fock spaces both having a single structure on $\emptyset$. Then the positive definite function associated to the vacuum state of $(F \times G, \omega_{F\times G})$ satisfies:*

$$t_{F\times G}(\mathcal{V}) = t_F(\mathcal{V}) \cdot t_G(\mathcal{V}) \tag{28}$$

for all $\mathcal{V} \in \mathcal{P}_2(\infty)$.

Proof. We construct a linear operator T from $\Gamma_{F\times G}(\mathcal{K})$ to $\Gamma_F(\mathcal{K}) \otimes \Gamma_G(\mathcal{K})$ with the property that its restriction to a certain subspace $\Gamma^{\text{ext}}_{F\times G}$ of $\Gamma_{F\times G}(\mathcal{K})$, is an isometry. The subspace $\Gamma^{\text{ext}}_{F\times G}$ is spanned by vectors $\delta_{[(f,g),c]}$ of the orthogonal basis $(F \times G)(J)$ of $\Gamma_{F\times G}(\mathcal{K})$ which have all colors different from each other, i.e. $c(i) \neq c(j)$ for $i \neq j$. We refer to Section 3 for the definitions related to the orthogonal basis of $\Gamma_{F\times G}(\mathcal{K})$.
The action of T on the basis vectors is:

$$\begin{aligned} T : \Gamma^{\text{ext}}_{F\times G}(\mathcal{K}) &\rightarrow \Gamma^{\text{ext}}_F(\mathcal{K}) \otimes \Gamma^{\text{ext}}_G(\mathcal{K}) \\ \delta_{[(f,g),c]} &\mapsto \delta_{[f,c]} \otimes \delta_{[g,c]} \end{aligned}$$

We check that the operator is well defined. Indeed the map

$$\begin{aligned} i : \sum_{n=0}^{\infty} (F\times G)[n] \times J^n &\rightarrow (\sum_{n=0}^{\infty} F[n] \times J^n) \times (\sum_{n=0}^{\infty} G[n] \times J^n) \\ ((f,g),c) &\mapsto ((f,c),(g,c)) \end{aligned}$$

commutes with the action of $\mathrm{S}(n)$ on the two sides, at each level and thus projects to a well defined map on the quotient:

$$\begin{aligned} i : (F\times G)(J) &\rightarrow F(J) \times G(J) \\ [(f,g),c] &\mapsto ([f,c],[g,c]) \end{aligned}$$

This means that T is well defined. But as we have shown in Section 3, the vectors $\delta_{[(f,g),c]}$, $\delta_{[f,c]}$ and $\delta_{[g,c]}$ for which $c(i) \neq c(j)$ if $i \neq j$, have norm one which implies that T is an isometry.
Let us now consider the vector

$$\varphi_F^{(p)} = \prod_{k=1}^{p} a_{F,i_k}^{\sharp_k} \ \Omega_F$$

the colors $(i_k)_{k=1,..,n}$ satisfying the condition that there are no three identical colors, and if there exists $k_1 < k_2$ such that $i_{k_1} = i_{k_2}$, then $a_{i_{k_1}}^{\sharp_{k_1}} = a_{i_{k_1}}$ and $a_{i_{k_2}}^{\sharp_{k_2}} = a_{i_{k_2}}^*$. It is clear that $\varphi_F^{(p)} \in \Gamma_F^{\text{ext}}(\mathcal{K})$. Analogously we define $\varphi_G^{(p)}$ and $\varphi_{F\times G}^{(p)}$. We want to prove by induction w.r.t. p that the action of the isometry T is such that

$$T : \varphi_{F\times G}^{(p)} \to \varphi_F^{(p)} \otimes \varphi_G^{(p)}. \tag{29}$$

This implies in particular (28), when the monomial $\prod_{k=1}^{p} a_{i_k}^{\sharp_k}$ contains equal number of creators and annihilators pairing each other according to color, no two pairs having the same color.
For $p = 0$ we have $T(\Omega_{F\times G}) = \Omega_F \otimes \Omega_G$. Suppose (29) holds for p. Then there are two possibilities for increasing the length of the monomial $\prod_{k=1}^{p} a_{i_k}^{\sharp_k}$ by one: either by adding on the first position a creation operator $a_{i_0}^*$ such that the color i_0 does not appear in the rest of the monomial, or by adding an annihilation operator a_{i_0} such that the term $a_{i_0}^*$ appears once in the rest of the monomial. We treat the two cases separately.
1.) suppose that we have $\varphi_{F\times G}^{(p)} = \prod_{k=1}^{p} a_{i_k}^{\sharp_k} \ \Omega_{F\times G}$, $i_0 \neq i_k$ which has the decomposition

$$\varphi_{F\times G}^{(p)} = \sum_{[(f,g),c]} \varphi([(f,g),c])\delta_{[(f,g),c]}$$

with $\varphi([(f,g),c]) \in \mathbb{C}$. Then

$$a_{F\times G,i_0}^* \varphi_{F\times G}^{(p)} = \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c]) \cdot \omega_{F\times G}((f,g),(f',g')) \ \delta_{[(f',g'),c_{i_0}^+]}$$

which implies

$$\begin{aligned} T(a_{F\times G,i_0}^* \varphi_{F\times G}^{(p)}) &= \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c]) \ a_{F,i_0}^* \delta_{[f,c]} \otimes a_{G,i_0}^* \delta_{[g,c]} \\ = a_{F,i_0}^* \otimes a_{G,i_0}^* T(\varphi_{F\times G}^{(p)}) &= a_{F,i_0}^* \varphi_F^{(p)} \otimes a_{G,i_0}^* \varphi_G^{(p)}. \end{aligned}$$

2.) suppose that we have $\varphi^{(p)}_{F\times G} = \prod_{k=1}^p a^{\sharp_k}_{i_k}\ \Omega_{F\times G}$ such that the term $a^*_{i_0}$ appears exactly one time in the the monomial $\prod_{k=1}^p a^{\sharp_k}_{i_k}$. We use again the Fourier decomposition

$$\varphi^{(p)}_{F\times G} = \sum_{[(f,g),c]} \varphi([(f,g),c])\delta_{[(f,g),c]}$$

and identify in each orbit $[(f,g),c] \in (F\times G)(J)$, a representant $((f,g),c) \in (F\times G)[n] \times J^n$ such that $c(n-1) = i_0$. Then

$$a_{F\times G,i_0}\varphi^{(p)}_{F\times G} = \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c]) \cdot \omega_{F\times G}((f',g'),(f,g))\ \delta_{[(f',g'),c^-_{i_0}]}$$

where $c^-_{i_0}$ is the restriction of c to the set $n-1$. Finally

$$\begin{aligned} T(a_{F\times G,i_0}\varphi^{(p)}_{F\times G}) &= \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c]) \cdot \omega_{F\times G}((f',g'),(f,g))\ T(\delta_{[(f',g'),c^-_{i_0}]}) \\ &= \sum_{[(f,g),c],(f'g')} \varphi([(f,g),c]) \cdot \omega_F(f,f') \cdot \omega_G(g,g')\delta_{[f',c^-_{i_0}]} \otimes \delta_{[g',c^-_{i_0}]} \\ &= a_{F,i_0}\varphi^{(p)}_F \otimes a_{G,i_0}\varphi^{(p)}_G \end{aligned}$$

which proves the induction hypothesis for $p+1$ and the proposition.

□

Application: Combining the result of the previous proposition and certain variations on the species of rooted trees, we investigate more general commutation relations of the type:

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2\rangle \cdot f(N)$$

with $f : \mathbf{N} \to \mathbf{R}$ and N the number operator characterized by

$$[N, a^*(h)] = a^*(h).$$

Theorem 5.1 *Let P be a real polynomial with positive coefficients. Then the commutation relations*

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2\rangle \cdot P(N)$$

are realizable on a symmetric Hilbert space.

We split the proof in a few lemmas.

Lemma 5.3 *Let* (F, ω_F) *and* (G, ω_G) *be two symmetric Hilbert spaces for which the commutation relations hold*

$$\begin{aligned} [a_F(h_1), a_F^*(h_2)] &= \langle h_1, h_2 \rangle \cdot a(N) \\ [a_G(h_1), a_G^*(h_2)] &= \langle h_1, h_2 \rangle \cdot b(N) \end{aligned}$$

where a, b *are real functions. Then on* $(F \times G, \omega_{F \times G})$ *we have*

$$[a_{F\times G}(h_1), a_{F\times G}^*(h_2)] = \langle h_1, h_2 \rangle \cdot (a \cdot b)(N)$$

Proof. This is a direct application of Lemmas 4.4, 4.5 and the following equations:

$$\begin{aligned} (\omega \cdot \overline{\omega})_k((f,g),(f',g')) &= (\omega \cdot \overline{\omega})_k(f,f') \cdot (\omega \cdot \overline{\omega})_k(g,g') \\ (\overline{\omega} \cdot \omega)_k((f,g),(f',g')) &= (\overline{\omega} \cdot \omega)_k(f,f') \cdot (\overline{\omega} \cdot \omega)_k(g,g') \end{aligned}$$

□

Lemma 5.4 *Let* $\mathcal{A}$ *be the species of rooted trees. Let* $f \in \mathcal{A}[U]$, $g \in \mathcal{A}[U + \{*\}]$ *and*

$$\tilde{\omega}_{\mathcal{A}}^c(f,g) = \omega_{\mathcal{A}}(f,g) + c^{\frac{1}{2}} \delta_{f_*,g}$$

a modification of the weight $\omega_{\mathcal{A}}$ *defined in section 4, with c, a positive constant. The structure* $f_* \in \mathcal{A}[U + \{*\}]$ *is defined by:*

$$f_*(u) = \begin{cases} f(u) & \text{if } u \neq root(f) \\ * & \text{if } u = root(f). \end{cases}$$

Then on $(\mathcal{A}, \tilde{\omega}_{\mathcal{A}}^c)$ *we have:*

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2 \rangle \cdot (N + c).$$

Proof. This is similar to the proof of Theorem 4.1, with an additional contribution to $(\omega \cdot \overline{\omega})_n(f,g)$ coming from the term $c^{\frac{1}{2}} \delta_{f_*,g}$ in $\tilde{\omega}_{\mathcal{A}}^c$.

□

Lemma 5.5 *Let* $\mathcal{A} \times \mathcal{A}$ *be the species of ordered pairs of rooted trees. We define the weight*

$$\omega_{\mathcal{A}\times\mathcal{A}}^c((f,g),(f',g')) = \omega_{\mathcal{A}}(f,g) \cdot \omega_{\mathcal{A}}(f',g') + c^{\frac{1}{2}} \delta_{f_*,f'} \cdot \delta_{g_*,g'}.$$

Then on $(\mathcal{A} \times \mathcal{A}, \omega_{\mathcal{A}\times\mathcal{A}}^c)$ *we have*

$$[a(h_1), a^*(h_2)] = \langle h_1, h_2 \rangle \cdot (N^2 + c).$$

Proof. Similar to the previous two lemmas.

□

Proof of Theorem 5.1. The polynomial $P(x)$ has a canonical expression as product of polynomials of the type $x+c$ and x^2+c with $c \geq 0$. The theorem follows by applying the previous 3 lemmas.

□

Remark. The result can be extended to power series with positive coefficients and infinite radius of convergence. In particular for $0 \leq q \leq 1$

$$s(x) = q^{-x} = \sum_{k=0}^{\infty} \frac{1}{n!} \cdot (-\log q)^n \cdot x^n$$

gives the commutation relations

$$[a_i, a_j^*] = q^{-N} \delta_{i,j}$$

which characterize the *q-deformations* [4], [7], up to a "rescaling" of the creation and annihilation operators with a function of N.

4) Compositions. Let (F, ω_F) and (G, ω_G) be two combinatorial Fock spaces. We recall that the composition of G in F is a species whose structures are F-assemblies of G-structures:

$$F \circ G[U] = \sum_{\pi} F[\pi] \times \prod_{p \in \pi} G[p].$$

We would like to define the annihilation and creation operators for the species $F \circ G$ by making use of the available weights ω_F and ω_G. Apart from the condition $|G[\emptyset]| = 0$ we require $|G[1]| = 1$. We consider an arbitrary structure $(f, \pi, (g_p)_{p\in\pi}) \in F \circ G[U]$ where π is a partition of the finite set U. Then we note that there are two essentially different possibilities to "add" a new point $*$, to the set U: one can enlarge the size of π by creating a partition of $U + \{*\}$ of the form $\pi^+ = \pi + \{\{*\}\}$, or one can keep the size of π constant by adding $*$ to one of the sets $p \in \pi$ and obtain the partition π_p^+. Between π and π_p^+ there is the bijection

$$\alpha_p : p' \to \begin{cases} p' & \text{if } p' \neq p \\ p + \{*\} & \text{if } p' = p \end{cases}$$

We recognize that in the first case the weight ω_F should play a role, while in the second, the weight ω_G. According to the properties of the species F, one can further distinguish among the subsets to which $*$ is added, by choosing (as we did for the creation and annihilation operators) a weight $\omega_{F,\epsilon}$ on the cartesian product $F \times \epsilon$ where ϵ is the species of elements: $\epsilon[U] = U$. Putting together the three data $(\omega_F, \omega_G, \omega_{F,\epsilon})$, we define:

$$\omega_{F\circ G}((f,\pi,(g_p)_{p\in\pi}),(f',\pi',(g'_{p'})_{p'\in\pi'})) = \omega_F(f,f') \cdot \prod_{p\in\pi} \delta_{g_p,g'_p}$$

$$+ \quad \sum_{p\in\pi} \delta_{f',F[\alpha_p](f)} \cdot \omega_{F,\epsilon}(f,p) \cdot \omega_G(g_p, g'_{p+\{*\}}).$$

where $f \in F[\pi],\ g_p \in G[p]$, etc.

Remark. We find this definition rather natural and broad enough to cover some interesting examples. One can easily check that $\omega_{F\circ G}$ satisfies the invariance property characterizing the weights.

Example: The species Bal of ordered partitions or Ballots is the composition of L (the species of linear orderings), with E_+ (the species of nonempty sets). A typical structure over a finite set U looks like: $s = (U_1,..,U_k)$ with $(U_p)_{p\in\{1,..,k\}}$, a partition of U. The vacuum is the empty sequence $\mathrm{Bal}[\emptyset] = \{\emptyset\}$. We use the weights ω_E and ω_L as defined in section 4. The action of the creation operator at the combinatorial level can be described as follows: we can add the point $*$ in the last subset of the sequence $s = (U_1,..,U_k)$ and obtain $s_k^+ = (U_1,..,U_k + \{*\})$, or we can create a new subset $U_{k+1} = \{*\}$ and position it at the end of the sequence s, producing $s^+ = (U_1,..,U_{k+1})$. We see that in this case the weight $\omega_{L,\epsilon}$ is simply identifying the last element of the sequence: $\omega_{L,\epsilon}(s,U_k) = \delta_{U_k,U_p}$. For the vacuum we set $\omega_{\mathrm{Bal}}(\{\emptyset\},\{*\}) = 1$. We use $0 \le q \le 1$ as an interpolation parameter:

$$\omega_{\mathrm{Bal}}(s,s') = q^{\frac{1}{2}}\delta_{s_k^+,s'} \ + \ (1-q)^{\frac{1}{2}}\delta_{s^+,s'} \tag{30}$$

Let us denote by t_{Bal} the positive definite function associated to the vacuum state of the combinatorial Fock space $(\mathrm{Bal},\omega_{\mathrm{Bal}})$, as defined in subsection 5.1. Following [3] we associate to any pair partition $\mathcal{V} \in \mathcal{P}_2(\infty)$ a set $B(\mathcal{V}) = \{\mathcal{V}_1,..,\mathcal{V}_k\}$ such that $\mathcal{V} = \mathcal{V}_1 \cup ... \cup \mathcal{V}_k$ is the decomposition of $\mathcal{V}$ into connected sub-partitions or *blocks*.

Theorem 5.2 *Let* $\mathcal{V} \in \mathcal{P}_2(\infty)$. *Then*

$$t_{\mathrm{Bal}}(\mathcal{V}) = q^{|\mathcal{V}|-|B(\mathcal{V})|} \tag{31}$$

Proof. We split the task of proving (31) into two simpler ones: first we prove the *strong multiplicativity* property for t:

$$t(\mathcal{V}) = \prod_{i=1}^{k} t(\mathcal{V}_i) \quad \text{if} \quad B(\mathcal{V}) = \{\mathcal{V}_1,..,\mathcal{V}_k\}$$

and then for $\mathcal{V}$ consisting of a single block, $t_{\mathrm{Bal}}(\mathcal{V}) = q^{|\mathcal{V}|-1}$. The proof of the strong multiplicativity is analogous to that of Proposition 5.2. We consider an orthogonal basis $(e_j)_{j\in J}$ of the Hilbert space $\mathcal{K}$ and a partition $J = J_1 + J_2$ of J with the corresponding relation $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. We define an isometry

$$S : \Gamma^{\mathrm{ext}}_{\mathrm{Bal}}(\mathcal{K}_1) \otimes \Gamma^{\mathrm{ext}}_{\mathrm{Bal}}(\mathcal{K}_2) \to \Gamma^{\mathrm{ext}}_{\mathrm{Bal}}(\mathcal{K}_1 \oplus \mathcal{K}_2)$$

and we will prove that it has a natural action on monomials of creation and annihilation operators:

$$S(\prod_k a^{\sharp_k}_{i_k}\Omega \otimes \prod_p a^{\sharp_p}_{j_p}\Omega) = \prod_k a^{\sharp_k}_{i_k} \cdot \prod_p a^{\sharp_p}_{j_p}\Omega. \tag{32}$$

We recall that the two monomials satisfy certain properties which are described in Proposition 5.2. The multiplicativity of t_{Bal} follows from equation (32) and the isometric property of S.
The action of S on the orthogonal bases defined in Section 3 is:

$$\delta_{[s_1,c_1]} \otimes \delta_{[s_2,c_2]} \to \sum_s q^{\frac{a(s)}{2}} \cdot (1-q)^{\frac{b(s)}{2}} \cdot \delta_{[s,c]}$$

where, for arbitrary $s_1 = (U_1, .., U_k)$ and $s_2 = (V_1, .., V_p)$, the sum runs over all $s = (V_1, .., V_{p-1},\ V,\ U,\ U_2, .., U_k)$ with $V_p \subset V$, $U \subset U_1$ and $U + V = U_1 + V_p$. The coloring c restricts to c_1 and c_2 on the sets $\bigcup_\alpha U_\alpha$ respectively $\bigcup_\beta V_\beta$. The coefficients appearing on the right side are $a(s) = |V| - |V_p|$ and $b(s) = |U|$. As $\|\delta_{[s,c]}\| = 1$ and $a(s) + b(s) = |U_1|$, we obtain

$$\begin{aligned} \|S(\delta_{[s_1,c_1]} \otimes \delta_{[s_2,c_2]})\|^2 &= \sum_{j=0}^{|U_1|} \binom{|U_1|}{j} \cdot q^k \cdot (1-q)^{|U_1|-k} \\ = 1 &= \|\delta_{[s_1,c_1]} \otimes \delta_{[s_2,c_2]}\|^2, \end{aligned}$$

which proves the isometry property. The equation (32) follows by induction w.r.t. k. For $k = 0$ is is obvious that

$$S\left(\Omega \otimes \prod_p a^{\sharp_p}_{j_p}\Omega\right) = \prod_p a^{\sharp_p}_{j_p}\Omega.$$

Then one can check on the basis vectors that

$$S \cdot (a^{\sharp_j}_j \otimes \mathbf{1}) = a^{\sharp_j}_j \cdot S$$

for $j \in J_1$, which provides the tool for the incrementation of k.

We pass now to the expression of $t_{\rm Bal}(\mathcal{V})$ for a one block partition $\mathcal{V}$. The basic observation is that the creation and annihilation operators have the following form, stemming from that of $\omega_{\rm Bal}$ (see (30)):

$$a_i^{\sharp_i} = q^{\frac{1}{2}} a_{E,i}^{\sharp_i} + (1-q)^{\frac{1}{2}} a_{L,i}^{\sharp_i}$$

with the choice $a_{E,i}^{*}\Omega = 0$. Let $M_{\mathcal{V}} = \prod_{l=1}^{2n} a_{i_l}^{\sharp_l}$ be a monomial associated to the pair partition $\mathcal{V} \in \mathcal{P}_2(2n)$. It is sufficient to prove that the only nonzero contribution to $M_{\mathcal{V}}\Omega$ is brought by the the term $a_{L,i_1} \prod_{l=2}^{2n-1} a_{E,i_l} \cdot a_{L,i_{2n}}^{*}\Omega = q^{n-1}\Omega$. Indeed the action of a_{L,i_1}^{*} at the combinatorial level is to increase the number of subsets in a sequence by 1. Thus the terms which are nonzero must contain an equal number of creation and annihilation operators of type L. Let us consider such a term. Then there exist $1 \leq l_1 \leq l_2 \leq 2n$ such that on the positions l_1 and l_2 we have annihilation respectively creation operators of type L and for $l_1 \leq l \leq l_2$ we have type E operators. We have identified a submonomial

$$m = a_{L,l_1} \cdot \prod_{l=l_1+1}^{l_2-1} a_{E,i_l}^{\sharp_l} \cdot a_{L,l_2}^{*}$$

which is nonzero only if it corresponds to a pair partition, that is if all creation and annihilation operators pair each other according to the color. But this is possible only when $l_1 = 1$ and $l_2 = 2n$ because $\mathcal{V}$ is a one-block pair partition.

□

4) Free Products. Inspired by the notion of freeness introduced by Dan Voiculescu [18] we make the following:
Definition. Let $(F_\alpha)_{\alpha\in J}$ be a finite set of species of structures with $F_\alpha[\emptyset] = \{\emptyset\}$ for all $\alpha \in J$. The *free product* of $(F_\alpha)_{\alpha\in J}$ is the species defined by:

$$\begin{aligned} *_{\alpha\in J}\,(F_\alpha)[U] \;&=\; \{(\pi,(s_1,..,s_p))|\pi = (U_1,..,U_p) \in {\rm Bal}[U], s_i \in F_{\alpha_i}[U_i], \\ &\qquad \alpha_i \neq \alpha_{i+1} \text{ for } i = 1,..p-1\} \end{aligned} \tag{33}$$

for $U \neq \emptyset$ and $*_{\alpha\in J}F_\alpha[\emptyset] = \{\emptyset\}$. The transport is induced from the species $(F_\alpha)_{\alpha\in J}$ and Bal. From the definition it is clear that we have the following combinatorial equation:

$$*_{\alpha\in J}(F_\alpha) = 1 + \sum_{p\geq 1} \sum_{\alpha_1\neq\alpha_2\neq..\neq\alpha_p} F_{\alpha_1+} \cdot F_{\alpha_2+} \cdot ... \cdot F_{\alpha_p+}$$

and using the property $\Gamma_{F \cdot G} = \Gamma_F \otimes \Gamma_G$ we obtain

$$\Gamma_{*_{\alpha \in J}(F_\alpha)}(\mathcal{K}) = *_{\alpha \in J}(\Gamma_{F_\alpha}, \Omega_\alpha)$$

where the last object is the Hilbert space free product [18].
The corresponding weight is similar to the one used for the species Bal. For $f_i \in F_{\alpha_i}[U_i]$ and $f_i' \in F_{\alpha_i'}[V_i]$, it has the expression:

$$\omega_{*_{\alpha \in J(F_\alpha)}}((\pi, (f_1, .., f_p)), (\pi', (f_1', .., f_q'))) =$$
$$\delta_{p,q} \cdot \delta_{\alpha_p, \alpha_p'} \prod_{i=1}^{p-1} \delta_{f_i, f_i'} \cdot \omega_{\alpha_p}(f_p.f_p') + \delta_{p+1,q} \cdot \prod_{i=1}^{p} \delta_{f_i, f_i'} \cdot \omega_{\alpha_q'}(\{\emptyset\}, f_q').$$

Moreover the creation and annihilation operators can be written like

$$a^\sharp_{*_{\alpha \in J(F_\alpha)}, i} = \sum_\alpha a^\sharp_{F_\alpha, i}$$

with the relations [18],

$$a_{F_\alpha, i} \cdot a^*_{F_\beta, j} = 0$$

for $\alpha \neq \beta$.

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